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Geometric Bogomolov conjecture in arbitrary characteristics

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Contents

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1 Introduction

The goal of this paper is to prove the full geometric Bogomolov conjecture. We first reduce it to the case that the extension of the base fields has transcendence degree 1, and then we prove the later case by intersection theory in algebraic geometry. The proof uses Yamaki's reduction theorem on the geometric Bogomolov conjecture and the Manin–Mumford conjecture proved by Raynaud and Hrushovski.

1.1 Abelian varieties and heights

Let k be an algebraically closed field. Let K/k be a finitely generated field extension of transcendence degree $\mathrm{trdeg}(K/k) \geq 1$. Let A be an abelian variety over K of dimension g. Let L be a symmetric and ample line bundle over A. To define canonical heights of subvarieties of A, we need to choose integral models.

Integral models

A *projective model of* K/k is a normal projective variety S over k with function field K. It follows that $\dim S = \mathrm{trdeg}(K/k)$.

A *polarization of* K/k is a pair $(S, \mathcal{M})$ consisting of a projective model S of K/k and an ample line bundle $\mathcal{M}$ over S.

An *integral model of* (A, L) over S is a triple $(\mathcal{A}, \pi, \mathcal{L})$ where:

- $\mathcal{A}$ is a projective variety over k;
- $\pi : \mathcal{A} \to S$ is a projective morphism whose generic fiber is A;
- $\mathcal{L}$ is a line bundle on $\mathcal{A}$ extending L.

We usually abbreviate $(\mathcal{A}, \pi, \mathcal{L})$ as $(\mathcal{A}, \mathcal{L})$.

Canonical heights

Let X be any closed subvariety of A over K. The **naive height** of X with respect to the polarization $(S, \mathcal{M})$ and the integral model $(\mathcal{A}, \mathcal{L})$ is defined as

$$h^{\mathcal{M}}_{(\mathcal{A},\mathcal{L})}(X) := \frac{\mathcal{L}^{\dim X+1} \cdot (\pi^*\mathcal{M})^{\dim S-1} \cdot \mathcal{X}}{(1 + \dim X) \deg_L(X)}, \tag{1.1}$$

where $\mathcal{X}$ is the Zariski closure of X in $\mathcal{A}$, the numerator is the intersection number in $\mathcal{A}$, and $\deg_L(X) = L^{\dim X} \cdot X$ is the intersection number in A.

By Tate's limiting argument, the **canonical height** of X with respect to the polarization $(S, \mathcal{M})$ and the line bundle L over A is the limit

$$\hat{h}_L^{\mathcal{M}}(X) := \lim_{n\to+\infty} \frac{1}{n^2} h_{(\mathcal{A},\mathcal{L})}^{\mathcal{M}}([n]X). \tag{1.2}$$

Here $[n]X$ is the image of X under the multiplication map $[n] : A \to A$. The limit exists and depends on $(S, \mathcal{M})$ and L, but is independent of the integral model $(\mathcal{A}, \mathcal{L})$. Moreover, $\hat{h}_L^{\mathcal{M}}(X) \geq 0$ by choosing $\mathcal{L}$ to be ample in the definition. Our definition of the canonical height differs from that of Gubler [10] (and of Yamaki [34]) by the normalizing denominator $(1+\dim X)\deg_L(X)$, by applying Tate's limiting argument with [10, Theorem 3.4(e), Theorem 3.6(d), Remark 3.7].

The definition of the heights can be easily extended to closed subvarieties of $A_{\overline{K}} = A \otimes_K \overline{K}$. In fact, let Y be a closed subvariety of $A_{\overline{K}}$. Let X be the minimal closed subvariety of A such that $X_{\overline{K}}$ contains Y in $A_{\overline{K}}$, or equivalently X is the image of the composition $Y \to A_{\overline{K}} \to A$ as schemes. Then we simply define

$$h_{(\mathcal{A},\mathcal{L})}^{\mathcal{M}}(Y) := h_{(\mathcal{A},\mathcal{L})}^{\mathcal{M}}(X), \qquad \hat{h}_L^{\mathcal{M}}(Y) := \hat{h}_L^{\mathcal{M}}(X). \tag{1.3}$$

Remark 1.1 The definition is compatible with base change in the sense that if K'/K is a finite extension (contained in $\overline{K}$), then

$$h_{(\mathcal{A}',\mathcal{L}')}^{\mathcal{M}'}(Y) = [K' : K]\, h_{(\mathcal{A},\mathcal{L})}^{\mathcal{M}}(Y), \qquad \hat{h}_{L'}^{\mathcal{M}'}(Y) = [K' : K]\, \hat{h}_L^{\mathcal{M}}(Y).$$

Here $(A', L') = (A_{K'}, L_{K'})$, $S' \to S$ is the normalization of $\operatorname{Spec} K' \to S$, $\mathcal{A}'$ is the Zariski closure of A' in $\mathcal{A} \times_S S'$, $\mathcal{M}'$ is the pull-back of $\mathcal{M}$ via the finite morphism $S' \to S$, and $\mathcal{L}'$ is the pull-back of $\mathcal{L}$ via the finite morphism $\mathcal{A}' \to \mathcal{A}$. In fact, the equality for the canonical heights follows for that for the naive heights, and the latter follows from the projection formula for the finite morphism $\mathcal{X}' \to \mathcal{X}$, where $\mathcal{X}'$ (resp. $\mathcal{X}$) is the Zariski closure of the image of the composition $Y \to A_{\overline{K}} \to A_{K'} \to \mathcal{A}'$ (resp. $Y \to A_{\overline{K}} \to A \to \mathcal{A}$). Here the degree of the finite morphism $\mathcal{X}' \to \mathcal{X}$ is equal to that of the finite morphism $X' \to X$, where X' is the image of the composition $Y \to A_{\overline{K}} \to A_{K'}$, so it is equal to $[K' : K]\deg_{L'}(X')/\deg_L(X)$ by the projection formula again.

Taking K'/K to be a finite extension such that Y is defined over K', we see that our definition is compatible with that of [38, (2.5.3)] up to the normalizing denominator $(1 + \dim X)\deg_L(X)$.

For simplicity, we usually omit the dependence on L and $(S, \mathcal{M})$ in the heights, which are usually fixed, so we write

$$\hat{h}(X) = \hat{h}_L^{\mathcal{M}}(X), \qquad \hat{h}(Y) = \hat{h}_L^{\mathcal{M}}(Y).$$

In the special case of $\dim Y = 0$, we get the canonical height for points

$$\hat{h} = \hat{h}_L^{\mathcal{M}} : A(\overline{K}) \to [0, +\infty)$$

with respect to L and $(S, \mathcal{M})$.

Remark 1.2 Let $\hat{h}_{L'}^{\mathcal{M}'}$ be a height function defined by another choice $(L', S', \mathcal{M}')$. Then $c^{-1}\,\hat{h}_L^{\mathcal{M}} \leq \hat{h}_{L'}^{\mathcal{M}'} \leq c\,\hat{h}_L^{\mathcal{M}}$ for some rational number $c > 1$. We can use a few reductions to prove this. In the process, we allow $\mathcal{M}$ and $\mathcal{M}'$ to be nef and big (instead of ample) over S and S' respectively. First, we can reduce to the case $S = S'$ by replacing S and S' by an integral model of K/k dominating both S and S', and replacing $\mathcal{M}, \mathcal{M}'$ by their pull-back's to the new integral model. Second, we can reduce to the case $\mathcal{M} = \mathcal{M}'$ since there is a positive integer $a > 0$ such that $a\mathcal{M} - \mathcal{M}'$ and $a\mathcal{M}' - \mathcal{M}$ are both big over S. This follows from Siu's inequality (cf. [19, §2.2, Theorem 2.2.15]). In this process, we use the inequality $D_1 \ldots D_n \geq D_1' \ldots D_n'$ for nef divisors $D_1, \ldots, D_n$ and $D_1', \ldots, D_n'$ over a projective variety Z of dimension n such that $D_1 - D_1', \ldots, D_n - D_n'$ are effective divisors over Z. Note that the inequality follows from the chain

$$\begin{aligned} D_1 \cdot D_2 \cdots D_n &\geq D_1' \cdot D_2 \cdots D_n \geq D_1' \cdot D_2' \cdot D_3 \cdots D_n \geq \ldots \\ &\geq D_1' \cdot D_2' \cdots D_n'. \end{aligned}$$

Third, we can reduce to the case $L = L'$ because there is a positive integer $b > 0$ such that $bL - L'$ and $bL' - L$ are both ample over A.

In the special case of $\mathrm{trdeg}(K/k) = 1$, the situation is much easier, since the the projective model S is unique and the line bundle $\mathcal{M}$ is not used in the definition of the heights.

Small points and special subvarieties

For any subvariety X of $A_{\overline{K}}$ and any $\epsilon > 0$, we set

$$X(\epsilon) := \{x \in X(\overline{K}) |\ \hat{h}(x) < \epsilon\}. \tag{1.4}$$

We say that X *contains a dense set of small points of* $A/K/k$ if $X(\epsilon)$ is Zariski dense in X for all $\epsilon > 0$. This notion is actually independent of the choice of L

and $(S, \mathcal{M})$ to define the canonical height function $\hat{h}_L^{\mathcal{M}} : A(\overline{K}) \to [0, +\infty)$, since any two such heights bound each other up to positive multiples (cf. Remark 1.2).

Let $(A^{\overline{K}/k}, \mathrm{tr})$ be the $\overline{K}/k$-trace of $A_{\overline{K}}$; it is the final object of the category of pairs (C, f), where C is an abelian variety over k and f is a morphism from $C \otimes_k \overline{K}$ to $A_{\overline{K}}$ (cf. [17, §8.3] or [4, §6]). If char $k = 0$, tr is a closed immersion and $A^{\overline{K}/k} \otimes_k \overline{K}$ can be naturally viewed as an abelian subvariety of $A_{\overline{K}}$; if char $k > 0$, tr is a purely inseparable isogeny to its image.

A **torsion subvariety** of $A_{\overline{K}}$ is a translate $a + C$ of an abelian subvariety $C \subset A_{\overline{K}}$ by a torsion point a of $A_{\overline{K}}$. A subvariety X of $A_{\overline{K}}$ is said to be **special** if

$$X = \mathrm{tr}(Y \otimes_k \overline{K}) + T \tag{1.5}$$

for some torsion subvariety T of $A_{\overline{K}}$ and some subvariety Y of $A^{\overline{K}/k}$. When X is special, $X(\epsilon)$ is Zariski dense in X for all $\epsilon > 0$ ([18, Theorem 5.4, Chapter 6]).

1.2 Geometric Bogomolov conjecture

The goal of this paper is to prove the following theorem, which is known as the geometric Bogomolov conjecture.

Theorem 1.3 (Geometric Bogomolov conjecture) *Let k be an algebraically closed field. Let K/k be a finitely generated field extension of transcendence degree at least 1. Let A be an abelian variety over K. Let X be a closed subvariety of $A_{\overline{K}}$. If X contains a dense set of small points of $A/K/k$, then X is special.*

The above geometric Bogomolov conjecture was proposed by Yamaki [34, Conjecture 0.3], but particular instances of it were studied earlier by Gubler in [10]. It is an analog over function fields of the original Bogomolov conjecture over number fields which was proved by Ullmo [31] and Zhang [43].

