Lecture: Duality of LP, SOCP and SDP

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Lagrangian

standard form problem (not necessarily convex)

$$\begin{array}{lll} \min & f_0(x) & \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

variable $x \in \mathbb{R}^n$, domain $\mathcal{D}$, optimal value p^*.
Lagrangian: $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, with $\mathbf{dom} \mathcal{L} = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$

$$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- weighted sum of objective and constraint functions
- λ_i is the Lagrange multiplier associated with $f_i(x) \leq 0$
- ν_i is the Lagrange multiplier associated with $h_i(x) = 0$

Lagrange dual function

Lagrange dual function: $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$
\begin{aligned}
g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} \quad \mathcal{L}(x, \lambda, \nu) \\
&= \inf_{x \in \mathcal{D}} \quad \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)
\end{aligned}
$$

g is concave, can be $-\infty$ for some λ, ν

- lower bound property: if $\lambda \geq 0$, then $g(\lambda, \nu) \leq p^*$
- Proof: If $\tilde{x}$ is feasible and $\lambda \geq 0$, then

$$f_0(\tilde{x}) \geq \mathcal{L}(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda, \nu) = g(\lambda, \nu).$$

minimizing over all feasible $\tilde{x}$ gives $p^* \geq g(\lambda, \nu)$.

The dual problem

Lagrange dual problem

$$\begin{array}{ll} \max & g(\lambda, \nu) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

- finds best lower bound on p^*, obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted d^*
- λ, ν are dual feasible if $\lambda \geq 0$, $(\lambda, \nu) \in \mathbf{dom} g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom} g$ explicit

Standard form LP

$$\textbf{(P)} \quad \begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \textbf{(D)} \quad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y + s = c \\ & s \geq 0 \end{array}$$

- Lagrangian is

$$\mathcal{L}(x, \lambda, \nu) = c^\top x + \nu^\top (Ax - b) - \lambda^\top x = -b^\top \nu + (c + A^\top v - \lambda)^\top x$$

- $\mathcal{L}$ is affine in x, hence

$$g(\lambda, \nu) = \inf_x \mathcal{L}(x, \lambda, \nu) = \begin{cases} -b^\top \nu, & A^\top \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

- lower bound property: $p^* \geq -b^\top v$ if $A^\top v + c \geq 0$
- Check the dual of **(D)**

Inequality form LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \end{aligned}$$

- Lagrangian is

$$\mathcal{L}(x, \lambda, \nu) \quad = \quad c^\top x + \nu^\top (Ax - b) = -b^\top \nu + (c + A^\top v)^\top x$$

- $\mathcal{L}$ is affine in x, hence

$$g(\lambda, \nu) = \inf_x \mathcal{L}(x, \lambda, \nu) = \begin{cases} -b^\top \nu, & A^\top \nu + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

- Dual problem

$$\begin{aligned} \max \quad & b^\top v \\ \text{s.t.} \quad & A^\top v = c, \quad v \le 0 \end{aligned}$$

Equality constrained norm minimization

$$\begin{array}{llcll} \min & \|x\| & \quad & \max & b^\top v \\ \text{s.t.} & Ax = b & & \text{s.t.} & \|A^\top v\|_* \leq 1 \end{array}$$

Dual function

$$g(\nu) = \inf_x (\|x\| - \nu^\top Ax + b^\top \nu) = \begin{cases} b^\top \nu & \|A^\top \nu\|_* \leq 1 \\ -\infty & \text{otherwise,} \end{cases}$$

where $\|v\|_* = \sup_{\|u\| \leq 1} u^\top v$ is the dual norm of $\| \cdot \|$

Proof: follows from $\inf_x(\|x\| - y^\top x) = 0$ if $\|y\|_* \leq 1$, $-\infty$ otherwise

- if $\|y\|_* \leq 1$, then $\|x\| - y^\top x \geq 0$ for all x, with equality if $x = 0$
- if $\|y\|_* > 1$, then choose $x = tu$, where $\|u\| \leq 1$, $u^\top y = \|y\|_* > 1$:

$$\|x\| - y^\top x = t(\|u\| - \|y\|_*) \to -\infty \quad \text{as } t \to \infty$$

for all x, with equality if $x = 0$

LP with box constraints

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & -1 \le x \le 1 \end{aligned} \qquad \begin{aligned} \max \quad & -b^\top v - 1^\top \lambda_1 - 1^\top \lambda_2 \\ \text{s.t.} \quad & c + A^\top v + \lambda_1 - \lambda_2 = 0 \\ & \lambda_1 \ge 0, \lambda_2 \ge 0 \end{aligned}$$

reformulation with box constraints made implicit

$$\begin{aligned} \min \quad & f_0(x) = \begin{cases} c^\top x & -1 \le x \le 1 \\ \infty & \text{otherwise} \end{cases} \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Dual function

$$g(\nu) = \inf_{-1 \le x \le 1} (c^\top x + \nu^\top (Ax - b)) = -b^\top \nu - \|A^\top v + c\|_1$$

