Homework 3 for "Algorithms for Big-Data Analysis"

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1. For each of the following function on $\mathbb{R}^n$, explain how to calculate a subgradient at a given x.
   A reference on subgradients is
   `http://bicmr.pku.edu.cn/~wenzw/bigdata/subgradients.pdf`

   (a) $f(x) = \|Ax - b\|_2 + \|x\|_2$ where $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$.

   (b) $f(x) = \inf_y \|Ay - x\|_\infty$ where $A \in \mathbb{R}^{n\times m}$ and $x \in \mathbb{R}^n$.

2. Give a formula or simple algorithm for evaluating the proximal operator

$$\mathrm{prox}_f(x) = \arg\min_u \left(f(u) + \frac{1}{2}\|u - x\|_2^2 \right).$$

   (a) $f(x) = \|x\|_1$ with domain $\mathrm{dom}(f) = \{x \mid \|x\|_\infty \le 1\}$

   (b) $f(x) = \max_k x_k$

   (c) $f(x) = \|Ax - b\|_1$ where $AA^\top = D$ and D is a diagonal matrix whose diagonal elements are positive.

3. Given $w \in \mathbb{R}^n$, $\alpha, \sigma > 0$, write down an algorithm for solving the problem

$$\min_{t,y} \quad \phi(t, y),$$

   where

$$\phi(t, y) := t + \frac{1}{(1-\alpha)n} \sum_{i=1}^{n} (y_i - t)_+ + \frac{\sigma}{2}\|y - w\|_2^2,$$

   where $x_+ := \max(x, 0)$.

4. Let $S^n = \{X \in \mathbb{R}^{n\times n} \mid X^\top = X\}$ and $S^n_{++} = \{X \in \mathbb{R}^{n\times n} \mid X^\top = X, X \text{ is positive definite }\}$. Find the proximal operator of the function $f(X) = -\log\det X$ where $X \in S^n$ and $\mathrm{dom} f = S^n_{++}$. Here, the proximal operator is defined as

$$\mathrm{prox}_f(X) = \arg\min_U \left(f(U) + \frac{1}{2}\|U - X\|_F^2 \right),$$

   where $\|\cdot\|_F$ is the Frobenius norm.