

Project on Railway Timetabling

June 22, 2022

Train timetable defines the departure/arrival time of each train $j \in J$ at the origin, destination and intermediate stations. For example, in the Beijing-Shanghai line, each train departs from Beijing and heads to Shanghai (called down direction) or departs from Shanghai and heads to Beijing (called up direction). In the timetable, the total running time of train j is defined by the elapsed time from origin station to destination station. There are so called ideal time schedule for some trains. However, they may be modified to meet practical constraints such as track capacity, interval times, etc.

Now we consider the macro-scope rail timetabling problem, in which we do not consider the internal operations within the station and assume that the station has only one track for up and down direction. At the same time, in order to take care of the inter-station operation requirements, we define several different types of time intervals to avoid any two trains being too close to each other. These assumptions simplifies the complexity of modelling. In meso-scope or micro-scope models, it's usually necessary to consider the inner-station structure to construct a feasible and practical train timetabling plan, which involves more complicated decision variables and constraints (more details in [Zhang et al., 2020]).

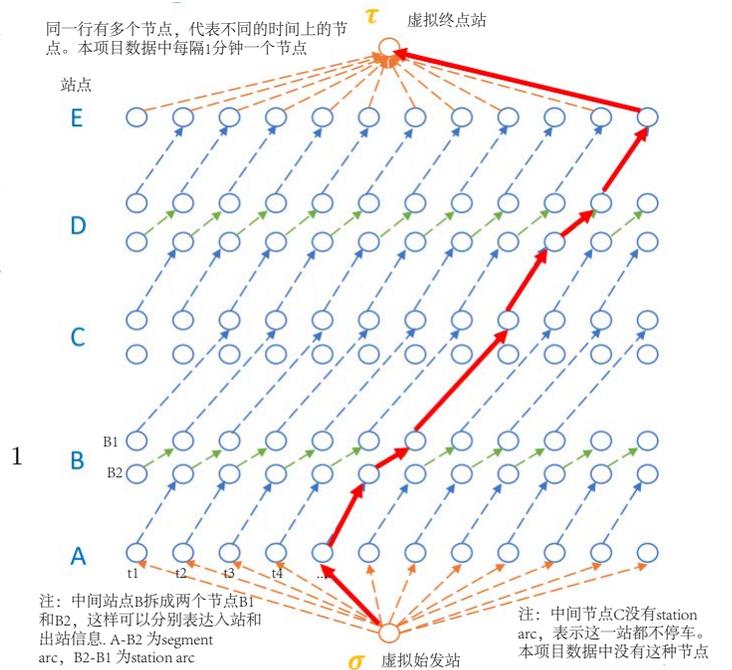
In this note, we create a space-time network model to solve discretized-time train timetabling problem based on the model [Caprara et al., 2002], we create a space-time network model, and apply the Lagrangian relaxation method to solve the problem. Results for a toy example involving 7 states and 16 trains are presented to show the effectiveness of our model and method.

1 Space-Time Network Model

We use a directed, acyclic and multiplicative graph $G = (V, E)$ to characterize the train timetabling problem. The node set V has the form $\{\sigma, \tau\} \cup U \cup W$, where σ, τ denote artificial origin and destination nodes, respectively. In addition to the artificial nodes, we further assume U denotes the set of arrival nodes and W denotes the set of departure nodes. Each normal node $v \in U \cup W$ is denoted by a binary vector $v = (s(v), t(v))$ where $s(v)$ denotes the station and $t(v)$ denotes the discrete time point.

The arcs E in graph G can be divided into the following categories:

1. starting arcs (σ, v) , where $s(v)$ denotes the starting station of certain train.
2. station arcs (u, w) , $u \in U, w \in W, s(u) = s(w)$ denotes that a certain train enters the station $s(u)$ at time $t(u)$ and leave the same station at time $t(w)$.
3. segment arcs (w, u) , $u \in U, w \in W$ denotes the route of some train, i.e., a train leaves the station $s(w)$ at time $t(w)$ and arrive at another station $s(u)$ at time $t(u)$.