Dual problem:

$$\max_{\nu} \quad -b^\top \nu - \|A^\top v + c\|_1$$

Lagrange dual and conjugate function

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & Ax \leq b \\ & Cx = d \end{array}$$

- dual function

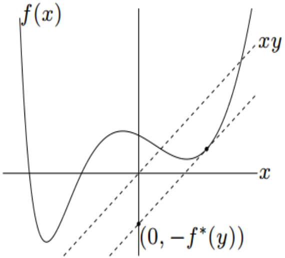
$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \textbf{dom} f_0} (f_0(x) + (A^\top \lambda + C^\top \nu)^\top x - b^\top \lambda - d^\top \nu) \\ &= -f^*(-A^\top \lambda - C^\top \nu) - b^\top \lambda - d^\top \nu \end{aligned}$$

- recall definition of conjugate $f^*(y) = \sup_{x \in \textbf{dom} f} (y^\top x - f(x))$
- simplifies derivation of dual if conjugate of f_0 is known

Conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



f^* is closed and

convex even if f is not **Fenchel's inequality**

$$f(x) + f(y) \geq x^T y \quad \forall x, y$$

(extends inequality $x^T x/2 + y^T y/2 \geq x^T y$ to non-quadratic convex f)

Quadratic function

$$f(x) = \frac{1}{2}x^T Ax + b^T x + c$$

strictly convex case ($A \succ 0$)

$$f^*(y) = \frac{1}{2}(y - b)^T A^{-1}(y - b) - c$$

general convex case ($A \succeq 0$)

$$f^*(y) = \frac{1}{2}(y - b)^T A^{\dagger}(y - b) - c, \quad \mathbf{dom}\, f^* = range(A) + b$$

Negative entropy and negative logarithm

negative entropy

$$f(x) = \sum_{i=1}^{n} x_i \log x_i \qquad f^*(y) = \sum_{i=1}^{n} e^{y_i - 1}$$

negative logarithm

$$f(x) = -\sum_{i=1}^{n} \log x_i \qquad f^*(y) = -\sum_{i=1}^{n} \log(-y_i) - n$$

matrix logarithm

$$f(x) = -\log\det X \quad (\mathbf{dom}\, f = \mathbf{S}^n_{++}) \quad f^*(Y) = -\log\det(-Y) - n$$

Indicator function and norm

indicator of convex set C: **conjugate** is support function of C

$$f(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases} \qquad f^*(y) = \sup_{x \in C} y^T x$$

norm: conjugate is indicator of unit dual norm ball

$$f(x) = ||x|| \qquad f^*(y) = \begin{cases} 0, & ||y||_* \leq 1 \\ +\infty, & ||y||_* > 1 \end{cases}$$

(see next page)

proof: recall the definition of dual norm:

$$||y||_* = \sup_{||x|| \leq 1} x^T y$$

to evaluate $f^*(y) = \sup_x (y^T x - ||x||)$ we distinguish two cases

- if $||y||_* \leq 1$, then (by definition of dual norm)

$$y^T x \leq ||x|| \quad \forall x$$

  and equality holds if $x = 0$; therefore $\sup_x (y^T x - ||x||) = 0$

- if $||y||_* > 1$, there exists an x with $||x|| \leq 1, x^T y > 1$; then

$$f^*(y) \geq y^T (tx) - ||tx|| = t(y^T x - ||x||)$$

  and $r.h.s.$ goes to infinity if $t \to \infty$

The second conjugate

$$f^{**}(x) = \sup_{y \in \mathbf{dom}\, f^*} (x^T y - f^*(y))$$

- $f^{**}(x)$ is closed and convex
- from Fenchel's inequality $x^T y - f^*(y) \leq f(x)$ for all y and x):

