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Weak Approximation for Cubic Hypersurfaces and Degree 4 del Pezzo Surfaces

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In this article we prove the following theorems about weak approximation of smooth cubic hypersurfaces and del Pezzo surfaces of degree 4 defined over global fields. (1) For cubic hypersurfaces of dimension at least two defined over global function fields, if there is a rational point, then weak approximation holds at places of good reduction whose residue field has at least eleven elements. (2) For del Pezzo surfaces of degree 4 defined over global function fields, if there is a rational point, then weak approximation holds at places of good reduction whose residue field has at least thirteen elements. (3) Weak approximation holds for cubic hypersurfaces of dimension at least ten defined over a global function field of characteristic not equal to $2, 3, 5$ or a purely imaginary number field.

1 Introduction

Let K be a number field or the function field of a smooth projective geometrically connected curve B over a finite field $\mathbb{F}_q$. Given a variety X defined over K, a natural question is to understand the set of K-rational points $X(K)$ of X. We can study the set $X(K)$ of K-rational points by inspecting its local behavior over the completion K_ν at each place ν of K. This naturally leads to the study of local-global problems (i.e., weak approximation)

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for X. The variety X is said to satisfy *weak approximation* if the set of rational points is dense in the set of adelic points.

Smooth cubic hypersurfaces have served as a good testing sample for people to investigate questions related to local-global principles. For smooth cubic hypersurfaces of dimension at least fifteen(or any other smooth hypersurfaces of sufficiently large dimensions) defined over the function field $\mathbb{F}_q(t)$, weak approximation is established in [17]. Recent work [2] of Browning and Vishe shows that weak approximation holds for cubic hypersurfaces of dimension at least six over $\mathbb{F}_q(t)$ with q not a power of 2 or 3. These results depend on the adaption of the circle method to the function field case.

Yong Hu [11] studies weak approximation for cubic hypersurfaces defined over global function fields $K = \mathbb{F}_q(B)$. He proves that if there is a rational point, then weak approximation holds at a single place of good reduction for $q > 47$ and the characteristic is not 2 or 3 using an argument of Swinnerton-Dyer [24], and weak approximation at zeroth order at places of good reduction for $q \geq 11$. In this article we improve his result as follows.

Theorem 1.1. Let X be a smooth cubic hypersurface of dimension at least two defined over the function field $K = \mathbb{F}_q(B)$ of a curve B defined over a finite field $\mathbb{F}_q$. Assume that X has a K-rational point. Then X satisfies weak approximation at finitely many places of good reduction whose residue fields have at least eleven elements. In particular, weak approximation at places of good reduction holds if $q \geq 11$. □

Here, we say that a place ν of K is a *place of good reduction* for X over K if there exists a smooth projective morphism $\widehat{\mathcal{X}} \to \operatorname{Spec}(\widehat{\mathcal{O}}_\nu)$, such that the generic fiber is isomorphic to $X \times_K K_\nu$ and the closed fiber is a smooth cubic hypersurface, where $\mathcal{O}_\nu$ is the ring of integers of the local field K_ν.

The proof of this theorem uses the deformation theory of rational curves, and is a refinement of Hu's approach in [11]. The general deformation techniques allow us to show that there are certain moduli spaces whose points parameterize sections that approximate given formal sections to a given order. This part is the new ingredient in this article in addition to Hu's approach [11] to weak approximation of cubic hypersurfaces. The remaining argument is the same as Hu's. Namely, using the Lang–Weil estimate we show that for all field extensions of large enough degrees, we can find a section defined over that field extension that approximates the given formal sections to a specific order. Finally a trick using the “addition law" on the rational points of cubic hypersurfaces solves the weak approximation problem over the original finite field.

With essentially the same ideas we obtain the following result for a smooth del Pezzo surface of degree 4.

Theorem 1.2. Let X be a smooth del Pezzo surface of degree 4 defined over $K = \mathbb{F}_q(B)$, the function field of a curve B defined over a finite field $\mathbb{F}_q$. Assume that X has a K-rational point. Then X satisfies weak approximation at finitely many places of good reduction whose residue fields have at least thirteen elements. In particular, weak approximation at finitely many places of good reduction holds if $q \geq 13$. □

Here, the definition of places of good reduction is similar to those of cubic hypersurfaces. Namely we require the existence of a smooth projective morphism $\widehat{\mathcal{X}} \to \operatorname{Spec} \widehat{\mathcal{O}}_v$ whose central fiber is a smooth del Pezzo surface of degree 4 (as opposed to any smooth projective surface).