4. ending arcs (u, τ) , where $s(u)$ is the terminal of some train.

Since the route and timetable plan of each train $j \in J$ are fairly different, the available nodes and arcs of each train are also different, so we define a subgraph $G_j \subseteq G$ for each train j . Any schedule of each train can be viewed as a path in the subgraph G_j (also in the original graph G). Finding the conflict-free paths for all the trains is defined as the train timetable problem.

1.1 A Binary Integer Programming Model

First, we introduce our model parameters, decision variables, objective functions, and various types of constraints.

Model Parameters

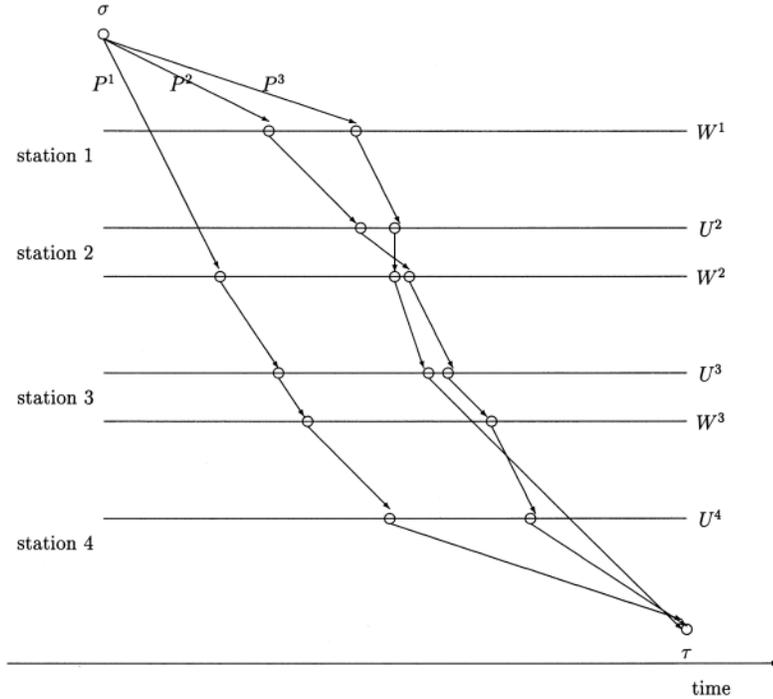
- p_e : the "profit" of using a certain arc e ;
- σ, τ : the artificial origin and destination node;
- J : the set of trains;
- $\delta_j^-(v)$: set of in arcs of node v in E^j ;
- $\delta_j^+(v)$: set of out arcs of node v in E^j ;
- E^j : set of available arcs of train j ;
- E : set of all arcs in graph G ;
- V^j : set of available nodes of train j ;
- V : set of all nodes in graph G ;
- $\mathcal{T}(v)$: set of trains may passing through node v ;
- $\mathcal{N}(v)$: set of nodes conflicted with node v .

Decision Variables

- $x_e = \{0, 1\}$: whether or not use the arc $e \in E$;
- y_v : whether or not use the node v ;
- z_{jv} : whether or not node v is occupied by train j .

Objective Function The objective defines as $\sum_{j \in J} \sum_{e \in E^j} p_e x_e$, which represents the sum of the “profits” of all occupied edges in a certain timetable. Although this objective is a simple linear function, we can greatly enrich the practical meaning by interpreting different definition of ”profit” of each edge. For example, if we set all $p_{(\sigma, v)}$ to 1 and all others to 0, the objective means we maximize the number of trains in the timetable; if we set p_e to the opposite of the block section running time, the objective means that we minimize the total running time of all trains; on top of that, some artificial adjustments are made to some p_e , such as assigning smaller values to those arcs which may be more congested, then the objective function indicates minimizing the total running time as well as considering congestion to some degree. This idea is especially crucial in the subsequent Lagrangian relaxation method, which in essence is to control the degree of congestion of arcs through adjusting the “profit” p_e of each arc in the space-time network.

