8. Conjugate functions

- closed functions
- conjugate function

Closed set

a set C is closed if it contains its boundary:

$$x^k \in C, \quad x^k \to \bar{x} \qquad \Longrightarrow \qquad \bar{x} \in C$$

operations that preserve closedness

- the intersection of (finitely or infinitely many) closed sets is closed
- the union of a finite number of closed sets is closed
- inverse under linear mapping: $\{x \mid Ax \in C\}$ is closed if C is closed

Image under linear mapping

the image of a closed set under a linear mapping is not necessarily closed

example (C is closed, $AC = \{Ax \mid x \in C\}$ is open):

$$C = \{(x_1, x_2) \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}, \qquad A = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad AC = \mathbf{R}_{++}$$

sufficient condition: AC is closed if

- C is closed and convex
- and C does not have a recession direction in the nullspace of A. *i.e.*,

$$Ay = 0, \quad \hat{x} \in C, \quad \hat{x} + \alpha y \in C \;\; \forall \alpha \geq 0 \qquad \Longrightarrow \qquad y = 0$$

in particular, this holds for any A if C is bounded

Closed function

definition: a function is closed if its epigraph is a closed set

examples

- $f(x) = -\log(1 - x^2)$ with $\mathbf{dom}\, f = \{x \mid |x| < 1\}$
- $f(x) = x \log x$ with $\mathbf{dom}\, f = \mathbf{R}_+$ and $f(0) = 0$
- indicator function of a closed set C: $f(x) = 0$ if $x \in C = \mathbf{dom}\, f$

not closed

- $f(x) = x \log x$ with $\mathbf{dom}\, f = \mathbf{R}_{++}$, or with $\mathbf{dom}\, f = \mathbf{R}_+$ and $f(0) = 1$
- indicator function of a set C if C is not closed

Properties

sublevel sets: f is closed if and only if all its sublevel sets are closed

minimum: if f is closed with bounded sublevel sets then it has a minimizer

common operations on convex functions that preserve closedness

- *sum:* $f + g$ is closed if f and g are closed (and $\mathbf{dom}\, f \cap \mathbf{dom}\, g \neq \emptyset$)
- *composition with affine mapping:* $f(Ax + b)$ is closed if f is closed
- *supremum:* $\sup_\alpha f_\alpha(x)$ is closed if each function f_α is closed

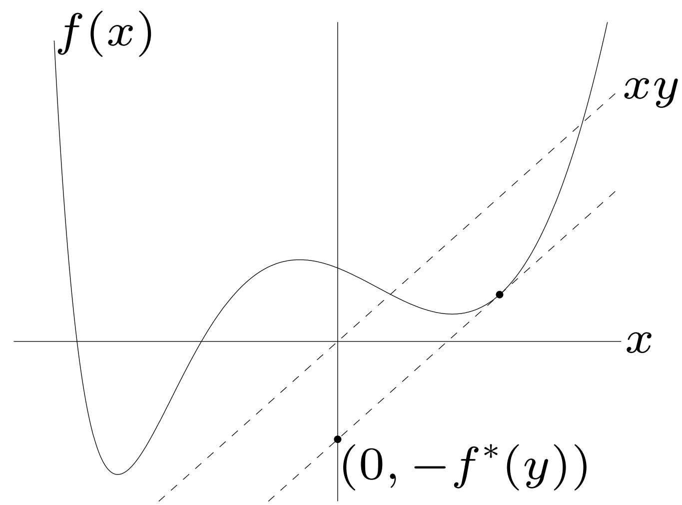
Outline

- closed functions
- **conjugate function**

Conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



f^* is closed and convex even if f is not

Fenchel's inequality

$$f(x) + f^*(y) \geq x^T y \qquad \forall x, y$$

(extends inequality $x^T x/2 + y^T y/2 \geq x^T y$ to non-quadratic convex f)

Quadratic function

$$f(x) = \frac{1}{2}x^T A x + b^T x + c$$

strictly convex case $(A \succ 0)$

$$f^*(y) = \frac{1}{2}(y - b)^T A^{-1}(y - b) - c$$

general convex case $(A \succeq 0)$

$$f^*(y) = \frac{1}{2}(y - b)^T A^{\dagger}(y - b) - c, \qquad \mathbf{dom}\, f^* = \mathrm{range}(A) + b$$

Negative entropy and negative logarithm

negative entropy

$$f(x) = \sum_{i=1}^{n} x_i \log x_i \qquad\qquad f^*(y) = \sum_{i=1}^{n} e^{y_i - 1}$$

negative logarithm

$$f(x) = -\sum_{i=1}^{n} \log x_i \qquad\qquad f^*(y) = -\sum_{i=1}^{n} \log(-y_i) - n$$

matrix logarithm

$$f(X) = -\log\det X \quad (\mathbf{dom}\, f = \mathbf{S}^n_{++}) \qquad f^*(Y) = -\log\det(-Y) - n$$