Colliot-Thélène has informed us that one could follow the method of [21] to deduce the result in Theorem 1.2 over a global function field of odd characteristic.

As a side remark, the anti-canonical system of a del Pezzo surface of degree 4 gives an embedding into $\mathbb{P}^4$ as the complete intersection of two quadrics. For a smooth complete intersection of two quadrics in $\mathbb{P}^n$, $n \geq 5$, it is recently proved in [26] that weak approximation at all places holds using a geometric argument due to Colliot-Thélène *et al.* [3, 4]. As for cubic hypersurfaces in $\mathbb{P}^n$ over a number field, partial results are known either by the descent and fibration methods [5, 7, 8] or the circle method [22, 23].

In this paper we prove the following result by a different argument (than the methods used above).

Theorem 1.3. Let X be a smooth cubic hypersurface of dimension at least ten defined over a global field K, which is either a purely imaginary number field or $\mathbb{F}_q(B)$, the function field of a smooth projective geometrically connected curve B defined over a finite field $\mathbb{F}_q$ whose characteristic is not $2, 3, 5$. Then X satisfies weak approximation. □

A smooth projective cubic hypersurface of dimension at least eight defined over a global field always has a rational point (the number field case is settled by [1]).

The proof of Theorem 1.3 is based on a method to construct rational curves over a global field using the geometry of cubic hypersurfaces, first used by Madore [18].

2 Preliminaries

In this section we will recall some important notions and constructions.

2.1 Formulation of weak approximation over function fields

Let k be a field, and let B be a smooth projective curve over k, with function field $K = k(B)$. We assume that X is a smooth proper variety over K.

Definition 2.1. A *model* of X is a flat proper morphism $\pi : \mathcal{X} \to B$ with generic fiber isomorphic to X. □

Each section of the model corresponds to a K-rational point of X, and vice versa. From now on we assume that the field k is either algebraically closed or finite and unwind the definition of weak approximation using a geometric formulation (cf. [9, Section 1]), which is equivalent to the definition in Section 1.

Given a finite sequence $(\nu_1, \ldots, \nu_l)$ of distinct places of K, each ν_i corresponds to a closed point $b_i \in B$. Let $\widehat{\mathcal{O}}_{B,b_i}$ denote the completion of the local ring $\mathcal{O}_{B,b_i}$ at the maximal ideal $\mathfrak{m}_{B,b_i}$, and let K_{ν_i} be the completion of $K = k(B)$ at b_i. Then X satisfies weak approximation at $(\nu_1, \ldots, \nu_l)$ if the image of the diagonal map $X(K) \to \prod_i X(K_{\nu_i})$ is dense, where $\prod_i X(K_{\nu_i})$ takes the product topology of the ν-adic topologies. Now we fix a model $\pi : \mathcal{X} \to B$, a place ν_i, and for each place ν_i a section $\widehat{s}_i$ of the restriction

$$\pi|_{\mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b_i}} : \mathcal{X} \times_B \mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b_i} \to \mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b_i}.$$

Note that $\widehat{s}_i$ corresponds to a point of $X(K_{\nu_i})$.

A basic ν_i-adic open neighborhood consists of $\widehat{s}_i$ in $X(K_{\nu_i})$ consists of sections of $\pi|_{\mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b_i}}$ which agree with $\widehat{s}_i$ modulo $\mathfrak{m}_{B,b_i}^{N+1}$ for some $N \in \mathbb{N}$. Then one can reformulate weak approximation in the following way.

Definition 2.2. We say that X satisfies *weak approximation at order* N if for any finite number of closed points $(b_1, \ldots, b_l)$ and any formal power series sections $(\widehat{s}_1, \ldots, \widehat{s}_l)$ of

$$\mathcal{X} \times_B \mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b_i} \to \mathrm{Spec}\, \widehat{\mathcal{O}}_{B,b_i},$$

there exists a regular section σ of π agreeing with $\widehat{s}_i$ modulo $\mathfrak{m}_{B,b_i}^{N+1}$ for each i. We say X satisfies *weak approximation* if X satisfies weak approximation at any order $N \geq 0$.

We say that X satisfies weak approximation (of order N for some fixed $N \geq 0$) at places of good reduction if the above condition holds when restricted to sequences $(b_1, \ldots, b_l)$ of places of good reduction for X over $k(B)$ (for the given N). □

2.2 Iterated blow-ups

Assume we have a section $\sigma : B \to \mathcal{X}^{sm}$ of the model $\pi : \mathcal{X} \to B$, where $\mathcal{X}^{sm}$ is the smooth locus of π.