  $$f^{**} \leq f(x) \quad \forall x$$

  equivalently, epi $f \subseteq$ epi f^{**} (for any f)
- if f is closed and convex, then

  $$f^{**}(x) = f(x) \quad \forall x$$

  equivalently, epi $f =$ epi f^{**} (if f is closed convex); proof on next page

Conjugates and subgradients

if f is closed and convex, then

$$y \in \partial f(x) \quad \Leftrightarrow \quad x \in \partial f^*(x) \quad \Leftrightarrow \quad x^T y = f(x) + f^*(y)$$

proof: if $y \in \partial f(x)$, then $f^*(y) = \sup_u (y^T u - f(u)) = y^T x - f(x)$

$$\begin{aligned} f^*(v) &= \sup_u (v^T u - f(u)) \\ &\geq v^T x - f(x) \\ &= x^T (v - y) - f(x) + y^T x \\ &= f^*(y) + x^T (v - y) \end{aligned} \tag{1}$$

for all v; therefore, x is a subgradient of f^* at y $(x \in \partial f^*(y))$
reverse implication $x \in \partial f^*(y) \Rightarrow y \in \partial f(x)$ follows from $f^{**} = f$

Some calculus rules

separable sum

$$f(x_1, x_2) = g(x_1) + h(x_2) \qquad f^*(y_1, y_2) = g^*(y_1) + h^*(y_2)$$

scalar multiplication: (for $\alpha > 0$)

$$f(x) = \alpha g(x) \qquad f^*(y) = \alpha g^*(y/\alpha)$$

addition to affine function

$$f(x) = g(x) + a^T x + b \qquad f^*(y) = g^*(y - a) - b$$

infimal convolution

$$f(x) = \inf_{u+v=x} (g(u) + h(v)) \qquad f^*(y) = g^*(y) + h^*(y)$$

Duality and problem reformulations

- equivalent formulations of a problem can lead to very different duals
- reformulating the primal problem can be useful when the dual is difficult to derive, or uninteresting

Common reformulations

- introduce new variables and equality constraints
- make explicit constraints implicit or vice-versa
- transform objective or constraint functions
  e.g., replace $f_0(x)$ by $\phi(f_0(x))$ with ϕ convex, increasing

Introducing new variables and equality constraints

$$\min \quad f_0(Ax+b)$$

- dual function is constant: $g = \inf_x \mathcal{L}(x) = \inf_x f_0(Ax+b) = p^*$
- we have strong duality, but dual is quite useless

reformulated problem and its dual

(P) $\begin{array}{ll} \min & f_0(y) \\ \text{s.t.} & Ax+b=y \end{array}$ **(D)** $\begin{array}{ll} \max & b^\top y - f_0^*(\nu) \\ \text{s.t.} & A^\top \nu = 0 \end{array}$

dual functions follows from

$$\begin{aligned} g(\nu) &= \inf_{x,y} \quad f_0(y) - \nu^\top y + \nu^\top Ax + b^\top \nu \\ &= \begin{cases} -f_0^*(\nu) + b^\top \nu, & A^\top \nu = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Norm approximation problem

$$\min \quad \|Ax - b\| \iff \begin{array}{ll} \min & \|y\| \\ \text{s.t.} & Ax - b = y \end{array}$$

can look up conjugate of $\|\cdot\|$, or derive dual directly

$$\begin{aligned} g(\nu) &= \inf_{x,y} \quad \|y\| + \nu^\top y - \nu^\top Ax + b^\top \nu \\ &= \begin{cases} b^\top \nu + \inf_y \quad \|y\| + \nu^\top y, & A^\top \nu = 0 \\ -\infty, & \text{otherwise} \end{cases} \\ &= \begin{cases} b^\top \nu, \quad A^\top \nu = 0, \quad \|v\|_* \leq 1 \\ -\infty, \quad \text{otherwise} \end{cases} \end{aligned}$$

Dual problem is

$$\begin{array}{ll} \max & b^\top \nu \\ \text{s.t.} & A^\top \nu = 0, \quad \|v\|_* \leq 1 \end{array}$$

Weak and strong duality

weak duality: $d^* \leq p^*$

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems for example, solving the maxcut SDP

strong duality: $d^* = p^*$

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**

Slater's constraint qualification

strong duality holds for a convex problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

If it is strictly feasible, i.e.,

$$\exists x \in \mathrm{int}\mathcal{D} : \quad f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

