5. Subgradient method

- subgradient method
- convergence analysis
- optimal step size when $f^\star$ is known
- alternating projections
- optimality

Subgradient method

to minimize a nondifferentiable convex function f: choose $x^{(0)}$ and repeat

$$x^{(k)} = x^{(k-1)} - t_k g^{(k-1)}, \quad k = 1, 2, \ldots$$

$g^{(k-1)}$ is **any** subgradient of f at $x^{(k-1)}$

step size rules

- fixed step: t_k constant
- fixed length: $t_k \|g^{(k-1)}\|_2$ constant (*i.e.*, $\|x^{(k)} - x^{(k-1)}\|_2$ constant)
- diminishing: $t_k \to 0$, $\sum_{k=1}^{\infty} t_k = \infty$

Assumptions

- f has finite optimal value $f^\star$, minimizer $x^\star$
- f is convex, $\mathbf{dom}\, f = \mathbf{R}^n$
- f is Lipschitz continuous with constant $G > 0$:

$$|f(x) - f(y)| \leq G\|x - y\|_2 \qquad \forall x, y$$

this is equivalent to

$$\|g\|_2 \leq G \qquad \forall g \in \partial f(x),\ \forall x$$

(see next page)

proof

- assume $\|g\|_2 \le G$ for all subgradients; choose $g_y \in \partial f(y)$, $g_x \in \partial f(x)$:

$$g_x^T(x-y) \ge f(x) - f(y) \ge g_y^T(x-y)$$

  by the Cauchy-Schwarz inequality

$$G\|x-y\|_2 \ge f(x) - f(y) \ge -G\|x-y\|_2$$

- assume $\|g\|_2 > G$ for some $g \in \partial f(x)$; take $y = x + g/\|g\|_2$:

$$\begin{aligned} f(y) &\ge f(x) + g^T(y-x) \\ &= f(x) + \|g\|_2 \\ &> f(x) + G \end{aligned}$$

Analysis

- the subgradient method is not a descent method
- the key quantity in the analysis is the distance to the optimal set

with $x^+ = x^{(i)}$, $x = x^{(i-1)}$, $g = g^{(i-1)}$, $t = t_i$:

$$\begin{aligned} \|x^+ - x^\star\|_2^2 &= \|x - tg - x^\star\|_2^2 \\ &= \|x - x^\star\|_2^2 - 2tg^T(x - x^\star) + t^2\|g\|_2^2 \\ &\leq \|x - x^\star\|_2^2 - 2t\left(f(x) - f^\star\right) + t^2\|g\|_2^2 \end{aligned}$$

combine inequalities for $i = 1, \ldots, k$, and define $f_{\rm best}^{(k)} = \min_{0 \leq i < k} f(x^{(i)})$:

$$\begin{aligned} 2(\sum_{i=1}^{k} t_i)\left(f_{\rm best}^{(k)} - f^\star\right) &\leq \|x^{(0)} - x^\star\|_2^2 - \|x^{(k)} - x^\star\|_2^2 + \sum_{i=1}^{k} t_i^2\|g^{(i-1)}\|_2^2 \\ &\leq \|x^{(0)} - x^\star\|_2^2 + \sum_{i=1}^{k} t_i^2\|g^{(i-1)}\|_2^2 \end{aligned}$$

fixed step size $t_i = t$

$$f_{\text{best}}^{(k)} - f^\star \le \frac{\|x^{(0)} - x^\star\|_2^2}{2kt} + \frac{G^2 t}{2}$$

- does not guarantee convergence of $f_{\text{best}}^{(k)}$
- for large k, $f_{\text{best}}^{(k)}$ is approximately $G^2 t/2$-suboptimal

fixed step length $t_i = s/\|g^{(i-1)}\|_2$

$$f_{\text{best}}^{(k)} - f^\star \le \frac{G\|x^{(0)} - x^\star\|_2^2}{2ks} + \frac{Gs}{2}$$