Given any jet data

$$J_N = (N; (b_1, \ldots, b_l); \widehat{s}_1, \ldots, \widehat{s}_l),$$

consisting of a number N, closed points $(b_1, \ldots, b_l)$, and formal sections $\widehat{s}_1, \ldots, \widehat{s}_l$ around each points, a section of the model π agreeing with $(\widehat{s}_1, \ldots, \widehat{s}_l)$ to the Nth order is the same as a section in the Nth iterated blowup associated with J (Proposition 11, [10]). To be precise, the *iterated blow-up*

$$\beta(J_N) : \mathcal{X}(J_N) \to \mathcal{X}$$

is obtained by performing a sequence of blow-ups as follows: blow up $\mathcal{X}$ successively N times, where at each stage the center is the point at which the strict transform of $\widehat{s}_i$ meets the fiber over b_i. We denote by $\mathcal{X}(J_k)$ the kth blow-up and denote by $\beta(J_k) : \mathcal{X}(J_k) \to \mathcal{X}$ the morphism. Observe that at each stage we blow up a smooth point of the fiber of the corresponding model and that the result does not depend on the order of the b_i. The procedure is depicted in Figure 2.

The fiber of $\mathcal{X}(J_N)$ over $b_i \in B$ decomposes into irreducible components

$$\mathcal{X}(J_N)_{b_i} = E_{i,0} \cup E_{i,1} \cup \cdots \cup E_{i,N},$$

where

- $E_{i,0}$ is the strict transform of $\mathcal{X}_{b_i}$.
- $E_{i,k} = \mathrm{Bl}_{r_{i,k}}(\mathbb{P}^n)$, $k = 1, \ldots, N-1$, is the blow-up of the intermediate exceptional divisor $\mathbb{P}^n$ at $r_{i,k}$, the point where the strict transform of $\widehat{s}_i$ meets the fiber of the kth blow-up over b_i. Here $n = \dim X$.
- $E_{i,N} \cong \mathbb{P}^n$ is the Nth exceptional divisor.
- The intersection $E_{i,k} \cap E_{i,k+1}$ is the exceptional divisor $\mathbb{P}^{n-1} \subset E_{i,k}$, and a strict transform of a hyperplane in $E_{i,k+1}$ for $k = 0, \ldots, N-1$.
- Each $E_{i,k}$ is a $\mathbb{P}^1$-bundle over the exceptional divisor $\mathbb{P}^{n-1}$ for $k = 1, \ldots, N-1$.

Let $r_i \in E_{i,N} \backslash E_{i,N-1}$ denote the intersection of the strict transform of $\widehat{s}_i$ with $E_{i,N}$. For each section σ' of $\pi \circ \beta(J_k) : \mathcal{X}(J_k) \to \mathcal{X} \to B$, the composition $\beta(J_k) \circ \sigma'$ is a section of

$\pi : \mathcal{X} \to B$. Given a section $\sigma' : \mathcal{X}(J_k) \to B$ with $\sigma'(b_i) = r_i$, composing with $\beta(J_k)$ yields a section σ of $\mathcal{X} \to B$ such that $\sigma \equiv \widehat{s}_i \mod \mathfrak{m}_{B,b_i}^{k+1}$, for $i = 1, \ldots, l$ and for $k = 0, \ldots, N$.

3 Proof of Theorem 1.1

One of the main ingredients in our proof is the deformation technique developed by Kollár *et al.* [14], which was later used by Hassett and Tschinkel [10] to prove weak approximation at places of good reduction of rationally connected varieties defined over $\mathbb{C}(B)$, the function field of a complex curve. Later Yong Hu [11] applied this method to prove weak approximation of order 0 at places of good reduction for cubic hypersurfaces over global function fields. Our approach is motivated by their works. The key to prove weak approximation of an arbitrary order is to show uniform boundedness results for the construction of combs, which we prove in the following section.

3.1 Construction of bounded family of combs

Definition 3.1. Let k be a field. A *comb with n teeth* over k is a nodal curve T over k, which is a union of two curves D and C over k, such that the following conditions hold:

- The curve D is a smooth and geometrically irreducible curve defined over k.
- The curve $\bar{C} := C \otimes_k \bar{k}$ is a union of n subcurves $\bar{T}_1, \ldots, \bar{T}_n$.
- Each $\bar{T}_i$ is a chain of $\mathbb{P}^1_{\bar{k}}$'s.
- Each $\bar{T}_i$ meets $\bar{D} := D \otimes_k \bar{k}$ transversally in a single smooth point of $\bar{D}$; however, the point may not be defined over k.
- $\bar{T}_i \cap \bar{T}_j = \emptyset$, for all $i \neq j$. □

Here the curve D is called the *handle* of the comb and each $\bar{T}_i$ a *tooth*.