Let us give a quick historical recall on results about the Bogomolov conjecture and its geometric version. The original Bogomolov conjecture over number fields was proved by Ullmo [31] and Zhang [43], where the major technique is the equidistribution theorem of Szpiro–Ullmo–Zhang [30]. The treatments are extended in terms of the Moriwaki height to finitely generated fields over number fields by Moriwaki [23].

For the geometric Bogomolov conjecture, it was first proved by Gubler [10] when A is totally degenerate at some place of K/k. Yamaki [35,38] reduced the geometric Bogomolov conjecture to the case of abelian varieties with good reduction everywhere and with a trivial trace; based on this reduction theorem, Yamaki [37] proved the conjecture for $\dim(X) = 1$ or $\mathrm{codim}(X) = 1$; he also proved the conjecture for abelian varieties of dimension 5, good reduction everywhere and with trivial trace in [36].

The works of Gubler [10] and Yamaki [35] are valid in arbitrary characteristics, and extend the strategy of Ullmo and Zhang applying equidistribution theorems. In fact, Gubler [10] considered tropical varieties of subvarieties of abelian varieties over non-archimedean fields, extended the equidistribution theorem of Szpiro–Ullmo–Zhang to the tropical setting, and studied the equilibrium measure in this setting. Yamaki [35] did more careful analysis of the situation using equidistribution over Berkovich spaces. The latter equidistribution theorem was proved by Gubler [11] and Faber [5], which generalized the equidistribution theorems over number fields of Szpiro–Ullmo–Zhang [30], Chambert-Loir [3], and Yuan [39].

Before Yamaki [37], various examples and partial results of the geometric Bogomolov conjecture with $\dim(X) = 1$ were previously obtained by [2,6,20–22,24,32,33]. In particular, Cinkir [2] proved the geometric Bogomolov conjecture for a curve embedded in its jacobian when $\mathrm{char}\, k = 0$ and $\mathrm{trdeg}(K/k) = 1$, based on a height identity of Zhang [44].

In the case $\mathrm{char}\, k = 0$, Gao–Habegger [8] proved the geometric Bogomolov conjecture for $\mathrm{trdeg}(K/k) = 1$. Recently Cantat–Gao–Habegger–Xie [1] proved the full geometric Bogomolov conjecture for $\mathrm{char}\, k = 0$. These proofs are based on the Betti map in the complex-analytic setting.

1.3 Plan of proof

Our proof of the geometric Bogomolov conjecture is based on Yamaki's reduction theorem, which reduces the conjecture to the case of good reduction and trivial trace. We also need the Manin–Mumford conjecture (in the case of trivial trace) proved by Raynaud and Hrushovski.

First, we reduce the conjecture to the case that K/k has transcendence degree 1. This is the main goal of Sect. 3. The idea is to take intermediate fields k' of K/k, algebraically closed in K and with transcendence degree 1 over k, such that the geometric Bogomolov conjecture for $(A/K/k, X)$ follows from those of $(A/K/k', X)$. For the construction, let $(S, \mathcal{M})$ be a polarization of K/k. Use $\mathcal{M}$ to get a pencil of hyperplane sections of S parametrized by $\mathbb{P}^1_k$. Take $k' = k(\mathbb{P}^1_k)$ to be the function field. There is a generic hyperplane section H over the generic point $\mathrm{Spec}\, k'$ of $\mathbb{P}^1_k$. In particular, $(H, \mathcal{M}|_H)$ is a polarization of K/k'. The process from $(S, \mathcal{M})$ to $(H, \mathcal{M}|_H)$ does not increase canonical heights of subvarieties of $A_{\overline{K}}$. Then we carry out careful analysis of the change of special subvarieties of A under this process.

We remark that a well-known procedure to reduce the transcendence degree is to take a closed hyperplane section of S over k. This process reduces K/k to K'/k (instead of our K/k'), and thus changes X and A by the corresponding reductions. Because of this, it is hard to track density of small points, and it is also hard to track the change of the trace of A. Our method does not change K and thus make everything trackable.

As a consequence, we can assume that K/k has transcendence degree 1. Then we apply Yamaki's reduction theorem. This reduces the problem to the essential case that K/k has transcendence degree 1, and A has good reduction over S and trivial K/k-trace. Here S is the unique smooth projective curve over k with function field K. Then $A \to \operatorname{Spec} K$ extends to an abelian scheme $\pi : \mathcal{A} \to S$.

The second step is to prove the Bogomolov conjecture in the essential case. Denote by $\mathcal{X}$ the Zariski closure of X in $\mathcal{A}$. Let $\mathcal{L}$ be a symmetric, relatively ample and rigidified line bundle over $\mathcal{A}$. Our key property is that any torsion multi-section $\mathcal{T}$ of $\mathcal{A} \to S$ is numerically equivalent to a multiple of the self-intersection $\mathcal{L}^{\dim A}$ in the Chow group of 1-cycles in $\mathcal{A}$. This is proved in Sect. 4.

Now we come to Sect. 5, which is the core of the proof of the essential case. To illustrate the idea, assume that there is a positive integer r such that the summation map $f : X^r \to A$ is surjective and generically finite. This happens, for example, if X is a curve and A is the Jacobian variety of X. We have a morphism $f : \mathcal{X}^r_{/S} \to \mathcal{A}$ over S. Note that $X^r \subset A^r$ and $\mathcal{X}^r_{/S} \subset \mathcal{A}^r_{/S}$. Moreover, X^r has canonical height 0 in A^r, by Zhang's fundamental inequality and the assumption that X has a dense set of small points.

Let $\mathcal{T}$ be a torsion multi-section of $\mathcal{A}$. Let us first assume that $\mathcal{T}' = f^{-1}(\mathcal{T})$ is finite and flat over S, and address the technical issue later. Denote by $h : \mathcal{A}^r_{/S} \to \mathcal{A}$ the summation morphism. For any symmetric, relatively ample and rigidified line bundle $\mathcal{L}_r$ over $\mathcal{A}^r_{/S}$, we have

$$[\mathcal{T}'] \cdot \mathcal{L}_r = [\mathcal{X}^r_{/S}] \cdot h^*[\mathcal{T}] \cdot \mathcal{L}_r = a\, [\mathcal{X}^r_{/S}] \cdot h^*(\mathcal{L}^{\dim A}) \cdot \mathcal{L}_r = 0.$$

Here $a > 0$ is a constant coming from the numerical equivalence mentioned above. The last equality follows from the fact that X^r has canonical height 0 in A^r. This implies that $T' = \mathcal{T}'_K$ has canonical height 0, and thus is a finite set of torsion points of A^r. When varying $\mathcal{T}$, we obtain a Zariski dense set of torsion points of X^r in A^r. By the Manin-Mumford conjecture, X^r is torsion, and thus X is torsion.

In general, we are not able to find r such that $f : X^r \to A$ is surjective and generically finite. But we can manage to find r such that f is surjective (up to replacing A by an abelian subvariety), and that the relative dimension e of f is strictly smaller than $\dim X$. If $\mathcal{T}' = f^{-1}(\mathcal{T})$ is flat over S of dimension

$e+1$, then the above process still implies that $T' = \mathcal{T}'_K$ has canonical height 0. Then we can conclude that T' is a torsion subvariety in A^r by induction, since $\dim T' < \dim X$. Torsion points of T' are also torsion points of X^r. Varying T', we obtain a dense set of torsion points of X^r in A^r. Then the Manin-Mumford conjecture implies that X is torsion.

In the above, we have made a technical assumption that $\mathcal{T}' = f^{-1}(\mathcal{T})$ is flat over S of dimension $e+1$. To remove this caveat, we will have to treat the case that $\mathcal{T}' = f^{-1}(\mathcal{T})$ has irreducible components of dimension bigger than the expected dimension. Note that it is easy to choose $\mathcal{T}$ so that the Zariski closure $\mathcal{T}^*$ of $T' = \mathcal{T}'_K$ in $\mathcal{T}'$ has the correct dimension. Then we prove that the difference $[\mathcal{X}^r_{/S}] \cdot h^*[\mathcal{T}] - [\mathcal{T}^*]$ is linearly equivalent to an effective Chow cycle of the correct dimension. It remedies the above argument. This is the content of Sect. 2, and was inspired by a result of Jia–Shibata–Xie–Zhang.

The idea of converting the Bogomolov conjecture to the Manin–Mumford conjecture was originally used by Zhang [40,41] in his proof of the Bogomolov conjecture for powers of $\mathbb{G}_m$. In particular, [41, Lemma 6.6] produced enough torsion hypersurfaces of X, and thus enough torsion points of X by induction. On the other hand, our situation is more complicated due to lack of torsion hypersufaces in A, and our solution is based on the crucial numerical identity between $\mathcal{T}$ and $\mathcal{L}^{\dim A}$. As these cycles are not of codimension 1, it also brings issues of non-proper intersection discussed above.

1.4 Notation and terminology

- For any field F, denote by $\overline{F}$ an algebraic closure.
- For a field extension K/k, denote by $\mathrm{trdeg}(K/k)$ the transcendence degree.
- A *variety* is an integral separated scheme of finite type over a field.
- For a Cartier divisor H on a scheme X, denote by $|H|$ the linear system associated to H.
- For an integral scheme X, denote by η_X its generic point.
- For a scheme X over a field, denote by X^{sm} its smooth locus.
- A *subvariety* of a variety is a closed integral subscheme.
- By a *line bundle* over a scheme, we mean an invertible sheaf over the scheme. We often write or mention tensor products of line bundles additively, so $aL - bM$ means $L^{\otimes a} \otimes M^{\otimes(-b)}$ for line bundles L, M and integers a, b.

2 Non-proper intersections

Let B be a smooth projective variety of dimension d over an algebraically closed field k. For a closed subvariety Z of B with $\dim Z = i$, denote by $[Z]$ its class in the Chow group $\mathrm{CH}_i(B)$. For $\alpha \in \mathrm{CH}_i(B)$, write $\alpha \geq 0$ if α can be represented by an effective i-cycle. The goal of this section is to prove the following technical result.

Proposition 2.1 *Let $g : B \to Y$ be a flat morphism between smooth projective varieties. Let X be a closed subvariety of B such that $f = g|_X : X \to Y$ is surjective. Denote $e = \dim X - \dim Y$. Let V be a closed subvariety of Y which is not contained in*

$$Y_{e+1} = \{y \in Y | \ \dim f^{-1}(y) \geq e + 1\}.$$

Let $Z_1, \ldots, Z_n$ be all irreducible components of $f^{-1}(V)$ satisfying $f(Z_i) = V$. Then $\dim Z_i = \dim V + e$ for $i = 1, \ldots, n$.

