Homework for "Algorithms for Big-Data Analysis"

Beijing International Center for Mathematical Research
Peking University

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Note: Please write up your solutions independently. If you get significant help from others, write down the source of references. A formal mathematical proof for all your claims is required.

1. This exercise shows that an efficient procedure for updating a tableau can be derived from the SMW formula in numerical linear algebra.

   (a) Let C be an $m \times m$ invertible matrix and let $u, v \in \mathbb{R}^m$ be two vectors. Show that

   $$(C + uv^\top)^{-1} = C^{-1} - \frac{C^{-1}uv^\top C^{-1}}{1 + v^\top C^{-1}u}.$$

   (b) Assuming that C^{-1} is available, explain how to obtain $(C + uv^\top)^{-1}$ using only $O(m^2)$ arithmetic operations.

   (c) Let B and $\bar{B}$ be basis matrices before and after an iteration of the simplex method. Let $A_{B(l)}$ and $A_{\bar{B}(l)}$ be the exiting and entering column, respectively. Show that

   $$\bar{B} - B = (A_{\bar{B}(l)} - A_{B(l)})e_l^\top,$$

   where e_l is the lth unit vector.

   (d) Note that $e_i^\top B^{-1}$ is the ith row of B^{-1} and $e_l^\top B^{-1}$ is the pivot row. Show that

   $$e_i^\top \bar{B}^{-1} = e_i^\top B^{-1} - g_i e_l^\top B^{-1}, \quad i = 1, \ldots, m,$$

   for suitable scalars g_i. Provide a formula for g_i. Interpret the above equation in terms of the mechanics for pivoting in the revised simplex method.

2. Let x be an element of the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. Prove that a vector $d \in \mathbb{R}^n$ is a feasible direction at x if and only if $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$.