Let B be a smooth projective curve over a field k and let $\pi : \mathcal{X} \to B$ be a flat projective morphism with smooth separably rationally connected generic fiber. To prove weak approximation of the generic fiber, we need to produce sections passing through some given points. To this end it is crucial to construct combs of bounded degree. The following lemmas will produce bounded family of sections over an algebraically closed field (cf. Lemma 3.2) and then over a finite field (cf. Lemma 3.3).

Lemma 3.2. Keep the notations introduced above and assume furthermore that the field k is algebraically closed. Let $\mathcal{H}$ be an ample divisor on $\mathcal{X}$. For any triple of positive integers (d, L, N), there exists an integer $r := r(d, L, N) > 0$ such that for any section $\sigma : B \to \mathcal{X}^{\mathrm{sm}}$ in the smooth locus of π, $\deg_{\mathcal{H}}(\sigma) \leq d$ and any sequence of distinct closed points $(b_1, \ldots, b_L) \subset B$, we can construct a comb T together with a morphism $f : T \to \mathcal{X}$ which satisfies the following conditions.

(1) The comb T takes B as its handle and the morphism f restricted to B is the section $\sigma : B \to \mathcal{X}$.
(2) The comb T has r-teeth mapping to r very free curves $\{T_1, \ldots, T_r\}$ with bounded $\mathcal{H}$-degree in distinct general fibers over points other than $(b_1, \ldots, b_L)$.
(3) The morphism $f : T \to \mathcal{X}$ is an immersion. That is, the map $f^*\Omega_{\mathcal{X}} \to \Omega_T$ is surjective and has locally free kernal. We define the normal sheaf $\mathcal{N}_T$ as the dual of the kernal of the surjection $f^*\Omega_{\mathcal{X}} \to \Omega_T$.
(4) If the fiber dimension is at least three, then we may choose the immersion $f : T \to \mathcal{X}$ to be an embedding.
(5) $H^1\left(B, \mathcal{N}_T|_B\left(-(N+1)\left(\sum b_i\right)\right)\right) = 0$.
(6) The sheaf $\mathcal{N}_T|_B\left(-(N+1)\left(\sum b_i\right)\right)$ is globally generated. □

This is proved in [6, Section 2.1], although the authors of [6] did not make explicit statement about the boundedness of the degree. Such boundedness result is immediate from proof in *loc. cit.*.

Then we can use the Lang–Weil estimate to prove the following boundedness result over finite fields.

Lemma 3.3. Keep the notations above. Assume that the field k is finite. Let $\mathcal{H}$ be an ample divisor on $\mathcal{X}$. For any triple of positive integers (d, L, N), there exist integers $r > 0$ and $m_0 > 0$ such that for any $m \geq m_0$, any section $\sigma : B \to \mathcal{X}^{\mathrm{sm}}$ defined over $\mathbb{F}_{q^m}$, $\deg_{\mathcal{H}}(\sigma) \leq d$ and any sequence of distinct closed points $(b_1, \ldots, b_l) \subset B$ defined over $\mathbb{F}_{q^m}$, where $\sum \deg[\kappa(b_i) : \mathbb{F}_{q^m}] = L$, we can construct a comb T with r teeth together with a morphism $f : T \to \mathcal{X}$ satisfying the following conditions.

(1) The comb T takes B as its handle and the morphism f restricted to B is the section $\sigma : B \to \mathcal{X}^{\mathrm{sm}}$.
(2) The comb T has r teeth mapping to r very free curves $\{T_1, \ldots, T_r\}$ with bounded $\mathcal{H}$-degree in general fibers outside $(b_1, \ldots, b_l)$.
(3) The morphism $f : T \to \mathcal{X}$ is an immersion.

(4) If the fiber dimension is at least three, then we may choose the immersion $f : T \to \mathcal{X}$ to be an embedding.

(5) $H^1\left(B, \mathcal{N}_T|_B\left(-(N+1)\left(\sum b_i\right)\right)\right) = 0$.

(6) The sheaf $\mathcal{N}_T|_B\left(-(N+1)\left(\sum b_i\right)\right)$ is globally generated. □

Proof. In our proof, all degrees are taken with respect to the ample divisor $\mathcal{H}$. Let S denote the set of places of bad reduction, and let (d, L, N) be a triple of positive integers. Let r be the number of very free curves obtained from Lemma 3.2 for the model $\pi : \mathcal{X} \to B$ over $\bar{\mathbb{F}}_q$ and the triple (d, L, N).