Assume furthermore that $V \cap Y_{e+1}$ is finite and contained in V^{sm}. Then we have

$$\sum_{i=1}^{n} m_i [Z_i] \leq g^*[V] \cdot [X]$$

in $\mathrm{CH}_{\dim V+e}(B)$. Here m_i is the multiplicity of Z_i in the scheme theoretical preimage $f^{-1}(V)$.

Note that if X is smooth, then the last inequality can be simplified as the inequality

$$\sum_{i=1}^{n} m_i [Z_i] \leq f^*[V]$$

in $\mathrm{CH}_{\dim V+e}(X)$.

The proposition will be used in Sect. 5. The remaining part of this section is to prove the proposition. Readers might skip the proof temporarily and move to the next section at the first time of reading this paper.

The idea to prove the proposition is to use complete intersections of hyperplane sections to bound the proper part of the intersection. See Proposition 2.4 for the result on complete intersection. In the following, we will start with a Bertini type of result to choose suitable hyperplane sections.

2.1 A Bertini type of result

Proposition 2.2 *Let Y be a projective variety over an algebraically closed field k. Let V be a closed subvariety of Y of codimension $r \geq 1$ such that Y is smooth at η_V. Let $Z_1, \ldots, Z_m$ be irreducible subvarieties of Y. Assume that*

- $(\cup_{i=1}^{m} Z_i) \cap V$ *is finite;*

- $(\cup_{i=1}^m Z_i) \cap V \subseteq Y^{\mathrm{sm}} \cap V^{\mathrm{sm}}$.

Let H be a Cartier divisor on Y such that $\mathcal{O}_Y(H) \otimes I_V$ is generated by global sections, where $I_V \subset \mathcal{O}_Y$ is the ideal sheaf associated to V. Every non-zero section $s \in H^0(\mathcal{O}_Y(H) \otimes I_V)$ defines a divisor $H_s \in |H|$ via the inclusion $H^0(\mathcal{O}_Y(H) \otimes I_V) \hookrightarrow H^0(\mathcal{O}_Y(H))$. Then for general elements

$$(s_1, \ldots, s_r) \in H^0(\mathcal{O}_Y(H) \otimes I_V)^r,$$

the following holds:

(i) $H_{s_1} \cap \cdots \cap H_{s_r}$ *is a proper intersection in* Y*;*
(ii) $H_{s_1} \cdots H_{s_r} = V + W$ *where* W *is an effective* $(\dim Y - r)$*-cycle such that* $V \not\subseteq \operatorname{Supp} W$*, and* $W \cap Z_i$ *is a proper intersection for every* $i = 1, \ldots, m$.

Proof Set $H_i := H_{s_i}$ for $i = 1, \ldots, r$. Because $\mathcal{O}_Y(H) \otimes I_V$ is generated by global sections and $s_1, \ldots, s_r$ are general in $H^0(\mathcal{O}_Y(H) \otimes I_V)$, $(H_1 \backslash V) \cap \cdots \cap (H_r \backslash V)$ is a proper intersection in $Y \backslash V$, and $(H_1 \backslash V) \cap \cdots \cap (H_r \backslash V) \cap (Z_i \backslash V)$ is proper intersection in $Y \backslash V$ for every $i = 1, \ldots, m$.

As $V \subseteq H_i$, the intersection

$$H_1 \cap \cdots \cap H_r = V \cup ((H_1 \backslash V) \cap \cdots \cap (H_r \backslash V))$$

is of pure codimension r in Y. So $H_1 \cap \cdots \cap H_r$ is a proper intersection. Write $H_1 \cdots H_r = V + W$ where W is an effective $(\dim Y - r)$-cycle.

We fix a point $x_V \in Y^{\mathrm{sm}}(k) \cap V^{\mathrm{sm}}(k)$ before we choose the general sections $s_1, \ldots, s_r$ and set $B := (V \cap (\cup_{i=1}^m Z_i)) \cup \{x_V\}$. It is a finite subset of $Y^{\mathrm{sm}}(k) \cap V^{\mathrm{sm}}(k)$. For every point $x \in B$, fix an isomorphism $\phi_x : (\mathcal{O}_Y(H) \otimes I_V)_x \to I_{V,x}$. Because $x \in (V^{\mathrm{sm}} \cap Y^{\mathrm{sm}})(k)$, by the Jacobi criterion for smoothness, $I_{V,x}/I_{V,x}I_x$ is a k-vector space of dimension r. Because $\mathcal{O}_Y(H) \otimes I_V$ is generated by global sections and $s_1, \ldots, s_r$ are general in $H^0(\mathcal{O}_Y(H) \otimes I_V)$, $\phi_x(s_1), \ldots, \phi_x(s_r)$ is a k-basis of $I_{V,x}/I_{V,x}I_x$. Then $\phi_x(s_1), \ldots, \phi_x(s_r)$ generate $I_{V,x}$. This implies that at every point $x \in B$, there is an open neighborhood U_x of x such that $H_1 \cap \cdots \cap H_r \cap U_x = V \cap U_x$ in the sense of schemes. Hence $(V + W)|_{U_x} = (H_1 \cdots H_r)|_{U_x} = V|_{U_x}$. It follows that $W \cap U_x = \emptyset$. Hence $V \not\subseteq \operatorname{Supp} W$, and $W \cap B = \emptyset$. Because $V \cap (\cup_{i=1}^m Z_i) \subseteq B$ for every $i = 1, \ldots, m$, the intersection

$$W \cap Z_i = W \cap (Z_i \setminus V) = (H_1 \setminus V) \cap \cdots \cap (H_r \setminus V) \cap (Z_i \setminus V)$$

is a proper intersection. □

2.2 Proper part of an intersection

Lemma 2.3 *Let B be a smooth projective variety of dimension d over an algebraically closed field k. Let H be a Cartier divisor on B such that $\mathcal{O}_B(H)$ is globally generated. Then for $\alpha \in \mathrm{CH}_i(B)$ with $\alpha \geq 0$, $\alpha \cdot [H] \geq 0$.*

Proof We may write $\alpha = [Z]$ for an effective i-cycle Z. We can further assume that Z is an integral subvariety of B. Because $\mathcal{O}_B(H)$ is globally generated, after replacing H by a general element in $|H|$, we may assume that $H \cap Z$ is a proper intersection. Then $\alpha \cdot [H] = [Z] \cdot [H] \geq 0$. □

Let B be a smooth projective variety of dimension d over an algebraically closed field k. Let $H_1, \ldots, H_m$ be effective Cartier divisors of B. Let X be a subvariety of B. A *proper component* Z of $H_1 \cap \cdots \cap H_m \cap X$ is an irreducible component of the underlying topological space of $H_1 \cap \cdots \cap H_m \cap X$ of dimension $\dim X - m$. Write $m(Z, H_1 \cap \cdots \cap H_m \cap X)$ for the multiplicity of Z in $H_1 \cap \cdots \cap H_m \cap X$, as defined by Serre using the tor-functor. Define

$$(H_1 \cdots H_m \cdot X)^{\mathrm{prop}} := \sum m(Z, H_1 \cap \cdots \cap H_m \cap X)[Z],$$

where the sum is taken over all proper components Z of $H_1 \cap \cdots \cap H_m \cap X$.

The following result is a generalization of [16, Lemma 3.3] with a similar proof.

Proposition 2.4 *Let $H_1, \ldots, H_r$ be effective Cartier divisors on B such that $\mathcal{O}_B(H_1), \ldots, \mathcal{O}_B(H_r)$ are generated by global sections. Let X be a subvariety of B. Then we have*

$$(H_1 \cdots H_r \cdot X)^{\mathrm{prop}} \leq H_1 \cdots H_r \cdot [X].$$

Proof Let $V_1, \ldots, V_s$ be the proper components of $H_1 \cap \cdots \cap H_r \cap X$ with multiplicities $m_1, \ldots, m_s$. For $i = 1, \ldots, s$, set $\eta_i := \eta_{V_i}$. Then $H_1 \cap \cdots \cap H_r \cap X$ has proper intersection at $\eta_1, \ldots, \eta_s$.

Let $X_1, \ldots, X_l$ be all irreducible components of $H_1 \cap \cdots \cap H_{r-1} \cap X$ passing through $\eta_1, \ldots, \eta_s$. For $j = 1, \ldots, l$, the variety X_j has dimension $\dim X - r + 1$ and $X_j \not\subseteq H_r$. Assume that X_j has multiplicity n_j in $H_1 \cap \cdots \cap H_{r-1} \cap X$.

If $r = 1$, by Lemma 2.3, this proposition is trivial. Now assume that $r \geq 2$. By induction,

$$\sum_{j=1}^{l} n_j[X_j] \leq H_1 \cdots H_{r-1} \cdot [X].$$

The previous paragraph shows that $(\sum_{j=1}^{l} n_j X_j) \cap H_r$ is a proper intersection and m_i is the multiplicity of V_i in $(\sum_{i=1}^{l} n_j X_j) \cap H_r$. By Lemma 2.3, we have

$$\begin{aligned}(H_1 \cdots H_r \cdot X)^{\text{prop}} = \sum_{i=1}^{s} m_i V_i &\leq \left(\sum_{j=1}^{l} n_j[X_j]\right) \cdot H_r \\ &\leq H_1 \cdots H_{r-1} \cdot [X] \cdot H_r.\end{aligned}$$

□

2.3 Strict transform

Let $f : X \to Y$ be a surjective morphism of projective varieties over an algebraically closed field k. Set $e := \dim X - \dim Y$.

For every integer $l \geq e$,

$$Y_l := \{y \in Y |\ \dim f^{-1}(y) \geq l\}$$

is a closed subset of Y. We have $Y_{l+1} \subseteq Y_l$. We note that $Y_e = Y$ and for $l \geq e + 1$, $\dim Y_l + l \leq \dim X - 1$. In particular, for $l \geq \dim X$, $Y_l = \emptyset$.

Lemma 2.5 *Let W be a subvariety of Y. Assume that $W \cap Y_l$ is a proper intersection for every $l \geq e + 1$. Then*

$$\dim f^{-1}(W) = \dim W + e.$$

Moreover, for every irreducible component Z of $f^{-1}(W)$ with $\dim Z = \dim f^{-1}(W)$, *we have $f(Z) = W$.*

Proof Write $W = \sqcup_{e \leq l \leq \dim X} W \cap (Y_l \backslash Y_{l+1})$. For every $l \geq e + 1$, if $W \cap (Y_l \backslash Y_{l+1}) \neq \emptyset$,

$$\begin{aligned}\dim W \cap (Y_l \backslash Y_{l+1}) &\leq \dim W \cap Y_l = \dim W + \dim Y_l - \dim Y \\ &\leq \dim W + \dim X - l - 1 - \dim Y = \dim W + e - l - 1.\end{aligned}$$

So for $l \geq e+1$, if $W \cap (Y_l \backslash Y_{l+1}) \neq \emptyset$,

$$\dim f^{-1}(W \cap (Y_l \backslash Y_{l+1})) \leq \dim W + e - 1.$$

Because $W \cap Y_{e+1}$ is a proper intersection, we have $W \backslash Y_{e+1} \neq \emptyset$. Then

$$\dim f^{-1}(W \backslash Y_{e+1}) = \dim W + e.$$

So we have

$$\dim f^{-1}(W) = \dim W + e.$$

Let Z be an irreducible component of $f^{-1}(W)$ of $\dim Z = \dim f^{-1}(W) = \dim W + e$. Then $\dim f(Z) \geq \dim Z - e = \dim W$. So $f(Z) = W$. $\square$

Proof of Proposition 2.1 Set $r := \dim Y - \dim V$. If $r = 0$, then $V = Y$ and $f^{-1}(V) = X$. In this case Proposition 2.1 trivially holds. Now assume that $r \geq 1$. For every $i = 1, \ldots, n$, since $f(Z_i) = V$ and Z_i is irreducible, $\dim Z_i = \dim(Z_i \backslash f^{-1}(Y_{e+1}))$. Since $f^{-1}(V \backslash Y_{e+1})$ is of pure dimension $e + \dim V$, $\dim Z_i = e + \dim V$.