- also guarantees that the dual optimum is attained (if $p^* > -\infty$)
- can be sharpened: e.g., can replace $\mathrm{int}\mathcal{D}$ with $\mathrm{relint}\mathcal{D}$ (interior relative to affine hull); linear inequalities do not need to hold with strict inequality, ...
- there exist many other types of constraint qualifications

Complementary slackness

assume strong duality holds, x^* is primal optimal, (λ^*, ν^*) is dual feasible

$$\begin{aligned}
f_0(x^*) = g(\lambda^*, \nu^*) &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\
&\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\
&\leq f_0(x^*)
\end{aligned}$$

hence, the two inequalities hold with equality

- x^* minimizes $\mathcal{L}(x, \lambda^*, \nu^*)$
- $\lambda_i^* f_i(x^*) = 0$ for $i = 1, \ldots, m$ (known as complementary slackness):

$$\lambda_i^* > 0 \Longrightarrow f_i(x^*) = 0, \quad f_i(x^*) < 0 \Longrightarrow \lambda_i^* = 0$$

Karush-Kuhn-Tucker (KKT) conditions

the following four conditions are called KKT conditions (for a problem with differentiable f_i, h_i):

1. primal constraints: $f_i(x) \leq 0, i = 1, \ldots, m, \quad h_i(x) = 0, i = 1, \ldots, p$
2. dual constraints: $\lambda \geq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0,\ i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

If strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

KKT conditions for convex problem

If $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal

- from complementary slackness: $f_0(\tilde{x}) = \mathcal{L}(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = \mathcal{L}(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

Hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

if **Slater's condition** is satisfied:
x is optimal if and only if there exist λ, ν that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

LP Duality

Strong duality: If a LP has an optimal solution, so does its dual, and their objective fun. are equal.

dual \ primal	finite	unbounded	infeasible
finite	$\surd$	$\times$	$\times$
unbounded	$\times$	$\times$	$\surd$
infeasible	$\times$	$\surd$	$\surd$

- If $p^* = -\infty$, then $d^* \le p^* = -\infty$, hence dual is infeasible
- If $d^* = +\infty$, then $+\infty = d^* \le p^*$, hence primal is infeasible
-

$$\begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3 \end{array} \qquad \begin{array}{ll} \max & p_1 + 3p_2 \\ \text{s.t.} & p_1 + 2p_2 = 1 \\ & p_1 + 2p_2 = 2 \end{array}$$

Problems with generalized inequalities

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \preceq_{\mathsf{K}_i} 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{aligned}$$

- $f_i(x) \preceq \mathsf{K}_i$ means $-f_i(x) \in \mathsf{K}_i$.
- Lagrangian: $\langle \cdot, \cdot \rangle_{\mathsf{K}_i}$ inner product in K_i

$$\mathcal{L}(x, \lambda_1, \ldots, \lambda_m, \nu) = f_0(x) + \sum_{i=1}^{m} \langle \lambda_i, f_i(x) \rangle_{\mathsf{K}_i} + \sum_{i=1}^{p} \nu_i h_i(x)$$

- dual function

$$g(\lambda_1, \ldots, \lambda_m, \nu) = \inf_{x \in \mathcal{D}} \quad \mathcal{L}(x, \lambda_1, \ldots, \lambda_m, \nu)$$

lower bound property: if $\lambda_i \in \mathsf{K}_i^*$, then $g(\lambda_1, \ldots, \lambda_m, \nu) \leq p^*$
proof: If $\tilde{x}$ is feasible and $\lambda \succeq_{\mathsf{K}_i^*} 0$, then

$$\begin{aligned} f_0(\tilde{x}) &\geq f_0(\tilde{x}) + \sum_{i=1}^{m} \langle \lambda_i, f_i(\tilde{x}) \rangle_{\mathsf{K}_i} + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \\ &\geq \inf_{x \in \mathcal{D}} \quad \mathcal{L}(x, \lambda_1, \ldots, \lambda_m, \nu) \\ &= g(\lambda_1, \ldots, \lambda_m, \nu) \end{aligned}$$

minimizing over all feasible $\tilde{x}$ gives $g(\lambda_1, \ldots, \lambda_m, \nu) \leq p^*$.
Dual problem

$$\begin{array}{ll} \max & g(\lambda_1, \ldots, \lambda_m, \nu) \\ \text{s.t.} & \lambda_i \succeq_{\mathsf{K}_i^*} 0, \quad i = 1, \ldots, m \end{array}$$