Consider the space $\mathcal{S}$ parameterizing the following data:

$$\left\{\begin{array}{l} \text{a section } \sigma : B \to \mathcal{X}^{\mathrm{sm}}, \deg(\sigma) \le d; \\ \text{a collection of } L \text{ distinct points } (b_1, \ldots, b_L) \subset \underbrace{B \times \ldots \times B}_{L}; \\ \text{a collection of } r \text{ distinct points } \{x_1, \ldots, x_r\} \in \underbrace{B \times \ldots \times B}_{r}; \\ \text{relative tangent directions } (v_1, \ldots, v_r) \in \left((\mathcal{T}^{\mathrm{rel}}_{\mathcal{X}})|_{\sigma(x_1)}, \ldots, (\mathcal{T}^{\mathrm{rel}}_{\mathcal{X}})|_{\sigma(x_r)}\right); \end{array}\right\}$$

subject to the following requirements:

- The fiber $\mathcal{X}_{b_i}$ over each b_i is smooth and separably rationally connected.
- The two sets of points $\{x_j\}_{j=1}^r$ and $\{b_i\}_{i=1}^L$ have empty intersection.
- One can attach r very free curves of bounded degree to $\sigma(B)$ at $\{x_j\}_{j=1}^r$ along the tangent directions $(v_1, \ldots, v_r)$ to obtain a comb T, such that

$$H^1\left(B, \mathcal{N}_T|_B\left(-(N+1)\left(\sum b_i\right)\right)\right) = 0$$

  and $\mathcal{N}_T|_B\left(-(N+1)\left(\sum b_i\right)\right)$ is globally generated.

Let $\mathcal{B}$ be the space parameterizing pairs consisting of a section σ of degree at most d and a collection of L distinct points $(b_1, \ldots, b_L)$ in $B \backslash S$, where fibers $\mathcal{X}_{b_i}$ are smooth and separably rationally connected. Lemma 3.2 ensures that fibers of $\mathcal{S}$ over any point $(\sigma, (b_1, \ldots, b_L)) \in \mathcal{B}$ is non-empty. The fiber of $\mathcal{S}$ over any point $(\sigma, (b_1, \ldots, b_L)) \in \mathcal{B}$ is geometrically irreducible.

Now we can complete the proof by applying the Lang and Weil estimate [16]. For a geometrically irreducible quasiprojective variety U, we embed it into a projective space of dimension n and take its closure $\bar{U}$ in $\mathbb{P}^n$. Denote by ∂U the complement of U in $\bar{U}$. In order to apply the Lang–Weil estimate to find a rational point in U for every large enough finite field, we need to bound the numbers $\deg \bar{U}$, $\deg \partial U$, $\dim U$, and the dimension of the ambient projective space n. If we have a family of geometrically irreducible varieties $p : \mathcal{U} \to \mathcal{T}$, with $\mathcal{U}$ and $\mathcal{T}$ quasi-projective, then we can realize the morphism p as an open immersion into some $\overline{\mathcal{U}}$ which is projective over $\mathcal{T}$. Thus there is a universal bound of the above numbers for each fiber over $\mathcal{T}$. We apply this to the family $\mathcal{S}$ over the base $\mathcal{B}$. Therefore, there exists an integer m_0' such that for any $m \geq m_0'$, for any section $\sigma : B \to \mathcal{X}^{\text{sm}}$ and $(b_1, \ldots, b_l)_L \subset B$ defined over $\mathbb{F}_{q^m}$, there is an $\mathbb{F}_{q^m}$-rational point of the fiber of $\mathcal{S}$ over $(\sigma, (b_1, \ldots, b_l)_L)$.

By [15, Theorem 2], there exist integers $a_0 > 0$ and $d_0 > 0$ such that for any $m \geq a_0$, $b \in B\backslash S$, any $\kappa(b)$-point x in $\mathcal{X}_b$, any tangent vector v in the relative tangent direction $\mathcal{T}_x^{\text{rel}}$, there exists an immersed very free rational curve C in $\mathcal{X}_b$ of degree at most d_0 passing through x over $\mathbb{F}_{q^m}$ whose tangent direction at x is v.

We take m_0 to be the larger of m_0' and a_0. ■

3.2 The induction and the reduction

Section 3.1 provides a construction of combs with bounded degree, and these combs will be used to construct sections for weak approximation of a prescribed order here. We will adopt notations from Section 2.