- does not guarantee convergence of $f_{\text{best}}^{(k)}$
- for large k, $f_{\text{best}}^{(k)}$ is approximately $Gs/2$-suboptimal

diminishing step size $t_i \to 0$, $\sum\limits_{i=1}^{\infty} t_i = \infty$

$$f_{\text{best}}^{(k)} - f^\star \le \frac{\|x^{(0)} - x^\star\|_2^2 + G^2 \sum\limits_{i=1}^{k} t_i^2}{2 \sum\limits_{i=1}^{k} t_i}$$

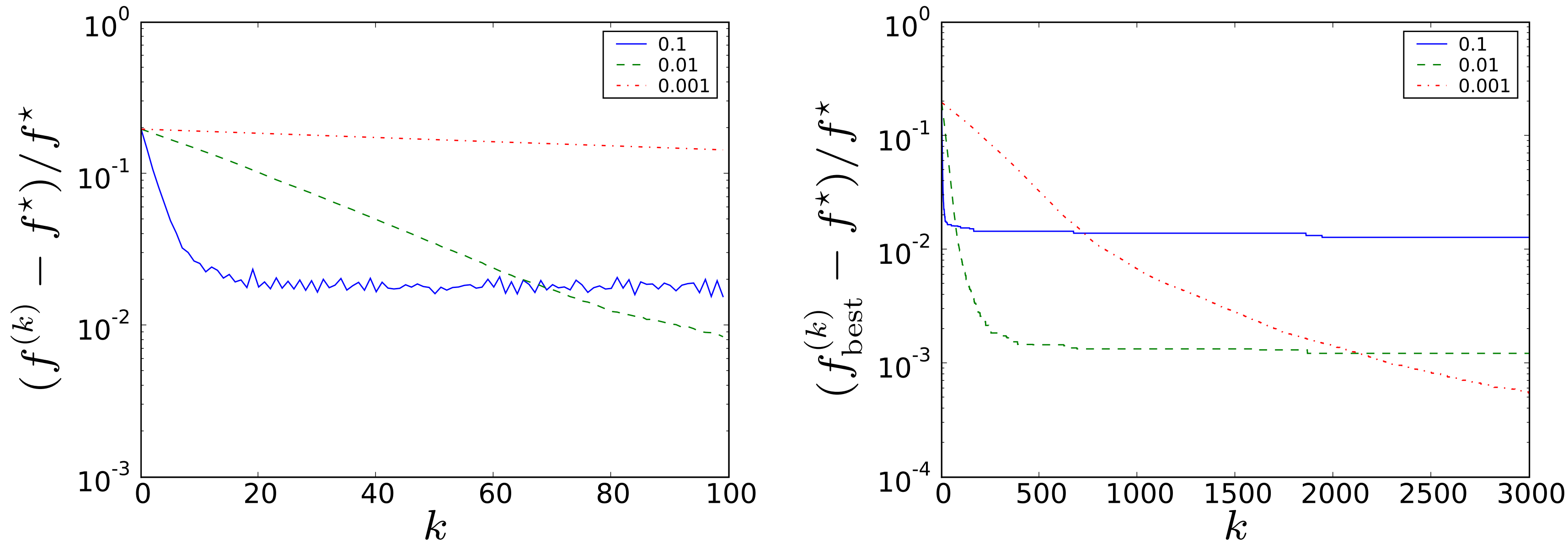
can show that $(\sum\limits_{i=1}^{k} t_i^2)/(\sum\limits_{i=1}^{k} t_i) \to 0$; hence, $f_{\text{best}}^{(k)}$ converges to $f^\star$