An example of train paths in graph G (with $s = 4, t = 3, f_1 = 2, l_1 = 4, f_2 = 1, l_2 = 4, f_3 = 1, l_3 = 3$).



Model Constraints Any feasible solution of the problem should satisfy the following constraints:

- For each train j , it can choose at most one starting/ending arcs. Some starting/ending occupied means that there exists some train j in the timetable (to occupy this arc):

$$\sum_{e \in \delta_j^+(\sigma)} x_e \leq 1, \quad \sum_{e \in \delta_j^-(\tau)} x_e \leq 1, \quad j \in J.$$

- Non-artificial nodes must have equal in and out degrees. Actually, the degree should be in $\{0, 1\}$.

$$\sum_{e \in \delta_j^-(v)} x_e = \sum_{e \in \delta_j^+(v)} x_e, \quad j \in J, \quad v \in V \setminus \{\sigma, \tau\}.$$

- Logic constraints: whether the node v is occupied by train j and whether the node v is occupied:

$$z_{jv} = \sum_{e \in \delta_j^-(v)} x_e, \quad j \in J, \quad v \in V^j, \quad \text{and} \quad y_v = \sum_{j \in \mathcal{T}(v)} z_{jv}, \quad v \in V^j.$$

- Headway constraints between trains, which means that only one of the conflicting nodes can be occupied. Headway constraints indicates that the trains departing/entering same station should satisfy certain time lower limit to avoid collision. For any node $v \in U \cup W$, the neighbourhood $\mathcal{N}(v) \subseteq V$ defines a clique constraint:

$$\sum_{v' \in \mathcal{N}(v)} y_{v'} \leq 1, \quad v \in V.$$

For any train $j \in J$, the sets or parameters with superscript or subscript notation corresponds to relevant object to j . The entire 0 – 1 integer programming model is given by

$$\max_x \sum_{j \in J} \sum_{e \in E^j} p_e x_e \tag{1}$$

$$\text{s.t.} \quad \sum_{e \in \delta_j^+(\sigma)} x_e \leq 1, \quad j \in J \tag{2}$$

$$\sum_{e \in \delta_j^-(v)} x_e = \sum_{e \in \delta_j^+(v)} x_e, \quad j \in J, \quad v \in V \setminus \{\sigma, \tau\}, \tag{3}$$

$$\sum_{e \in \delta_j^-(\tau)} x_e \leq 1, \quad j \in J \tag{4}$$

$$z_{jv} = \sum_{e \in \delta_j^-(v)} x_e, \quad j \in J, \quad v \in V^j \tag{5}$$

$$y_v = \sum_{j \in \mathcal{T}(v)} z_{jv}, \quad v \in V^j \tag{6}$$

$$\sum_{v' \in \mathcal{N}(v)} y_{v'} \leq 1, \quad v \in V \tag{7}$$

$$x_e \in \{0, 1\} \quad e \in E, \tag{8}$$

where (2), (3), (4) denotes the arcs of train j should form a valid path in G , (5), (6) represent the logical relationship of x, y, z , (7) represents headway constraints.

This model is a pure binary programming problem with many variables and constraints, and may take a long time to solve directly by a mathematical optimization solver (e.g. Gurobi or COPT).

1.2 The Lagrangian Relaxation Method

Note that constraint (3) is a flow conservation constraint, which means the in and out degree of v must be balanced. Constraints (2) and (4) denote whether a certain train is in the timetable or not. If we only consider constraints (2), (3), (4) and (8), then the model is separable respect to each train and the model for train j is:

$$\max_x \sum_{e \in E^j} p_e x_e \quad (9)$$

$$\text{s.t.} \quad \sum_{e \in \delta_j^+(\sigma)} x_e \leq 1, \quad (10)$$

$$\sum_{e \in \delta_j^-(v)} x_e = \sum_{e \in \delta_j^+(v)} x_e, \quad v \in V \setminus \{\sigma, \tau\}, \quad (11)$$

$$\sum_{e \in \delta_j^-(\tau)} x_e \leq 1, \quad (12)$$

$$x_e \in \{0, 1\} \quad e \in E, \quad (13)$$

The above problem is a shortest path problem which can be solved efficiently in a polynomial time.