Lemma 3.4. Let $\pi : \mathcal{X} \to B$ be a flat projective family over a finite field $\mathbb{F}_q$. Assume that there is a section $\sigma : B \to \mathcal{X}^{sm}$ in the smooth locus of π, and the generic fiber is smooth projective and separably rationally connected. Let $\mathcal{H}$ be an ample divisor on $\mathcal{X}$. Let $S \subset B$ denote the finite set of points the fibers over which are either singular or not separably rationally connected. Then for any two positive integers N and L, there exist two integers $C_{N,L}, d_{N,L}$, both of which only depend on the numbers N and L, and a bounded family $\mathcal{M}_{N,L}$ parameterizing sections in the smooth locus of π with $\mathcal{H}$-degree at most $d_{N,L}$, such that the following is true:

For every $m \geq C_{N,L}$, every collection of closed points $(b_1, \ldots, b_l) \subset B\backslash S$ defined over $\mathbb{F}_{q^m}$, with the property that $\sum_i \deg[\kappa(b_i) : \mathbb{F}_{q^m}] = L$, and every sequence $(\widehat{s}_1, \ldots, \widehat{s}_l)$ of formal power series sections of π over $\operatorname{Spec} \widehat{\mathcal{O}}_{B,b_1}, \ldots, \operatorname{Spec} \widehat{\mathcal{O}}_{B,b_l}$, there is a section $s : B \to \mathcal{X}^{sm}$, parameterized by an $\mathbb{F}_{q^m}$-point of $\mathcal{M}_{N,L}$ such that the section s of π is congruent to $\widehat{s}_i$ modulo $\mathfrak{m}_{B,b_i}^{N+1}$ for $i = 1, \ldots, l$. □

Proof. Unless otherwise specified, all degrees discussed in our proof are taken with respect to $\mathcal{H}$. We use induction on N and adopt notations of iterated blow-ups from Section 2.2.

The case where $N = 0$ follows from Proposition 3 [11], that is there exists a lower bound $C_{0,L}$, an upper bound $d_{0,L}$, and a bounded family $\mathcal{M}_{0,L}$ of sections $\sigma_0 : B \to \mathcal{X}^{\mathrm{sm}}$ of degree at most $d_{0,L}$, such that for every $m \geq C_{0,L}$, weak approximation at order $N = 0$ holds at any finite sequence of distinct closed points $(b_1, \ldots, b_l)_L \subset B \backslash S$ defined over $\mathbb{F}_{q^m}$.

Assume the statement is true for $n = N - 1$.

For $n = N$ and $m \geq C_{N-1,L}$, consider the jet data,

$$J_N = (N; (b_1, \ldots, b_l)\,; \widehat{s}_1, \ldots, \widehat{s}_l)\,,$$

where $(b_1, \ldots, b_l) \subset B\backslash S$ are closed points defined over $\mathbb{F}_{q^m}$, $\sum_i \deg[\kappa(b_i) : \mathbb{F}_{q^m}] = L$, and $(\widehat{s}_1, \ldots, \widehat{s}_l)$ is a sequence formal power series sections of π over Spec $\widehat{\mathcal{O}}_{B,b_i}$ over $\mathbb{F}_{q^m}$. As in Section 2.2, we consider the iterated blow-up

$$\beta(J_N) : \mathcal{X}(J_N) \to \mathcal{X}(J_{N-1}) \to \ldots \to \mathcal{X}$$

associated with the jet data.

By the assumption of induction, we have a bounded family $\mathcal{M}_{N-1,L}$ of sections $\sigma_{N-1} : B \to \mathcal{X}^{\mathrm{sm}}$ of degree at most $d_{N-1,L}$, such that

$$\sigma_{N-1} \equiv \widehat{s}_i \mod \mathfrak{m}_{B,b_i}^{N-1+1} \text{ for } i = 1, \ldots, l.$$

In terms of the iterated blow-up shown in Figure 1, this is equivalent to the following: in the $(N-1)$th iterated blow-up $\mathcal{X}(J_{N-1}) \to \mathcal{X}$, the strict transform of σ_{N-1} meets the strict transforms of $\widehat{s}_i$ in the exceptional divisors $E'_{i,N-1} \cong \mathbb{P}^n$, for $i = 1, \ldots, l$. We denote these intersection points by $r_{i,N-1}$ for $i = 1, \ldots, l$. Then $\mathcal{X}(J_N)$ is obtained by blowing up $\mathcal{X}(J_{N-1})$ at $r_{i,N-1} \in E'_{i,N-1}$ for $i = 1, \ldots, l$.