- weak duality: $p^* \geq d^*$
- strong duality: $p^* = d^*$ for convex problem with constraint qualification (for example, Slater's: primal problem is strictly feasible)

Semidefinite program

primal SDP ($A_i, C \in \mathbb{S}^n$)

$$\begin{array}{ll} \min & b^T y \\ \text{s.t.} & y_1 A_1 + \cdots + y_m A_m \preceq C \end{array}$$

- Lagrange multiplier is matrix $Z \in \mathbb{S}^n$
- Lagrangian $L(y, Z) = b^T y + \mathrm{tr}(Z(y_1 A_1 + \cdots + y_m A_m - C))$
- dual function

$$g(Z) = \inf_y L(y, Z) = \begin{cases} -\mathrm{tr}(CZ) & \mathrm{tr}(A_i Z) + b_i = 0, \quad i = 1, \dots, m \\ -\infty & \text{otherwise} \end{cases}$$

dual SDP

$$\begin{array}{ll} \max & -\mathrm{tr}(CZ) \\ \text{s.t.} & Z \succeq 0, \mathrm{tr}(A_i Z) + b_i = 0, \quad i = 1, \dots, m \end{array}$$

$p^* = d^*$ if primal SDP is strictly feasible ($\exists y$ with $y_1 A_1 + \cdots + y_m A_m \prec C$)

SDP Relaxtion of Maxcut

$$\begin{array}{ll} \min & x^\top W x \\ \text{s.t.} & x_i^2 = 1 \end{array} \implies \begin{array}{ll} \max & -1^\top \nu \\ \text{s.t.} & W + \mathrm{diag}(\nu) \succeq 0 \end{array} \iff \begin{array}{ll} \min & \mathrm{Tr}(WX) \\ \text{s.t.} & X_{ii} = 1 \\ & X \succeq 0 \end{array}$$

- a nonconvex problem; feasible set contains 2n discrete points
- interpretation: partition $\{1, \ldots, n\}$ in two sets; W_{ij} is cost of assigning i, j to the same set;; $-W_{ij}$ is cost of assigning to different sets

dual function

$$g(\nu) = \inf_x \left(x^\top W x + \sum_i \nu_i (x_i^2 - 1) \right) = \inf_x x^\top (W + \mathrm{diag}(\nu)) x - 1^\top \nu$$

$$= \begin{cases} -1^\top \nu & W + \mathrm{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

SOCP/SDP Duality

(P) $\min \; c^\top x$

s.t. $Ax = b, x_Q \succeq 0$

(D) $\max \; b^\top y$

s.t. $A^\top y + s = c, s_Q \succeq 0$

(P)
$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_1, X \rangle = b_1 \\ & \ldots \\ & \langle A_m, X \rangle = b_m \\ & X \succeq 0 \end{aligned}$$

(D)
$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & \sum_i y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

Strong duality

- If $p^* > -\infty$, (P) is **strictly** feasible, then (D) is feasible and $p^* = d^*$
- If $d^* < +\infty$, (D) is **strictly** feasible, then (P) is feasible and $p^* = d^*$
- If (P) and (D) has **strictly** feasible solutions, then both have optimal solutions.

Failure of SOCP Duality

$$\begin{array}{llll} \inf & (1,-1,0)x & \sup & y \\ \text{s.t.} & (0,0,1)x = 1 & \text{s.t.} & (0,0,1)^\top y + z = (1,-1,0)^\top \\ & x_{\mathcal{Q}} \succeq 0 & & z_{\mathcal{Q}} \succeq 0 \end{array}$$

- primal: $\min\ x_0 - x_1,$ s.t. $x_0 \geq \sqrt{x_1^2 + 1}$; It holds $x_0 - x_1 > 0$ and $x_0 - x_1 \to 0$ if $x_0 = \sqrt{x_1^2 + 1} \to \infty$. Hence, $p^* = 0$, no finite solution
- dual: $\sup\ y$ s.t. $1 \geq \sqrt{1 + y^2}$. Hence, $y = 0$

$p^* = d^*$ but primal is not attainable.

Failure of SDP Duality

Consider

$$\begin{aligned} \min \quad & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \right\rangle = 0 \\ & \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}, X \right\rangle = 2 \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 2y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} y_2 \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

- primal: $X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $p^* = 1$
- dual: $y^* = (0, 0)$. Hence, $d^* = 0$

Both problems have finite optimal values, but $p^* \neq d^*$