By Proposition 2.2, there are effective and very ample divisors $H_1, \ldots, H_r$ on Y such that $H_1 \cap \cdots \cap H_r$ is a proper intersection and

$$H_1 \cdots H_r = V + W$$

where W is an effective $(\dim Y - r)$-cycle such that $V \not\subseteq \operatorname{Supp} W$, and such that $W \cap Y_l$ is a proper intersection for $l \geq e+1$. We have

$$g^*[V] + g^*[W] = [g^* H_1 \cdots g^* H_r].$$

By Lemma 2.5, $\dim f^{-1}(W) = \dim V + e$. Then $f^{-1}(W) = g^{-1}(W) \cap X$ is a proper intersection. Since B is smooth, $f^{-1}(W)$ is equidimensional. By Lemma 2.5 again, for every irreducible component R of $f^{-1}W$, the image $f(R)$ is an irreducible component of W.

We claim that $[f^{-1}W] = g^*[W] \cdot [X]$ as algebraic cycles over B. In fact, it suffices to prove the restriction of the equality to $U = B \backslash g^{-1}(V)$. Over $Y \backslash V$, W is the proper intersection $H_1 \cap \cdots \cap H_r$. Then the result follows by the fact that the intersection multiplicities are given by length of the local rings. This can be obtained by the vanishing of the higher tor-functors in Serre's intersection formula, or as an example of [7, Proposition 7.1 (b)].

Similarly, we have the Zariski closure

$$[\overline{f^{-1}(V) \cap f^{-1}(Y \backslash Y_{e+1})}] = \sum_{i=1}^{n} m_i [Z_i]$$

as algebraic cycles over B. The sum gives

$$[\overline{f^{-1}(H_1 \cap \cdots \cap H_r) \cap f^{-1}(Y \backslash Y_{e+1})}] = \sum_{i=1}^{n} m_i[Z_i] + g^*[W] \cdot [X].$$

Note that every irreducible component of $f^{-1}(H_1 \cap \cdots \cap H_r) \cap f^{-1}(Y \backslash Y_{e+1})$ has dimension $\dim V + e$. So we have

$$[\overline{f^{-1}(H_1 \cap \cdots \cap H_r) \cap f^{-1}(Y \backslash Y_{e+1})}] \leq (g^* H_1 \cdots g^* H_r \cdot X)^{\text{prop}}$$

in $\mathrm{CH}_{\dim V+e}(B)$.

Finally, by Proposition 2.4, since $\mathcal{O}_B(g^* H_i)$ for $i = 1, \ldots, r$ are generated by global sections, we get

$$(g^* H_1 \cdots g^* H_r \cdot X)^{\text{prop}} \leq g^* H_1 \cdots g^* H_r \cdot [X].$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n} m_i[Z_i] + g^*[W] \cdot [X] &\leq g^* H_1 \cdots g^* H_r \cdot [X] \\ &= g^*[V] \cdot X + g^*[W] \cdot [X]. \end{aligned}$$

This concludes the proof. □

3 Lowering the transcendence degree

The geometric Bogomolov conjecture concerns a finitely generated extension K/k. The goal of this section is to lower $\operatorname{trdeg}(K/k)$ to 1 in the conjecture. The main result of this section is as follows.

Proposition 3.1 *Let k be an algebraically closed field. Let K/k be a finitely generated field extension of transcendence degree at least 2. Let A be an abelian variety over K. Then there are two intermediate fields k_1, k_2 of K/k, algebraically closed in K and of transcendence degree 1 over k, such that for any closed subvariety X of $A_{\overline{K}}$, the geometric Bogomolov conjecture holds for $(A/K/k, X)$ if the geometric Bogomolov conjecture holds for $(A_{K\overline{k}_1}/K\overline{k}_1/\overline{k}_1, X)$ and $(A_{K\overline{k}_2}/K\overline{k}_2/\overline{k}_2, X)$.*

Note that the intermediate fields k_1, k_2 do not depend on the subvariety X. Applying the theorem repeatedly, we reduce the geometric Bogomolov conjecture to the case $\operatorname{trdeg}(K/k) = 1$.

3.1 Field of definition

For two abelian schemes A_1, A_2 over a base scheme, denote $A_1 \sim A_2$ if A_1 is isogenous to A_2. For an abelian variety over a field K and a subfield k of K, we say A is defined over k up to isogeny if there is an abelian variety A' over k such that $A \sim A' \otimes_k K$.

Proposition 3.2 *Let k be an algebraically closed field. Let K be a field extension of k with $\mathrm{trdeg}(K/k) < \infty$. Let k_1, k_2 be algebraically closed intermediate fields of K/k with $k_1 \cap k_2 = k$. Let A_1, A_2 be abelian varieties over k_1, k_2 respectively. If $A_1 \otimes_{k_1} K \sim A_2 \otimes_{k_2} K$, then there is an abelian variety A over k, such that $A_1 \sim A \otimes_k k_1$ and $A_2 \sim A \otimes_k k_2$. Moreover, the abelian variety A is unique up to isogeny.*

For the uniqueness in the proposition, we have the following result.

Lemma 3.3 *Let A_1, A_2 be two abelian varieties over an algebraically closed field k. Let K be any field extension of k. If $A_1 \otimes_k K \sim A_2 \otimes_k K$, then $A_1 \sim A_2$.*

Proof There is an isogeny $\Phi : A_1 \otimes_k K \to A_2 \otimes_k K$ over K. There is a subfield K' of K, finitely generated over k, such that Φ is defined over K'. After replacing K by K', we may assume that K is a finitely generated extension over k. There is a k-variety S such that $K = k(S)$. After shrinking S, we may assume that there is an isogeny $\Phi_S : A_1 \times_k S \to A_2 \times_k S$ over S such that Φ is the generic fiber of Φ_S (cf. [25, Theorem 3.2.1]) Pick a point $b \in S(k)$. The restriction of Φ_S to the fiber at b induces an isogeny $\Phi_b : A_1 \to A_2$, which concludes the proof. □

Proof of Proposition 3.2 We have noted that Lemma 3.3 implies the uniqueness of A. For the existence, we only need to show that both A_1 and A_2 are defined over k up to isogeny. Indeed, if $A_1 \sim A_1' \otimes_k k_1$ and $A_2 \sim A_2' \otimes_k k_2$ for abelian varieties A_1', A_2' over k, then $A_1' \otimes_k K \sim A_2' \otimes_k K$, which implies $A_1' \sim A_2'$ by Lemma 3.3 again.

Now we prove that A_1 and A_2 are defined over k up to isogeny. By Lemma 3.3, there is an isogeny $A_1 \otimes_{k_1} K \sim A_2 \otimes_{k_2} K$ defined over the algebraic closure of $k_1 k_2$. Thus we may assume that $K = \overline{k_1 k_2}$.

For $i = 1, 2$, there is a k-variety S_i, an abelian scheme $\pi : \mathcal{A}_i \to S_i$ such that $\overline{k(S_i)} = k_i$ and $A_i \to \mathrm{Spec}\, k_i$ is the geometric generic fiber of $\pi : \mathcal{A}_i \to S_i$.

Because $A_1 \otimes_{k_1} K \sim A_2 \otimes_{k_2} K$ and $K = \overline{k(S_1)k(S_2)}$, there is k-variety U, a flat and quasi-finite morphism $\psi : U \to S_1 \times S_2$, and an isogeny

$$\Phi_U : \mathcal{A}_{1,U} = \mathcal{A}_1 \times_{S_1} U \longrightarrow \mathcal{A}_{2,U} = \mathcal{A}_2 \times_{S_2} U.$$

Pick a point $s = (s_1, s_2) \in \psi(U)(k) \subseteq S_1 \times S_2$. Consider $S_1 \times s_2 \subseteq S_1 \times S_2$. Pick an irreducible component V of $\psi^{-1}(S_1 \times s_2) \subseteq U$. Then Φ_U induces an isogeny

$$\Phi_V = \Phi_U \times_U V : \mathcal{A}_{1,U} \times_U V \longrightarrow \mathcal{A}_{2,U} \times_U V.$$

The isomorphism $S_1 \simeq S_1 \times s_2$ induces an isomorphism

$$\mathcal{A}_{1,U} \times_U V \simeq \mathcal{A}_1 \times_{S_1} V. \tag{3.1}$$

On the other hand,

$$\mathcal{A}_{2,U} \times_U V \simeq \mathcal{A}_{2,s_2} \times V, \tag{3.2}$$

where $\mathcal{A}_{2,s_2} := \mathcal{A}_2 \times_{S_2} s_2$ is an abelian variety over k. Because $k(V)/k(S_1)$ is finite, taking the geometric generic fibers in (3.1) and (3.2), we get

$$A_1 \sim \mathcal{A}_{s_2} \otimes_k k_1.$$

Thus A_1 is defined over k up to isogeny. By symmetry, A_2 is defined over k up to isogeny. This concludes the proof. □

Lemma 3.4 *Let k_1, k_2 be two intermediate fields of a field extension K/k such that $k_1 \cap k_2 = k$. Let Y be a variety over k. Let X be a closed subvariety of $Y_K = Y \otimes_k K$ over K. Assume that as a subvariety of Y_K, X is defined over k_1 and also defined over k_2. Then X is defined over k.*

Proof By taking an open affine cover of Y, it suffices to treat the case that $Y = \operatorname{Spec} R_0$ is affine. Denote by I the ideal of $R_{0,K} = R_0 \otimes_k K$ associated to X. We only need to show $I = I_0 \otimes_k K$ for $I_0 := I \cap R_0$. Here we identify R_0 with its image in $R_{0,K} = R_0 \otimes_k K$.

This converts to a basic result in linear algebra. Namely, let K, k, k_1, k_2 be as before, let V_0 be a vector space over k, and let $V = V_0 \otimes_k K$ be the vector space over K. Let W be a K-subspace of V. Assume that W can be descended to k_i for $i = 1, 2$; i.e., $W = W_i \otimes_{k_i} K$ for a k_i-subspace W_i of $V_0 \otimes_k k_i$. Then W can be descended to k.