The fiber $\mathcal{X}(J_N)_{b_i}$ decomposes into irreducible components

$$\mathcal{X}(J_N)_{b_i} = E_{i,0} \cup E_{i,1} \cup \ldots \cup E_{i,N-1} \cup E_{i,N}.$$

Let $r_i \in E_{i,N} \backslash E_{i,N-1}$ be the intersection of the strict transform of $\widehat{s}_i$ and $E_{i,N}$, let $\tilde{\sigma}_{N-1}$ denote the strict transform of σ_{N-1} in $\mathcal{X}(J_N)$, and let $\widetilde{\mathcal{H}} \subset \mathcal{X}(J_N)$ denote the pullback of the ample divisor $\mathcal{H}$.

Over $\mathbb{F}_{q^m}, m \geq \max(C_{N-1,L}, m_0)$, where $m_0 = m_0(d_{N-1}, N, L)$ is the number m_0 given in Lemma 3.3, we can construct a comb T_{J_N} with an immersion $f_{J_N} : T_{J_N} \to \mathcal{X}(J_N)$ as follows.

Figure 2 depicts the construction of a teeth of T_{J_N}.

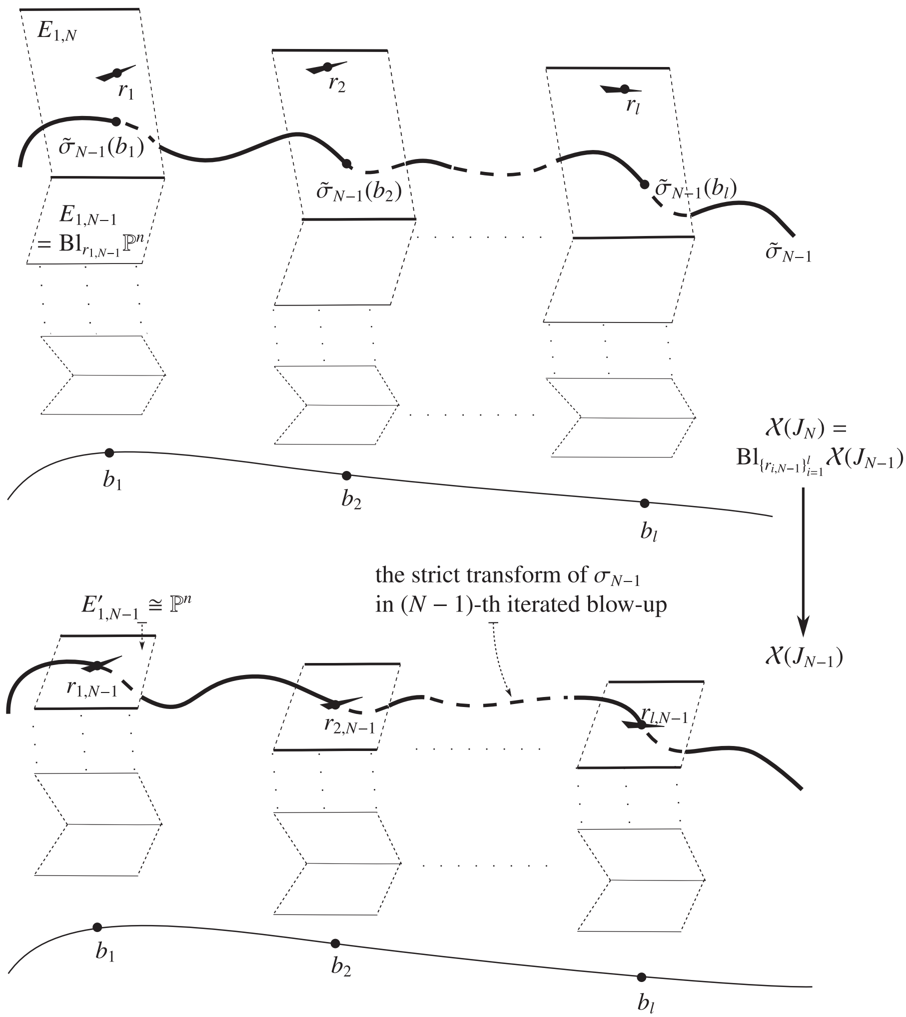

Fig. 1. From $(N-1)$th iterated blow-up to the Nth iterated blow-up.

Step 1: Take B as the handle and f restricted to B is the section $\tilde{\sigma}_{N-1}(B)$.

Step 2: For the pair of points r_i and $\tilde{\sigma}_{N-1}(b_i)$, connect them with a line $L_{i,N} \subset E_{i,N} \cong \mathbb{P}^n$, and $L_{i,N}$ intersects the exceptional divisor $\mathbb{P}^{n-1} \subset E_{i,N-1}$ at a point $P_{i,N-1}$.