Indicator function and norm

indicator of convex set C: conjugate is support function of C

$$f(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} \qquad\qquad f^*(y) = \sup_{x \in C} y^T x$$

norm: conjugate is indicator of unit dual norm ball

$$f(x) = \|x\| \qquad\qquad f^*(y) = \begin{cases} 0 & \|y\|_* \le 1 \\ +\infty & \|y\|_* > 1 \end{cases}$$

(see next page)

proof: recall the definition of dual norm:

$$\|y\|_* = \sup_{\|x\| \leq 1} x^T y$$

to evaluate $f^*(y) = \sup_x \left(y^T x - \|x\|\right)$ we distinguish two cases

- if $\|y\|_* \leq 1$, then (by definition of dual norm)

$$y^T x \leq \|x\| \quad \forall x$$

  and equality holds if $x = 0$; therefore $\sup_x (y^T x - \|x\|) = 0$

- if $\|y\|_* > 1$, there exists an x with $\|x\| \leq 1$, $x^T y > 1$; then

$$f^*(y) \geq y^T(tx) - \|tx\| = t(y^T x - \|x\|)$$

  and r.h.s. goes to infinity if $t \to \infty$

The second conjugate

$$f^{**}(x) = \sup_{y \in \mathbf{dom}\, f^*} (x^T y - f^*(y))$$

- f^{**} is closed and convex
- from Fenchel's inequality ($x^T y - f^*(y) \leq f(x)$ for all y and x):

$$f^{**}(x) \leq f(x) \qquad \forall x$$

  equivalently, $\mathbf{epi}\, f \subseteq \mathbf{epi}\, f^{**}$ (for any f)

- if f is closed and convex, then

$$f^{**}(x) = f(x) \qquad \forall x$$

  equivalently, $\mathbf{epi}\, f = \mathbf{epi}\, f^{**}$ (if f is closed convex); proof on next page

proof ($f^{**} = f$ if f is closed and convex): by contradiction

suppose $(x, f^{**}(x)) \notin \mathbf{epi}\, f$; then there is a strict separating hyperplane:

$$\begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} z - x \\ s - f^{**}(x) \end{bmatrix} \leq c < 0 \qquad \forall (z, s) \in \mathbf{epi}\, f$$

for some a, b, c with $b \leq 0$ ($b > 0$ gives a contradiction as $s \to \infty$)

- if $b < 0$, define $y = a/(-b)$ and maximize l.h.s. over $(z, s) \in \mathbf{epi}\, f$:

$$f^*(y) - y^T x + f^{**}(x) \leq c/(-b) < 0$$

  this contradicts Fenchel's inequality

- if $b = 0$, choose $\hat{y} \in \mathbf{dom}\, f^*$ and add small multiple of $(\hat{y}, -1)$ to (a, b):

$$\begin{bmatrix} a + \epsilon \hat{y} \\ -\epsilon \end{bmatrix}^T \begin{bmatrix} z - x \\ s - f^{**}(x) \end{bmatrix} \leq c + \epsilon \left(f^*(\hat{y}) - x^T \hat{y} + f^{**}(x) \right) < 0$$

  now apply the argument for $b < 0$

Conjugates and subgradients

if f is closed and convex, then

$$y \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^*(y) \quad \Longleftrightarrow \quad x^T y = f(x) + f^*(y)$$

proof: if $y \in \partial f(x)$, then $f^*(y) = \sup_u (y^T u - f(u)) = y^T x - f(x)$

$$\begin{array}{rcl} f^*(v) & = & \sup\limits_u \, (v^T u - f(u)) \\ & \geq & v^T x - f(x) \\ & = & x^T (v - y) - f(x) + y^T x \\ & = & f^*(y) + x^T (v - y) \end{array}$$

for all v; therefore, x is a subgradient of f^* at y $(x \in \partial f^*(y))$

reverse implication $x \in \partial f^*(y) \Longrightarrow y \in \partial f(x)$ follows from $f^{**} = f$

Some calculus rules

separable sum

$$f(x_1, x_2) = g(x_1) + h(x_2) \qquad\qquad f^*(y_1, y_2) = g^*(y_1) + h^*(y_2)$$

scalar multiplication: (for $\alpha > 0$)

$$f(x) = \alpha g(x) \qquad\qquad f^*(y) = \alpha g^*(y/\alpha)$$

addition to affine function

$$f(x) = g(x) + a^T x + b \qquad\qquad f^*(y) = g^*(y - a) - b$$

infimal convolution

$$f(x) = \inf_{u+v=x} (g(u) + h(v)) \qquad\qquad f^*(y) = g^*(y) + h^*(y)$$

References

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- D.P. Bertsekas, A. Nedić, A.E. Ozdaglar, *Convex Analysis and Optimization* (2003), chapter 7.

- R. T. Rockafellar, *Convex Analysis* (1970)