If V_0 is finite-dimensional, the result is an easy consequence of the existence of the Grassmannian variety, but we give an elementary proof as follows. Write $n = \dim_k V_0$ and $m = \dim_K W$. Take a k-basis of V_0, and use it to identify $V_0 = k^n$ and $V = K^n$. Then W is represented by a $m \times n$ matrix with coefficients in K. In fact, take a basis of W, each element of which gives a row of the matrix. By row operations, we can convert the matrix to a reduced row echelon form E over K. We can also get a reduced row echelon form E_i over k_i for $W_i \subset k_i^n$. By the uniqueness of the reduced row echelon from, we have

$E = E_1 = E_2$. Then the coefficients of E are in $k_1 \cap k_2 = k$. It follows that W can be descended to k.

If V_0 is infinite-dimensional, let V_0' be a finite-dimensional subspace over k and write $V' = V_0' \otimes_k K$. Apply the above results to the subspace $W \cap V'$ of V'. We see that $W \cap V'$ can be descended to k. Vary V_0'. Note that any vector of W is contained in some V'. It follows that W is contained in the K-subspace of V spanned by $W \cap V_0$. Then W can be descended to k. This concludes the proof. □

Corollary 3.5 *Let K/k be an extension of algebraically closed fields.*

(i) *Let A be an abelian variety over K. Then there is a unique algebraically closed intermediate field k_A of K/k, such that for every algebraically closed intermediate field k' of K/k, A is defined over k' up to isogeny if and only if $k_A \subseteq k'$. Moreover, we have* $\mathrm{trdeg}(k_A/k) < \infty$.

(ii) *Let Y be a variety over k. Let X be a subvariety of $Y_K = Y \otimes_k K$. Then there is a unique algebraically closed intermediate field $k_{X \subseteq Y_K}$ of K/k, such that for every algebraically closed intermediate field k' of K/k, $X \subseteq Y_K$ is defined over k' if and only if $k_{X \subseteq Y_K} \subseteq k'$. Moreover, we have* $\mathrm{trdeg}(k_{X \subseteq Y_K}/k) < \infty$.

Proof We only prove (i). The proof for (ii) is almost the same, except replacing Proposition 3.2 by Lemma 3.4.

Let I be the set of algebraically closed intermediate fields k' of K/k such that A is defined over k' up to isogeny. Because A is of finite type, for every $k' \in I$, there is $k'' \in I$ contained in k' and satisfying $\mathrm{trdeg}(k''/k) < \infty$.

So there is $k_A \in I$, such that $\mathrm{trdeg}(k_A/k)$ is the smallest. It is clear that for every algebraically closed intermediate field k' of K/k, if $k_A \subseteq k'$, $k' \in I$. So we only need to show that for every $k' \in I$, $k_A \subseteq k'$. One may assume that $\mathrm{trdeg}(k'/k) < \infty$. By Proposition 3.2, $k' \cap k_A \in I$. This proves $k_A \subseteq k'$ because $\mathrm{trdeg}(k_A/k)$ is the smallest. □

3.2 Special subvarieties of abelian varieties

Let K/k be a field extension such that k is algebraically closed in K. Denote by $I(K/k)$ the set of intermediate fields k' of K/k which are algebraically closed in K. If $\mathrm{trdeg}(K/k) > 1$, then $I(K/k)$ is infinite.

Let A be an abelian variety over K. Recall that a subvariety X of $A_{\overline{K}}$ is said to be special in $A/K/k$ if

$$X = \mathrm{tr}(Y \otimes_{\bar{k}} \overline{K}) + T \tag{3.3}$$

for some torsion subvariety T of $A_{\overline{K}}$ and some subvariety Y of $A^{\overline{K}/\bar{k}}$.

Proposition 3.6 *Let K/k be a field extension such that k is algebraically closed in K. Let A be an abelian variety over K. Let X be a subvariety of $A_{\overline{K}}$. Then there is $k_{A,X} \in I(K/k)$ such that for every $k' \in I(K/k)$, X is special for $A/K/k'$ if and only if $k_{A,X} \subseteq k'$.*

Proof There is a bijection $\phi : I(\overline{K}/\overline{k}) \to I(K/k)$ sending k' to $k' \cap K$. Because ϕ preserves the ordering and for $k' \in I(\overline{K}/\overline{k})$, X is special for $A_{\overline{K}}/\overline{K}/k'$ if and only if X is special for $A/K/\phi(k')$, we may assume that k and K are algebraically closed.

Let $\mathrm{Stab}^0(X)$ be the identity component of the closed subgroup scheme

$$\mathrm{Stab}(X) := \{g \in A|\ g + X = X\}.$$

We note that, for every $k' \in I(K/k)$, X is special for $A/K/k'$ if and only if $X/\mathrm{Stab}^0(X)$ is special for $(A/\mathrm{Stab}^0(X))/K/k'$, where $X/\mathrm{Stab}^0(X)$ is the image of X under the quotient morphism $A \to A/\mathrm{Stab}^0(X)$. After replacing (A, X) by $(A/\mathrm{Stab}^0(X), X/\mathrm{Stab}^0(X))$, we may assume that $\mathrm{Stab}^0(X) = 0$.

Let T_X be the minimal torsion subvariety of A containing X. Let a be a torsion point in T_X, so that $T_X - a$ is an abelian subvariety of A. Then for every $k' \in I(K/k)$, X is special for $A/K/k'$ if and only if $X - a$ is special for $(T_X - a)/K/k'$. After replacing (A, X) by $(T_X - a, X - a)$, we may assume that $T_X = A$.

By Corollary 3.5(i), there is $k_A \in I(K/k)$ such that for any $k' \in I(K/k)$, $\mathrm{tr}(A^{K/k'})_K = A$ if and only if $k_A \subseteq k'$. Since $\mathrm{Stab}^0(X) = 0$ and $T_X = A$, if X is special for $A/K/k'$ then $k_A \subseteq k'$.

After replacing k by k_A, we may assume that $\mathrm{tr}(A^{K/k})_K = A$. Pick an isogeny $\Phi : A \to (A^{K/k})_K$. Then for every $k' \in I(K/k)$, X is special for $A/K/k'$ if and only if $\Phi(X)$ is special for $(A^{K/k'})_K/K/k'$. Moreover, we still have $\mathrm{Stab}^0(\Phi(X)) = 0$ and $T_{\Phi(X)} = (A^{K/k})_K$. After replacing (A, X) by $((A^{K/k})_K, \Phi(X))$, we may assume that A is defined over k. In this case, for any $k' \in I(K/k)$, X is special in $A/K/k'$ if and only if X as a subvariety of A is defined over k'. By Corollary 3.5(ii), there is $k_{A,X} := k_{X \subseteq A} \in I(K/k)$ such that for every $k' \in I(K/k)$, X is special in $A/K/k'$ if and only if $k_{A,X} \subseteq k'$. □

3.3 Proof of Proposition 3.1

We start with the following general result.

Lemma 3.7 *Let k be an algebraically closed field. Let K/k be a finitely generated extension of transcendence degree* $\mathrm{trdeg}(K/k) > 1$. *Let $(S, \mathcal{M})$ be a polarization of K/k. Assume that $\mathcal{M}$ is very ample over S.*

Then there are infinitely many intermediate fields k' of K/k, algebraically closed in K and of transcendence degree 1 over k, together with a geometrically integral subvariety H in $S_{k'}$ of codimension 1 satisfying the following properties:

(1) *the divisor H of $S_{k'}$ is linearly equivalent to $\mathcal{M}_{k'}$;*
(2) *the composition $H \to S_{k'} \to S$ induces an isomorphism between the function fields of H and S.*

As a consequence of (1) and (2), the pair (k', H) satisfies the following property. Let A be any abelian variety over K, and let L be any symmetric and ample line bundle over A. Then under the polarization $(S, \mathcal{M})$ of K/k and the polarization of $(H, \mathcal{M}_{k'}|_H)$ of K/k', we have the inequality

$$\hat{h}_L^{\mathcal{M}}(X) \geq \hat{h}_L^{\mathcal{M}_{k'}|_H}(X)$$

of canonical heights for any subvariety X of $A_{\overline{K}}$.

Proof By a Bertini type of theorem (cf. [15, Theorem 6.10]), for a general section $s_0 \in \Gamma(S, \mathcal{M})$, $H_0 = \mathrm{div}(s_0)$ is geometrically integral. Set $H_1 = \mathrm{div}(s_1)$ for some $s_1 \in \Gamma(S, \mathcal{M})$ which is not a multiple of s_0. Then s_0 and s_1 determine a plane in $\Gamma(S, \mathcal{M})$, and thus a pencil in S with base locus $H_0 \cap H_1$. This gives a rational map $S \dashrightarrow \mathbb{P}^1_k$ sending x to $(s_0(x), s_1(x))$. Let $\pi : \widetilde{S} \to S$ be the blowing-up along $H_0 \cap H_1$. Then the rational map becomes a flat morphism $\psi : \widetilde{S} \to \mathbb{P}^1_k$ as in [13, II, Example 7.17.3].

For any point $(a_0, a_1) \in \mathbb{P}^1_k(k)$, the fiber $\psi^{-1}(a_0, a_1) \subset \widetilde{S}$ is isomorphic to the hyperplane section $\mathrm{div}(a_0 s_1 - a_1 s_0)$ of S. In fact, $\psi^{-1}(a_0, a_1) \subset \widetilde{S}$ is the strict transform of $\mathrm{div}(a_0 s_1 - a_1 s_0)$ in $\widetilde{S}$, and thus isomorphic to the blowing-up of $\mathrm{div}(a_0 s_1 - a_1 s_0)$ along $H_0 \cap H_1$. Note that $H_0 \cap H_1$ is a Cartier divisor in $\mathrm{div}(a_0 s_1 - a_1 s_0)$, by $H_0 \cap H_1 = \mathrm{div}(a_0 s_1 - a_1 s_0) \cap \mathrm{div}(s_0)$ if $a_0 \neq 0$ and $H_0 \cap H_1 = \mathrm{div}(a_0 s_1 - a_1 s_0) \cap \mathrm{div}(s_1)$ if $a_1 \neq 0$. Then the blowing-up of $\mathrm{div}(a_0 s_1 - a_1 s_0)$ along $H_0 \cap H_1$ does not change $\mathrm{div}(a_0 s_1 - a_1 s_0)$.

As a consequence, H_0 and H_1 are closed fibers of ψ. As H_0 is geometrically integral, the generic fiber $H' \to \mathrm{Spec}\, k'$ of $\psi : \widetilde{S} \to \mathbb{P}^1_k$ is geometrically integral, which follows from a general property in [9, 3, Theorem 12.2.4(viii)]. Here $k' = k(\mathbb{P}^1_k)$ is set to be the function field of $\mathbb{P}^1_k$, and identified with a subfield of K via ψ. Because $H' \to \mathrm{Spec}\, k'$ is geometrically integral, k' is algebraically closed in K.