Compared to the constraints (2)-(4), the constraints (5)-(7) are all coupling constraints involved with multiple trains. Let $\{\lambda_{v'}\}$ be the Lagrangian multiplier associated to the constraints (7). At the k -th iteration, the Lagrangian relaxation method is to relax the constraints (7) and solves the subproblem

$$x^{k+1} = \arg \max_x \sum_{j \in J} \sum_{e \in E^j} p_e x_e - \sum_{v \in V} \lambda_v^k \left(\sum_{v' \in \mathcal{N}(v)} y_{v'} - 1 \right) \quad (14)$$

$$\text{s.t.} \quad \sum_{e \in \delta_j^+(\sigma)} x_e \leq 1, \quad j \in J \quad (15)$$

$$\sum_{e \in \delta_j^-(v)} x_e = \sum_{e \in \delta_j^+(v)} x_e, \quad j \in J, \quad v \in V \setminus \{\sigma, \tau\}, \quad (16)$$

$$\sum_{e \in \delta_j^-(\tau)} x_e \leq 1, \quad j \in J \quad (17)$$

$$z_{jv} = \sum_{e \in \delta_j^-(v)} x_e, \quad j \in J, \quad v \in V^j \quad (18)$$

$$y_v = \sum_{j \in \mathcal{T}(v)} z_{jv}, \quad v \in V^j \quad (19)$$

$$x_e \in \{0, 1\} \quad e \in E. \quad (20)$$

Although the constraints (18) and (19) are still kept in the model but they are eventually eliminated. Thus, the model (14)-(20) only has variables x_e , and both the objective function and constraints can be decomposed into shortest path sub-problems for each train. Then, the Lagrangian multiplier is updated as

$$\lambda_v^{k+1} = \max\{0, \lambda_v^k + \eta \left(\sum_{v' \in \mathcal{N}(v)} y_{v'} - 1 \right)\}.$$

One important drawback of Lagrangian relaxation is the violation of the relaxing constraints. It is often necessary to obtain a feasible solution by some primal heuristic algorithm.

Primal heuristic algorithm The primal heuristic algorithm module is important for the success of the Lagrangian relaxation method. Since the Lagrangian relaxation method can only obtain pairwise solutions and cannot guarantee to satisfy the relaxation constraints, the heuristic algorithm directly determines the quality of the final output feasible solution. One commonly used heuristic method is Algorithm 1. It is based on the dual solution of Lagrangian relaxation, and constructs a primal solution by scheduling

the congested train first. Note that this heuristic works well when the timetable is not very crowded, otherwise the problem needs to be solved with the aid of a mathematical optimization solver (e.g., gurobi and COPT).

Algorithm 1 Ranking based SPP Primal Heuristic

Require: reordering trains by dual objective function (including multipliers) in descending order.
priority_list \leftarrow sort by descending order of dual obj(more congested train first)
while *priority_list* Not Empty **do**
 Step 1. $j \leftarrow$ first train in *priority_list*
 Step 2. run the SPP algorithm in the origin graph, and remove all conflicting nodes and arcs. If the algorithm succeed, then keep the train in the timetable, otherwise skip the train.
end while
Output all trains with feasible paths, which defines a timetable.

2 A Toy Example

In this section, we present a toy example involving 7 states and 16 trains. The object is to schedual a timetable for these trains withing 160 minutes. The time interval is 1 minute, i.e., there are 161 time nodes at each train station in the space-time network model.