Example: 1-norm minimization

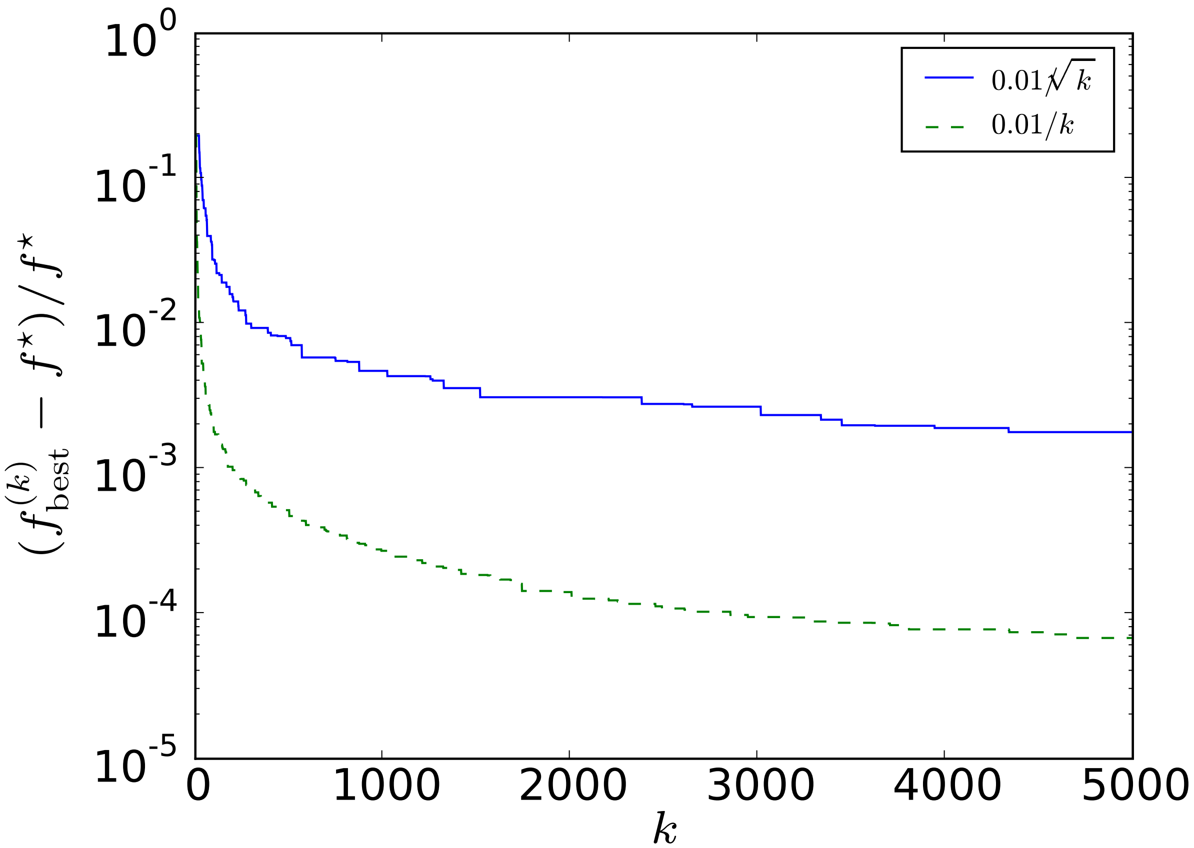
$$\text{minimize} \quad \|Ax - b\|_1 \qquad\qquad (A \in \mathbf{R}^{500\times 100}, b \in \mathbf{R}^{500})$$

subgradient is given by $A^T \mathbf{sign}(Ax - b)$

fixed steplength $t_k = s/\|g^{(k-1)}\|_2$, $s = 0.1,\ 0.01,\ 0.001$



diminishing step size $t_k = 0.01/\sqrt{k}$, $t_k = 0.01/k$



Optimal step size for fixed number of iterations

from page 5-5: if $s_i = t_i \|g^{(i-1)}\|_2$ and $\|x^{(0)} - x^\star\|_2 \le R$:

$$f_{\rm best}^{(k)} - f^\star \le \frac{R^2 + \sum_{i=1}^k s_i^2}{2\sum_{i=1}^k s_i / G}$$