The field k' above depends on the choice of sections s_0, s_1 in the following way. For another choice of sections $s_{0,1}, s_{1,1}$, let k'_1 be the field associated to them. Then $k' \neq k'_1$ if $V = ks_0 + ks_1$ and $V_1 = ks_{0,1} + ks_{1,1}$ are different subspaces of $\Gamma(S, \mathcal{M})$. Because k is algebraically closed, there are infinitely many choices of sections $s_{0,n}, s_{1,n},\ n \geq 1$ such that the spaces $V_n = ks_{0,n} +$

$ks_{1,n}$, $n \geq 1$ of $\Gamma(S, \mathcal{M})$ are pairwise distinct. Hence we get infinitely many choices of k'.

Let H be the image of H' under the morphism $\pi \times \psi : \widetilde{S} \to S \times \mathbb{P}^1_k$. Let $\pi_1 : S \times \mathbb{P}^1_k \to S$ be the projection to the first factor and $\pi_2 : S \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the projection to the second factor. Then $H \subseteq \pi_2^{-1}(\operatorname{Spec} k') = S_{k'}$. The generic point of H is also the generic point of $S_{k'}$, so H satisfies condition (2).

Now we check that H satisfies condition (1). As above, for any field extension F/k and any point $t \in \mathbb{P}^1_k(F) = \mathbb{P}^1_F(F)$, the fiber $\psi^{-1}(t) \subset \widetilde{S}_F$ is isomorphic to a hyperplane section of S_F corresponding to $\mathcal{M}_F$. Take $F = k'$ and take $t \in \mathbb{P}^1_k(F)$ to be the generic point $\operatorname{Spec} k' \to \mathbb{P}^1_k$. This implies that H' is isomorphic to a hyperplane section of $S_{k'}$. The same result holds for H, since it is the image of the composition $H' \to S_{k'} \to S \times \mathbb{P}^1_k$.

It remains to check the height inequality. We can assume that X is a closed subvariety of A. Let$(\mathcal{A}, \mathcal{L})$ be an integral model of (A, L) over S with structure morphism $\pi : \mathcal{A} \to S$. We can further assume that $\mathcal{L}$ is ample, which can be achieved by passing to a tensor power of L (cf. [1, Section 1.2.1]). Denote by $\mathcal{X}$ the Zariski closure of X in $\mathcal{A}$. It suffices to prove the inequality for the relevant naive heights, which becomes the inequality

$$\mathcal{L}^{\dim X+1} \cdot (\pi^*\mathcal{M})^{\dim S-1} \cdot \mathcal{X} \geq (\mathcal{L}_H)^{\dim X+1} \cdot (\pi_H^*\mathcal{M}_H)^{\dim S-2} \cdot \mathcal{X}_H^*.$$

Here the right-hand side is an intersection in $\mathcal{A}_H = \mathcal{A} \times_S H$, and $\pi_H, \mathcal{L}_H, \mathcal{M}_H, \mathcal{X}_H$ denote the base change of $\pi, \mathcal{L}, \mathcal{M}, \mathcal{X}$ via the morphism $H \to S$. Moreover, $\mathcal{X}_H^*$ denotes the Zariski closure of X in $\mathcal{A}_H$.

Denote by $\pi_{k'} : \mathcal{A}_{k'} \to S_{k'}, \mathcal{L}_{k'}, \mathcal{M}_{k'}, \mathcal{X}_{k'}$ the base change of $\pi : \mathcal{A} \to S, \mathcal{L}, \mathcal{M}, \mathcal{X}$ via the morphism $\operatorname{Spec} k' \to \operatorname{Spec} k$. By base change, the left-hand side of the inequality is equal to

$$\mathcal{L}_{k'}^{\dim X+1} \cdot (\pi_{k'}^*\mathcal{M}_{k'})^{\dim S-1} \cdot \mathcal{X}_{k'},$$

which is an intersection in $\mathcal{A}_{k'}$. Note that $\mathcal{M}_{k'}$ is linearly equivalent to the hyperplane section H of $S_{k'}$. The intersection number is further equal to

$$(\mathcal{L}_H)^{\dim X+1} \cdot (\pi_H^*\mathcal{M}_H)^{\dim S-2} \cdot (\mathcal{X}_{k'})_H.$$

Here $(\mathcal{X}_{k'})_H = \mathcal{X}_{k'} \cap \pi_{k'}^{-1}(H)$ is a proper intersection in $\mathcal{A}_{k'}$, so it is equidimensional.

The Zariski closure $\mathcal{X}_H^*$ of X in $\mathcal{A}_H$ is an irreducible component of $(\mathcal{X}_{k'})_H$. Then we have $(\mathcal{X}_{k'})_H - \mathcal{X}_H^*$ is an effective cycle. As $\mathcal{L}$ and $\mathcal{M}$ are ample, we have

$$(\mathcal{L}_H)^{\dim X+1} \cdot (\pi_H^* \mathcal{M}_H)^{\dim S-2} \cdot (\mathcal{X}_{k'})_H \\ \geq (\mathcal{L}_H)^{\dim X+1} \cdot (\pi_H^* \mathcal{M}_H)^{\dim S-2} \cdot \mathcal{X}_H^*.$$

This proves the inequality. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1 Let $A/K/k$ be as in the proposition. Let L be an ample and symmetric line bundle over A, and let $(S, \mathcal{M})$ be a polarization of K/k. We can assume that $\mathcal{M}$ is very ample by passing to a positive tensor power.

Choose (H_1, k_1), (H_2, k_2) as in Lemma 3.7 such that $k_1 \neq k_2$. For $i = 1, 2$, denote by $\mathcal{M}_i$ the pull-back of $\mathcal{M}$ via $(H_i)_{\overline{k}_i} \to S$. Note that $K\overline{k}_i = \overline{k}_i((H_i)_{\overline{k}_i})$. For any $x \in A(\overline{K})$, we have the height inequality

$$\widehat{h}_L^{\mathcal{M}}(x) \geq \widehat{h}_L^{\mathcal{M}_i}(x).$$

Let X be a subvariety of $A_{\overline{K}}$ which contains a dense set of small points of $A/K/k$. Then X contains a dense set of small points of $A_{K\overline{k}_i}/K\overline{k}_i/\overline{k}_i$ for $i = 1, 2$ by the height inequality. By assumption, the geometric Bogomolov conjecture holds for $(A_{K\overline{k}_i}/K\overline{k}_i/\overline{k}_i, X)$, so X is special in $A/K/k_i$ for both $i = 1, 2$.

If X is not special for $A/K/k$, let $k_{A,X}$ as in Proposition 3.6. We have $k_{A,X} \neq k$. By $\mathrm{trdeg}(k_i/k) = 1$, we have $k_{A,X} \not\subseteq k_i$ for at least one $i \in \{1, 2\}$. Then X is not special for $A/K/k_i$ for that i. This is a contradiction. □

4 Line bundles over abelian schemes

In this section, we introduce some preliminary results for abelian schemes over curve, which will be used in the proof the geometric Bogomolov conjecture in Sect. 5. These results are well-known, but we collect them here for convenience.

4.1 Rigidified line bundles

Let S be a smooth projective curve over a field k. Let $\pi : \mathcal{A} \to S$ be an abelian scheme over S. Let K be the function field of S and A be the generic fiber of π. This will be the basic setup of this section.

For any $m \in \mathbb{Z}$, denote by $[m] : \mathcal{A} \to \mathcal{A}$ the homomorphism of multiplication by m, and denote by $\mathcal{A}[m]$ the kernel of this homomorphism.

By a *multi-section* of $\pi : \mathcal{A} \to S$, we mean a closed integral subscheme $\mathcal{T}$ of $\mathcal{A}$ such that the induced morphism $\mathcal{T} \to S$ is finite and flat. The multi-section $\mathcal{T}$ is called *torsion* if its corresponding element in the abelian group $\mathcal{A}_{\mathcal{T}}(\mathcal{T})$ is torsion. The *order* of the torsion multi-section $\mathcal{T}$ is defined to be the order of the corresponding element in $\mathcal{A}_{\mathcal{T}}(\mathcal{T})$.

Any torsion multi-section $\mathcal{T}$ is necessarily the Zariski closure of a torsion point of $A(\overline{K})_{\mathrm{tor}}$ in $\mathcal{A}$. If the order of $\mathcal{T}$ is m, then $\mathcal{T}$ is an open and closed subscheme of $\mathcal{A}[m]$. If furthermore m is not divisible by $\mathrm{char}(k)$, then $\mathcal{A}[m]$ is finite and étale over S, and thus $\mathcal{T}$ is also finite and étale over S. In this case, $\mathcal{T}$ is actually smooth over k.

Let $\mathcal{L}$ be a line bundle over $\mathcal{A}$. We say that $\mathcal{L}$ is *symmetric* if $[-1]^*\mathcal{L} \simeq \mathcal{L}$. We say that $\mathcal{L}$ is *anti-symmetric* if $[-1]^*\mathcal{L} \simeq \mathcal{L}^\vee$. We say that $\mathcal{L}$ is *rigidified* if it is endowed with an isomorphism $e^*\mathcal{L} \simeq \mathcal{O}_S$ via the identity section $e : S \to \mathcal{A}$. The isomorphism is called a *rigidification*. The following results are well-known to experts, but we sketch proofs for convenience of readers.

Lemma 4.1 *Let $\mathcal{L}$ be a rigidified line bundle over $\mathcal{A}$. Then the following hold.*

(1) *If $\mathcal{L}$ is symmetric, then $[m]^*\mathcal{L} \simeq m^2\mathcal{L}$ for any $m \in \mathbb{Z}$; if $\mathcal{L}$ is anti-symmetric, then $[m]^*\mathcal{L} \simeq m\mathcal{L}$ for any $m \in \mathbb{Z}$.*
(2) *For any torsion multi-section $\mathcal{T} \subset \mathcal{A}$, the line bundle $\mathcal{L}|_{\mathcal{T}}$ is torsion in* $\mathrm{Pic}(\mathcal{T})$.
(3) *If $\mathcal{L}$ is symmetric and π-ample, then $\mathcal{L}$ is nef over $\mathcal{A}$.*
(4) *If $\mathcal{L}$ is symmetric and π-ample, and $\mathcal{L}'$ is another symmetric and rigidified line bundle over $\mathcal{A}$, then there is a positive integer a such that $a\mathcal{L} - \mathcal{L}'$ is nef over $\mathcal{A}$.*

Proof For (1), we only consider the symmetric case as the anti-symmetric case is similar. Note that $[m]^*\mathcal{L} - m^2\mathcal{L}$ is trivial over every fiber of $\pi : \mathcal{A} \to S$, so it lies in $\pi^*\mathrm{Pic}(S)$. Consider the pull-back by the identity section. Then $[m]^*\mathcal{L} - m^2\mathcal{L}$ is trivial by the rigidification.

For (2), by $2\mathcal{L} = (\mathcal{L} + [-1]^*\mathcal{L}) + (\mathcal{L} - [-1]^*\mathcal{L})$, we can assume that $\mathcal{L}$ is either symmetric or anti-symmetric. Let m be the order of $\mathcal{T}$. It suffices to prove that $\mathcal{L}|_{\mathcal{A}[m]}$ is torsion. Note that $\mathcal{A}[m] \to S$ is the base change of $[m] : \mathcal{A} \to \mathcal{A}$ by the identity section. It follows that $([m]^*\mathcal{L})|_{\mathcal{A}[m]}$ is trivial over $\mathcal{A}[m]$ by the rigidification. Then $\mathcal{L}|_{\mathcal{A}[m]}$ is torsion over $\mathcal{A}[m]$ by (1).