Station Data Each column of Table 1 indicates the name of the station and the distance between each station and starting station A.

station	mile
A	0
B	50
C	100
D	170
E	210
F	250
G	300

Table 1: information of the train stations

Trains Data Each column of Table 2 indicates, from left to right, the train number, the train speed, and the status of the train at this station. Specifically, 0 means that the train will pass through the station directly and 1 means that this train must stop at the station. These information will determine the available arcs E^j of train j .

trainNO	speed	A	B	C	D	E	F	G
G1	350	1	0	0	0	0	1	1
G3	350	1	1	1	0	1	0	1
G5	350	1	1	0	1	0	1	1
G7	350	1	1	0	0	1	0	1
G9	350	1	1	0	0	0	1	1
G11	350	1	1	0	0	1	0	1
G13	300	1	1	1	1	1	1	1
G15	300	1	1	1	1	1	1	1
G17	300	1	1	1	1	1	1	1
G19	300	1	1	1	1	1	1	1
G21	300	1	1	1	1	1	1	1
G23	300	1	1	1	1	1	1	1
G25	300	1	1	1	1	1	1	1
G27	300	1	1	1	1	1	1	1
G29	300	1	1	1	1	1	1	1
G31	300	1	1	1	1	1	1	1

Table 2: status at train stations

Block Section Data Each column of Table 3 indicates, from left to right, the name of the blocked interval, the running time (minutes) of the train at the speed of 300 km/h, and the running time (minutes) of the train at the speed of 350 km/h.

station	runtime(300)	runtime(350)
A-B	10	9
B-C	20	18
C-D	14	12
D-E	8	7
E-F	8	7
F-G	10	8

Table 3: Running time between stations

Other Parameter All kinds of headway time lower bound are set as 5 minutes. If a train stop at a station, it needs to stop at least 2 mins and at most 15 mins.

Simulation Results We run the Lagrangian relaxation method on the sample data with stopping criteria as $ub - lb \leq 0.1ub$. Figure 1 shows the bound updated through iterations and Figure 2 shows the output timetable.