- for given k, bound is minimized by fixed step length $s_i = s = R/\sqrt{k}$
- resulting bound after k steps is

$$f_{\rm best}^{(k)} - f^\star \le \frac{GR}{\sqrt{k}}$$

- guarantees accuracy $f_{\rm best}^{(k)} - f^\star \le \epsilon$ in $k = O(1/\epsilon^2)$ iterations

Optimal step size when $f^\star$ is known

right-hand side in first inequality of page 5-5 is minimized by

$$t_i = \frac{f(x^{(i-1)}) - f^\star}{\|g^{(i-1)}\|_2^2}$$

optimized bound is

$$\frac{\left(f(x^{(i-1)}) - f^\star\right)^2}{\|g^{(i-1)}\|_2^2} \leq \|x^{(i-1)} - x^\star\|_2^2 - \|x^{(i)} - x^\star\|_2^2$$

applying recursively (with $\|x^{(0)} - x^\star\|_2 \leq R$ and $\|g^{(i)}\|_2 \leq G$) gives

$$f_{\mathrm{best}}^{(k)} - f^\star \leq \frac{GR}{\sqrt{k}}$$

Exercise: find point in intersection of convex sets

to find a point in the intersection of m closed convex sets $C_1, \ldots, C_m$,

$$\text{minimize} \quad f(x) = \max\{d_1(x), \ldots, d_m(x)\}$$

where $d_j(x) = \inf_{y \in C_j} \|x - y\|_2$ is Euclidean distance of x to C_j

- $f^\star = 0$ if the intersection is nonempty
- (from p. 4-15): $g \in \partial f(\hat{x})$ if $g \in \partial d_j(\hat{x})$ and C_j is farthest set from $\hat{x}$
- (from p. 4-21) subgradient $g \in \partial d_j(\hat{x})$ from projection $P_j(\hat{x})$ on C_j:

$$g = 0 \quad (\text{if } \hat{x} \in C_j), \qquad g = \frac{1}{d(\hat{x}, C_j)}(\hat{x} - P_j(\hat{x})) \quad (\text{if } \hat{x} \notin C_j)$$

note that $\|g\|_2 = 1$ if $\hat{x} \notin C_j$

subgradient method with optimal step size

- optimal step size for $f^\star = 0$ and $\|g^{(i-1)}\|_2 = 1$ is $t_i = f(x^{(i-1)})$.

- at iteration k, find farthest set C_j (with $f(x^{(k-1)}) = d_j(x^{(k-1)})$); take

$$
\begin{aligned}
x^{(k)} &= x^{(k-1)} - \frac{f(x^{(k-1)})}{d_j(x^{(k-1)})}(x^{(k-1)} - P_j(x^{(k-1)})) \\
&= P_j(x^{(k-1)})
\end{aligned}
$$

- a version of the *alternating projections* algorithm

- at each step, project the current point onto the farthest set

- for $m = 2$, projections alternate onto one set, then the other

Example: Positive semidefinite matrix completion

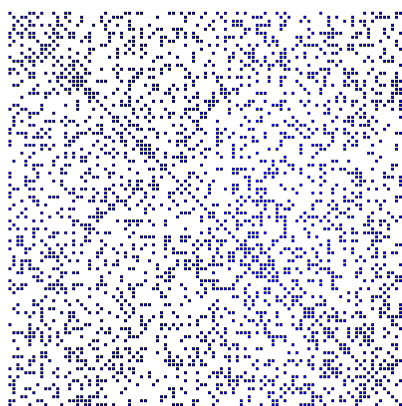
some entries of $X \in \mathbf{S}^n$ fixed; find values for others so $X \succeq 0$