For (3), since $\mathcal{L}$ is π-ample, there is an ample line bundle $\mathcal{L}'$ over S such that $\mathcal{L}+\pi^*\mathcal{L}'$ is ample over $\mathcal{A}$. Then $[m]^*(\mathcal{L}+\pi^*\mathcal{L}') \simeq m^2\mathcal{L}+\pi^*\mathcal{L}'$ is ample over $\mathcal{A}$. The $\mathbb{Q}$-line bundle $\mathcal{L} + m^{-2}\pi^*\mathcal{L}'$ is ample over $\mathcal{A}$, and thus its limit $\mathcal{L}$ is nef over $\mathcal{A}$.

For (4), note that $a\mathcal{L} - \mathcal{L}'$ is π-ample for sufficiently large a. Apply (3). □

4.2 Numerical classes of torsion multi-sections

The following actually holds for rational equivalence and for high dimensional base S (cf. [12, Theorem 2.1]). We will be content to numerical equivalence for dim $S = 1$, which has the following quick proof and is sufficient for our application.

Proposition 4.2 *Let S be a smooth projective curve over a field k. Let $\pi : \mathcal{A} \to S$ be an abelian scheme over S of relative dimension $g \geq 1$. Let $\mathcal{L}$ be a symmetric and rigidified line bundle over $\mathcal{A}$. Let $\mathcal{T}$ be a torsion multi-section of $\mathcal{A}$ over S. Then there is a numerical equivalence*

$$[\mathcal{L}]^g \equiv \frac{\deg(\mathcal{L}_\eta)}{\deg(\mathcal{T}/S)}[\mathcal{T}]$$

of 1-cycles over $\mathcal{A}$ with rational coefficients. Here η is the generic point of S, and $\deg(\mathcal{L}_\eta)$ *is the degree of $\mathcal{L}_\eta$ over the abelian variety $\mathcal{A}_\eta$.*

Proof It suffices to prove that

$$\mathcal{L}^g \cdot \mathcal{M} = \frac{\deg(\mathcal{L}_\eta)}{\deg(\mathcal{T}/S)}\mathcal{T} \cdot \mathcal{M}$$

for any line bundle $\mathcal{M}$ over $\mathcal{A}$.

We first prove that if $\mathcal{M}$ is rigidified, then both sides are 0. The right-hand side is 0 by Lemma 4.1(2). For the left-hand side, by the decomposition $2\mathcal{M} = (\mathcal{M} + [-1]^*\mathcal{M}) + (\mathcal{M} - [-1]^*\mathcal{M})$ again, we can assume that $\mathcal{M}$ is either symmetric or anti-symmetric. Then $[m]^*\mathcal{M} \simeq m^i\mathcal{M}$ for $i = 1, 2$. By the projection formula,

$$([m]^*\mathcal{L})^g \cdot ([m]^*\mathcal{M}) = \deg([m])\mathcal{L}^g \cdot \mathcal{M}.$$

This is just

$$m^{2g+i}\,\mathcal{L}^g \cdot \mathcal{M} = m^{2g}\,\mathcal{L}^g \cdot \mathcal{M}.$$

It follows that $\mathcal{L}^g \cdot \mathcal{M} = 0$.

To prove the identity for general $\mathcal{M}$, write $\mathcal{M} = (\mathcal{M} - \pi^*e^*\mathcal{M}) + \pi^*e^*\mathcal{M}$. Note that $\mathcal{M} - \pi^*e^*\mathcal{M}$ is canonically rigidified. It is reduced to prove the result for $\mathcal{M} = \pi^*\mathcal{N}$ for line bundles $\mathcal{N}$ over S. This follows from the simple equalities

$$\mathcal{L}^g \cdot \pi^*\mathcal{N} = \deg(\mathcal{L}_\eta)\deg(\mathcal{N}), \quad \mathcal{T} \cdot \pi^*\mathcal{N} = \deg(\mathcal{T}/S)\deg(\mathcal{N}).$$

This finishes the proof. □

4.3 Canonical height

Let S be a smooth projective curve over a field k. Let $K = k(S)$ be the function field. Let A be an abelian variety over K, and L a symmetric and ample line bundle over A. Let X be a closed subvariety of A. As in (1.2) of Sect. 1.1, we have the canonical height $\hat{h}_L(X)$ associated to L.

Assume that A has everywhere *good reduction* over S. In this case, we have the following well-known easy interpretation of the canonical height of closed subvarieties of A.

In fact, let $\pi : \mathcal{A} \to S$ be the unique abelian scheme with generic fiber $A \to \operatorname{Spec} K$. Let $\mathcal{L}$ be a symmetric and rigidified line bundle over $\mathcal{A}$ extending L.

For the existence of $\mathcal{L}$, we first take any line bundle $\mathcal{L}'$ over $\mathcal{A}$ extending L, which can be obtained by passing to divisors and taking Zariski closures. By replacing $\mathcal{L}'$ by $\mathcal{L}' - \pi^*e^*\mathcal{L}'$, we can assume that $\mathcal{L}'$ is rigidified. Here $e : S \to \mathcal{A}$ is the identity section. Then $\mathcal{L}$ is automatically symmetric. In fact, $\mathcal{L} - [-1]^*\mathcal{L}$ is trivial over A, so it is isomorphic to $\pi^*\mathcal{M}$ for some line bundle $\mathcal{M}$ over S. The rigidification implies that $\mathcal{M}$ is trivial.

Denote by $\mathcal{X}$ the Zariski closure of X in $\mathcal{A}$. By Lemma 4.1(1) and the projection formula, the naive height

$$h_{\mathcal{L}}(X) = \frac{\mathcal{L}^{\dim X+1} \cdot \mathcal{X}}{(\dim X + 1) \deg_L(X)}$$

already satisfies $h_{\mathcal{L}}([n]X) = n^2 h_{\mathcal{L}}(X)$. As a consequence, $h_{\mathcal{L}}(X)$ is exactly equal to the canonical height $\hat{h}_L(X)$ associated to L.

5 Proof of the geometric Bogomolov conjecture

The goal of this section is to prove Theorem 1.3. Let $A/K/k$ be as in the theorem. By Proposition 3.1, we can assume the following condition.

(a) the transcendence degree of K over k is 1.

Let S be the unique (normal) projective model of K/k. We can further make the following assumptions.

(b) S is a smooth projective curve over k;
(c) A has semi-stable reduction over S.

The second condition is a consequence of the semistable reduction theorem.

Our next step is to apply Yamaki's result to make the following further assumption:

(d) A has trivial $\overline{K}/k$-trace and good reduction over S.

Denote by A' the maximal abelian subvariety of A that has everywhere good reduction over S. Yamaki [38, Theorem 1.5] asserts that the geometric Bogomolov conjecture holds for A if and only if it holds for $A'/\mathrm{tr}(A^{\overline{K}/k} \otimes_k K)$. This gives the condition.

The major contribution of this section is to prove the geometric Bogomolov conjecture under (a)–(d), based on the Manin–Mumford conjecture in this case.

Recall that the Manin–Mumford conjecture was proved by Raynaud [28, 29] over number fields, and proved by Hrushovski [14] over arbitrary fields. Hrushovski's proof relies on the model theory of difference fields. Inspired by Hrushovski's proof, Pink–Rössler [26,27] gave a new proof using classical algebraic geometry. We will only need the conjecture under assumption (a)-(d). For convenience, we state it in the following case of trivial $\overline{K}/k$-trace (cf. [27, Theorem 3.6]).

Theorem 5.1 (Manin–Mumford conjecture) *Let K be a finitely generated field over an algebraically closed field k. Let A be an abelian variety over K of trivial $\overline{K}/k$-trace. Let X be a closed subvariety of $A_{\overline{K}}$. Assume that $X(\overline{K}) \cap A(\overline{K})_{\mathrm{tor}}$ is Zariski dense in X. Then X is a torsion subvariety of $A_{\overline{K}}$.*

We also need the following well-known consequence of Zhang's fundamental inequality.

Lemma 5.2 *Let K/k be a finitely generated field extension of transcendence degree* 1. *Let A be an abelian variety over K. Let L be a symmetric and ample line bundle over A. Let X be a closed subvariety of $A_{\overline{K}}$. Then X contains a dense set of small points of $A/K/k$ if and only if* $\widehat{h}_L(X) = 0$.

Proof It is a direct consequence of the fundamental inequalities for successive minima, which asserts that

$$(\dim X + 1)h_L(X) \geq \sup_{U \subset X} \inf_{x \in U(\overline{K})} \widehat{h}_L(x) \geq \widehat{h}_L(X).$$

Here the supremum goes through all non-empty open subvarieties U of X. The fundamental inequality was proved over number fields by Zhang [42, Theorem 1.10], based on his previous works [40,41] and as a part of his theorem of successive minima. It was transferred to function fields by Gubler [10, Lemma 4.1 and Proposition 4.3]. □

5.1 Subvarieties generated by addition

With these preparations, we are ready to prove Theorem 1.3. Let $A/K/k$ and X be as in the theorem. By the above reduction, we can assume that conditions (a)-(d) hold. By extending K if necessary, we can assume that $X = X^* \otimes_K \overline{K}$ for a subvariety X^* of A over K. By abuse of notations, we will write $X = X^*$, viewed as a subvariety of A.

In this step, we consider the sum X with itself in A. The key assumption we are going to use is that A has a trivial $\overline{K}/k$-trace.

For any integer $m \geq 1$, denote by X_m the image of the addition morphism

$$f_m : X^m \longrightarrow A, \quad (x_1, \cdots, x_m) \longmapsto x_1 + \cdots + x_m.$$

Set $X_0 = 0$ to be the identity point of A. Since X_m is the image of the addition morphism $X_{m-1} \times X \to A$, we have

$$\dim X_{m-1} \leq \dim X_m \leq \dim X_{m-1} + \dim X, \quad m \geq 1.$$

Lemma 5.3 *There is a unique integer $r \geq 1$ such that* $\dim X_{r-1} < \dim X_r$ *and X_r is a torsion subvariety of A.*

Proof Fix a point $x_0 \in X(K)$, which exists by replacing K by a finite extension. Denote by $X'_m = X_m - mx_0$ the translation of X_m by $-mx_0$ in A. Note that X'_m is an (irreducible) subvariety of A, and that the sequence $\{X'_m\}_m$ is increasing. As a consequence, there is $r \geq 1$ such that $X'_m \subsetneq X'_r$ for all $m < r$ and $X'_m = X'_r$ for all $m \geq r$.