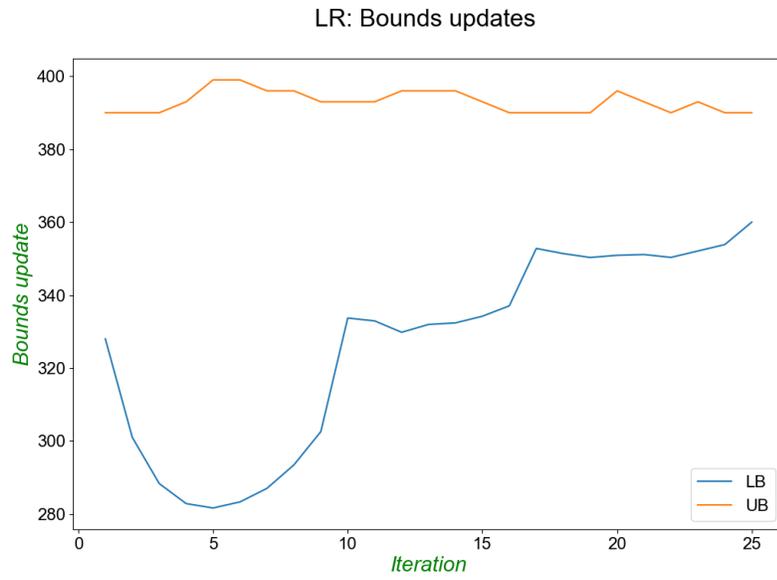


Figure 1: Bound update through iterations

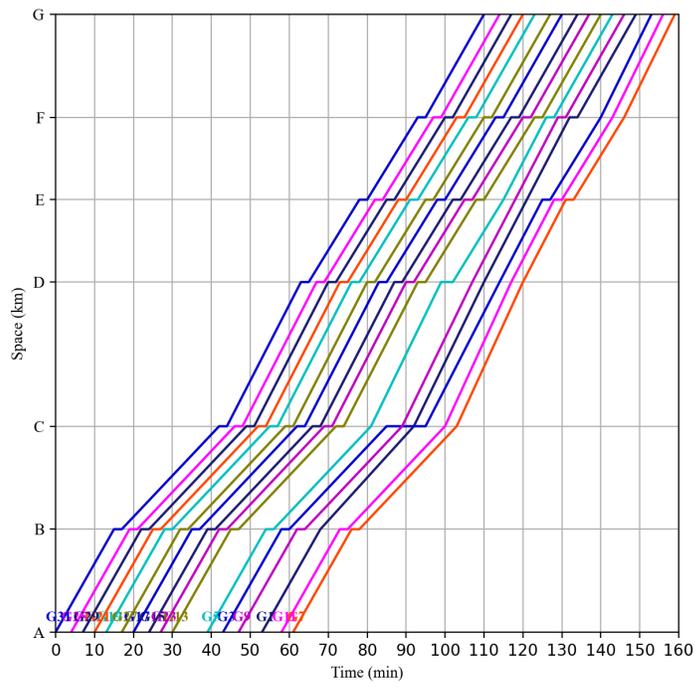


Figure 2: timetable generated by the Lagrangian relaxation method

3 Questions

Submission requirement:

1. Prepare a report including
 - detailed answers to each question
 - numerical results and their interpretation
2. The programming language can be either matlab, Python or c/c++.
3. 6月22日晚12点前将书面报告(包括latex源文件, 程序等等) 打包, 发邮件给助教(pkupt@163.com). 提交的文件请全部打包, 文件名为“proj2-name1-name2.zip”. 提交word的同学需要提供word原文件并将其转换成pdf文件。
4. 请勿大量将代码粘在报告中, 涉及到实际结果需要打表或者作图, 不要截图或者直接从命令行拷贝结果。
5. If you get significant help from others on one routine, write down the source of references at the beginning of this routine.

Project Questions:

1. Read the description of the problem (1)-(8) carefully. Create the space-time network model and write a code to construct the data of the toy example in section 2. Then solve the problem and its LP relaxation by using either Gurobi or Mosek or COPT. Report the number of the variables and constraints as well the CPU time. Plot the timetable similar to Figure 2.
2. Write down and implement an algorithm for solving the shortest path problem (9)-(13). Checking a textbook or a suitable reference if necessary. Note that the constraints (10) and (12) are inequalities.
3. Implement the Lagrangian relaxation method for solving problem (1)-(8). Write down more detailed description the Lagrangian relaxation method if your implementation is different from 1.2. Report the CPU time and the violation of constraints:

$$\text{feas} := \sum_{v \in V} \max \left\{ 0, \sum_{v' \in \mathcal{N}(v)} y_{v'} - 1 \right\}. \quad (21)$$

Plot the timetable similar to Figure 2.

Requirements: feas should be zero. Otherwise, this solution is not meaningful.

4. Write down and implement either the augmented Lagrangian method or the alternating direction method of multipliers for solving problem (1)-(8). Report the CPU time and the violation of constraints defined in (21). Plot the timetable similar to Figure 2.

Requirements: feas should be zero. Otherwise, this solution is not meaningful.

Hints: The objective function of the subproblem with respect to the variable x is a general quadratic function. A possible strategy is to linearize the objective function and add a proximal term. Since $x_e^2 = x_e$ when $x_e \in \{0, 1\}$, the resulted subproblem is still linear and can be solved the same as the shortest path problem (9)-(13).

5. (Optional) Write down and implement a better method (either a classical method or deep learning/reinforcement learning method) to solve problem (1)-(8). It is preferred that this method can be much faster than Gurobi/Mosek/COPT.
6. (Optional) What are the major algorithmic difficulties in items 1-3? Is it possible to reduce the size of problem (1)-(8) significantly? Is there a better model to describe the railway timetabling problem?

References

- [Caprara et al., 2002] Caprara, A., Fischetti, M., and Toth, P. (2002). Modeling and Solving the Train Timetabling Problem. *Operations Research*, 50(5):851–861.
- [Zhang et al., 2020] Zhang, Q., Lusby, R. M., Shang, P., and Zhu, X. (2020). Simultaneously re-optimizing timetables and platform schedules under planned track maintenance for a high-speed railway network. *Transportation Research Part C: Emerging Technologies*, 121:102823.