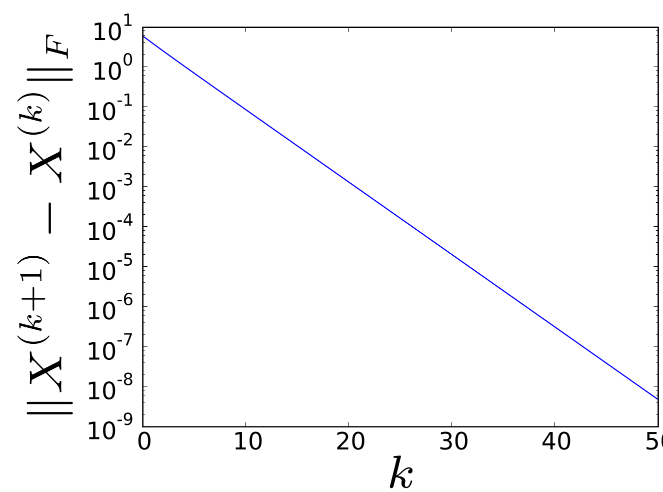
- $C_1 = \mathbf{S}^n_+$, C_2 is (affine) set in $\mathbf{S}^n$ with specified fixed entries
- projection onto C_1 by eigenvalue decomposition, truncation

$$P_1(X) = \sum_{i=1}^n \max\{0, \lambda_i\} q_i q_i^T \qquad \text{if } X = \sum_{i=1}^n \lambda_i q_i q_i^T$$

- projection of X onto C_2 by re-setting specified entries to fixed values

100×100
matrix missing
71% entries





Optimality of the subgradient method

can the $f_{\text{best}}^{(k)} - f^\star \leq GR/\sqrt{k}$ bound on page 5-10 be improved?

problem class

- f is convex, with a minimizer $x^\star$
- we know a starting point $x^{(0)}$ with $\|x^{(0)} - x^\star\|_2 \leq R$
- we know the Lipschitz constant G of f on $\{x \mid \|x - x^{(0)}\|_2 \leq R\}$
- f is defined by an oracle: given x, oracle returns $f(x)$ and a subgradient

algorithm class: k iterations of any method that chooses $x^{(i)}$ in

$$x^{(0)} + \text{span}\{g^{(0)}, g^{(1)}, \ldots, g^{(i-1)}\}$$

test problem and oracle

$$f(x) = \max_{i=1,\ldots,k} x_i + \frac{1}{2}\|x\|_2^2, \qquad x^{(0)} = 0$$

- solution: $x^\star = -\frac{1}{k}(\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{n-k})$ and $f^\star = -\frac{1}{2k}$
- $R = \|x^{(0)} - x^\star\|_2 = 1/\sqrt{k}$ and $G = 1 + 1/\sqrt{k}$
- oracle returns subgradient $e_{\hat{\jmath}} + x$ where $\hat{\jmath} = \min\{j \mid x_j = \max_{i=1,\ldots,k} x_i\}$

iteration: for $i = 0,\ldots,k-1$, entries $x_{i+1}^{(i)}, \ldots, x_k^{(i)}$ are zero

$$f_{\text{best}}^{(k)} - f^\star = \min_{i<k} f(x^{(i)}) - f^\star \geq -f^\star = \frac{GR}{2(1+\sqrt{k})}$$

conclusion: $O(1/\sqrt{k})$ bound cannot be improved

Summary: subgradient method

- handles general nondifferentiable convex problem
- often leads to very simple algorithms
- convergence can be very slow
- no good stopping criterion
- theoretical complexity: $O(1/\epsilon^2)$ iterations to find ϵ-suboptimal point
- an 'optimal' 1st-order method: $O(1/\epsilon^2)$ bound cannot be improved

References

- S. Boyd, lecture notes and slides for EE364b, Convex Optimization II
- Yu. Nesterov, *Introductory Lectures on Convex Optimization. A Basic Course* (2004)

  §3.2.1 with the example on page 5-16 of this lecture