It follows that there is a morphism $\sigma : X'_r \times X'_r \to X'_{2r} = X'_r$ induced by the addition morphism of A. This is sufficient to imply that $B = X'_r$ is an abelian subvariety of A. In fact, the relative dimension of σ is equal to $\dim B$. By semi-continuity, the dimension of the fiber $\sigma^{-1}(0) := \{(x, y) \in B^2 : x + y = 0\}$ is at least $\dim B$. On the other hand, the first projection $\sigma^{-1}(0) \to B$ is injective on $\overline{K}$-points, and thus bijective on $\overline{K}$-points by comparing dimensions. This implies that for any $x \in B(\overline{K})$, $y = -x \in B(\overline{K})$. Thus we have that the inverse morphism satisfies $[-1]B \subset B$. It follows that B is an abelian subvariety of A. As a consequence, $X_r = B + t$ for the abelian subvariety B of A and the point $t = rx_0 \in A(K)$.

By assumption, X contains a dense set of small points of $A/K/k$. Since the canonical height is quadratic and positive definite up to torsion, we have

$$\hat{h}(x_1 + \cdots + x_m) \leq m(\hat{h}(x_1) + \cdots + \hat{h}(x_m)), \quad x_i \in A(\overline{K}).$$

Then "small points" of X^r transfer to "small points" of X_r. As a consequence, X_r contains a dense set of small points of $A/K/k$.

By [36, Lemma 2.4], X_r is torsion. The assumption that the $\overline{K}/k$-trace of A is trivial is missing in [36, Lemma 2.4], but such an assumption is satisfied in our situation. We may also prove it as follows. Denote $C = A/B$, which is an abelian variety over K. It suffices to prove that the image t' of t in $C(K)$

is a torsion point. There is a surjective homomorphism $A \to B$ such that the composition $B \to A \to B$ is an isogeny. This induces an isogeny $A \to A'$ with $A' = B \times C$. Since X_r contains a dense set of small points of $A/K/k$, its image $X' = B + t'$ in A' contains a dense set of small points of $A'/K/k$.

Take symmetric and ample line bundles L_1 over B and L_2 over C. Choose $L = p_1^* L_1 + p_2^* L_2$, which is a symmetric and ample line bundle over A', where $p_1 : B \times C \to B$ and $p_2 : B \times C \to C$ are the projections. For any point $x \in B(\overline{K})$, we have the heights

$$\hat{h}_L(x + t') = \hat{h}_{L_1}(x) + \hat{h}_{L_2}(t') \geq \hat{h}_{L_2}(t') \geq 0.$$

This forces $\hat{h}_{L_2}(t') = 0$ as $B + t'$ contains a dense set of small points of $A'/K/k$.

By assumption, A has a trivial $\overline{K}/k$-trace, so B and C have trivial $\overline{K}/k$-traces (cf. [37, Lemma 5.6]). Then $\hat{h}_{L_2}(t') = 0$ implies that $t' \in C(K)_{\mathrm{tor}}$. This follows from the Northcott property (cf. [4, Theorem 9.15]). It finishes the proof. □

By the lemma, X_r is a torsion subvariety of A. Replacing X by a translation by a suitable torsion point, we can assume that X_r is an abelian subvariety of A. In this process, to make X to be defined over K, we may need to replace K by a finite extension again. To summarize, we can furthermore make the following assumption.

(e) $A' = X_r$ is an abelian subvariety of A and $\dim X_{r-1} < \dim X_r$.

5.2 Fibers of the addition map

By assumption (e), the image of the summation map $f_r : X^r \to A$ is an abelian subvariety A' of A. This induces a surjective summation morphism

$$f : X_{r-1} \times X \longrightarrow A'.$$

Denote by

$$e = \dim X_{r-1} + \dim X - \dim A'$$

the relative dimension of this morphism. By assumption (e), we have

$$e < \dim X.$$

Denote

$$A'_{e+1} = \{y \in A' : \dim f^{-1}(y) \geq e + 1\}.$$

Then A'_{e+1} is a closed subset of A' of codimension at least 2. Note that $X_{r-1} \times X$ is a natural subvariety of the abelian variety $A^2 = A \times A$. The goal of this step is to prove the following result.

Proposition 5.4 *For any $t \in A'(\overline{K})_{\mathrm{tor}} \backslash A'_{e+1}(\overline{K})$ with order of t non-divisible by* char K*, every irreducible component of the fiber $f^{-1}(t) \subset (X_{r-1} \times X)_{\overline{K}}$ has canonical height 0 in $(A \times A)_{\overline{K}}$.*

Proof Denote by B the preimage of A' under the summation map $A \times A \to A$. Since the preimage of 0 under $A \times A \to A$ is geometrically integral, B is also geometrically integral, and thus an abelian subvariety of $A \times A$ over K. We have the following commutative diagram.

$$\begin{tikzcd} X_{r-1} \times X \arrow[r] \arrow[dr, "f"] & B \arrow[r] \arrow[d, "h"] & A \times A \arrow[d] \\ & A' \arrow[r] & A \end{tikzcd}$$

Let S/k with $K = k(S)$ be as in assumption (b). Denote by $\pi : \mathcal{A} \to S$ the unique abelian scheme extending the abelian variety $A \to \operatorname{Spec} K$, which exists by assumption (d). Then $\mathcal{A} \times_S \mathcal{A} \to S$ is the unique abelian scheme extending $A \times A \to \operatorname{Spec} K$. Denote by $\mathcal{X}$ the Zariski closure of X in $\mathcal{A}$. Denote by $\mathcal{A}'$ (resp. $\mathcal{B}$) the Zariski closure of A' in $\mathcal{A}$ (resp. B in $\mathcal{A} \times_S \mathcal{A}$). Both $\mathcal{A}'$ and $\mathcal{B}$ are abelian schemes over S. Denote by $\mathcal{Y}$ is the Zariski closure of $Y = X_{r-1} \times X$ in $\mathcal{A} \times_S \mathcal{A}$. This gives an integral version of the above diagram.

$$\begin{tikzcd} f^{-1}(\mathcal{T}) \arrow[r] \arrow[dr] & \mathcal{Y} \arrow[r] \arrow[dr, "f"] & \mathcal{B} \arrow[r] \arrow[d, "h"] & \mathcal{A} \times_S \mathcal{A} \arrow[d] \\ & \mathcal{T} \arrow[r] & \mathcal{A}' \arrow[r] & \mathcal{A} \end{tikzcd}$$

Let $\mathcal{T}$ be a torsion multi-section of $\mathcal{A}' \to S$ of order non-divisible by char K. Then $\mathcal{T} \to S$ is finite and étale, and $\mathcal{T}$ is a smooth projective curve over k. Assume that $T = \mathcal{T}_K$ is not contained in A'_{e+1}. Apply Proposition 2.1 to the triangle of the second diagram. We obtain that

$$\sum_{i=1}^{n} m_i[\mathcal{Z}_i] \leq h^*[\mathcal{T}] \cdot [\mathcal{Y}]$$

in $\mathrm{CH}_{e+1}(\mathcal{B})$. Here $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ are the irreducible components of $f^{-1}(\mathcal{T})$ satisfying $f(\mathcal{Z}_i) = \mathcal{T}$, and m_i is the multiplicity of $\mathcal{Z}_i$ in $f^{-1}(\mathcal{T})$.

Let $\mathcal{L}_{\mathcal{A}'}$ (resp. $\mathcal{L}_{\mathcal{B}}$) be a symmetric, relatively ample and rigidified line bundle over $\mathcal{A}'$ (resp. $\mathcal{B}$). By Proposition 4.2, there is a numerical equivalence

$$[\mathcal{T}] \equiv a\,[\mathcal{L}_{\mathcal{A}'}]^{\dim A'}$$

in $\mathrm{CH}_1(\mathcal{A}')_{\mathbb{Q}}$ for some $a > 0$.

By these two results, we have

$$\begin{aligned}\sum_{i=1}^{n} m_i[\mathcal{Z}_i]\cdot[\mathcal{L}_{\mathcal{B}}]^{e+1} &\leq h^*[\mathcal{T}]\cdot[\mathcal{Y}]\cdot[\mathcal{L}_{\mathcal{B}}]^{e+1}\\ &= a\,[h^*\mathcal{L}_{\mathcal{A}'}]^{\dim A'}\cdot[\mathcal{L}_{\mathcal{B}}]^{e+1}\cdot[\mathcal{Y}].\end{aligned}$$

By Lemma 4.1(4), there is a constant $b > 0$ such that $b\,\mathcal{L}_{\mathcal{B}} - h^*\mathcal{L}_{\mathcal{A}'}$ is a nef line bundle over $\mathcal{B}$. As a consequence,

$$[h^*\mathcal{L}_{\mathcal{A}'}]^{\dim A'}\cdot[\mathcal{L}_{\mathcal{B}}]^{e+1}\cdot[\mathcal{Y}] \leq b^{\dim A'}\,[\mathcal{L}_{\mathcal{B}}]^{\dim \mathcal{Y}}\cdot[\mathcal{Y}].$$

As in the proof of Lemma 5.3, "small points" of X transfer to "small points" of $Y = X_{r-1}\times X$. Then Y has a dense set of small points in B. This also follows from [34, Lemma 2.1 and Lemma 2.4]. By Lemma 5.2, the height $h_{\mathcal{L}_{\mathcal{B}}}(Y) = 0$. Then we have $[\mathcal{Y}]\cdot[\mathcal{L}_{\mathcal{B}}]^{\dim \mathcal{Y}} = 0$. This forces $[\mathcal{Z}_i]\cdot[\mathcal{L}_{\mathcal{B}}]^{e+1}$, and thus $h_{\mathcal{L}_{\mathcal{B}}}(Z_i) = 0$. Here $Z_i = \mathcal{Z}_{i,K}$ for $i = 1,\ldots,n$ are exactly the irreducible components of $f^{-1}(\mathcal{T}_K)$. This finishes the proof. □

5.3 Lowering the dimension

Now we prove that X is torsion by induction on $\dim X$. If $\dim X = 0$, we must have $\hat{h}(X) = 0$, and then X is a torsion point of A by the Northcott theorem (cf. [4, Theorem 9.15]).

If $\dim X > 0$, consider the morphism $f : X_{r-1}\times X \to A'$ obtained in assumption (e). By Proposition 5.4, for any $t \in A'(\overline{K})_{\mathrm{tor}}\backslash A'_{e+1}(\overline{K})$, every irreducible component of the fiber $f^{-1}(t) \subset (X_{r-1}\times X)_{\overline{K}}$ has canonical height 0 in $A\times A$. Note that every irreducible component of $f^{-1}(t)$ has dimension $e < \dim X$. By induction, it is torsion in $A\times A$. Therefore, $f^{-1}(t)$ contains a Zariski dense set of torsion points of $A\times A$. As $A'(\overline{K})_{\mathrm{tor}}\backslash A'_{e+1}(\overline{K})$ is Zariski dense in A', $X_{r-1}\times X$ contains a Zariski dense set of torsion points of $A\times A$.

By the Manin–Mumford conjecture (cf. Theorem 5.1), $X_{r-1}\times X$ is a torsion subvariety of $A\times A$. Then X is a torsion subvariety of A, since it is the image of the composition of $X_{r-1}\times X \to A\times A$ with $p_2 : A\times A \to A$. This finishes the proof of Theorem 1.3.

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