Step 3: Since $E_{i,N-1}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1} = E_{i,N-1} \cap E_{i,N}$, we have a $\mathbb{P}^1$ passing through $P_{i,N-1}$ and intersects the exceptional divisor $\mathbb{P}^{n-1} \subset E_{i,N-2}$ at a point $P_{i,N-2}$.

Step 4: Rep eating the previous step till $E_{i,1}$, we obtain a chain of $\mathbb{P}^1$'s starting from $E_{i,N-1}$ and arriving in $E_{i,1}$. Let P_i be the intersection of the $\mathbb{P}^1$ in $E_{i,1}$ and the exceptional divisor of $E_{i,0}$, the proper transform of $\mathcal{X}_{b_i}$.

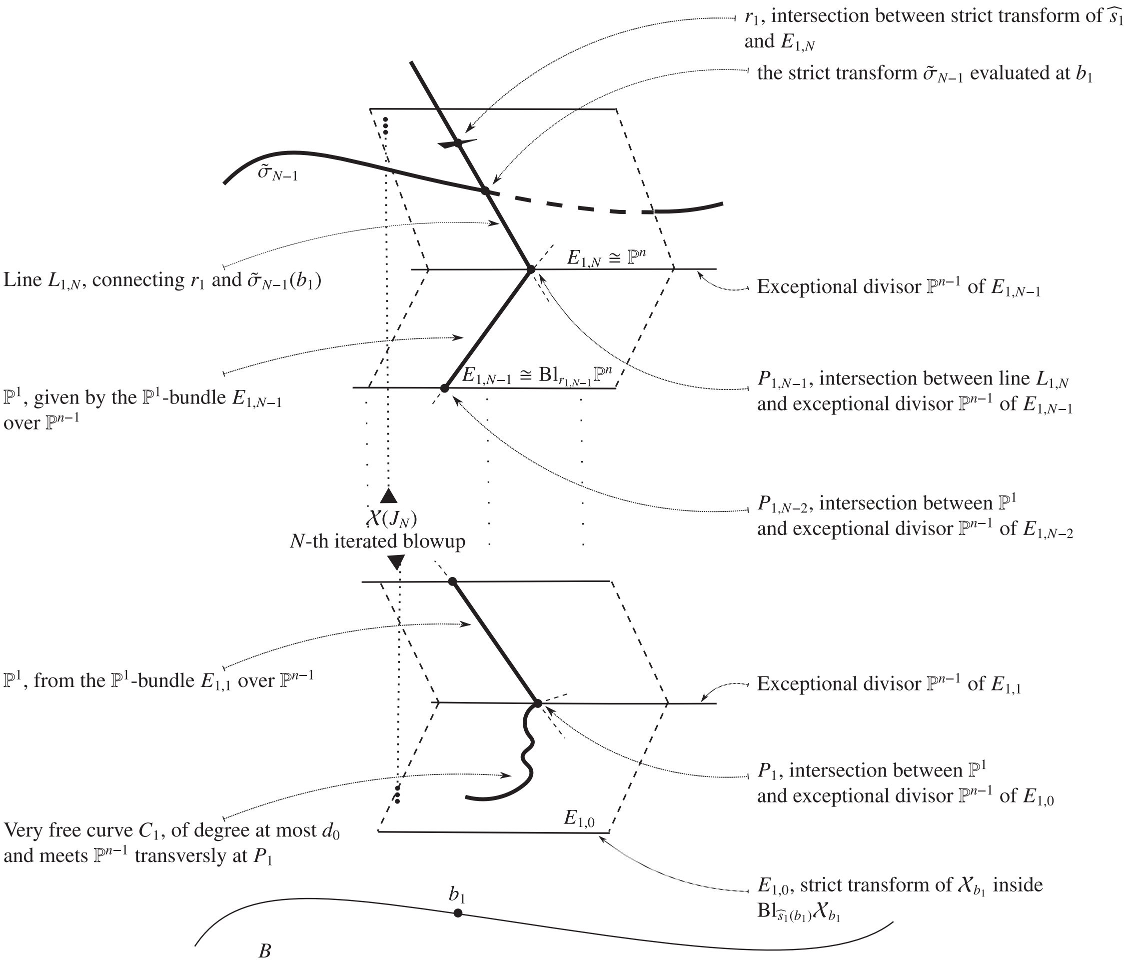

Fig. 2. Consider the fiber $\mathcal{X}(J_N)_{b_1}$ of the Nth iterated blow of $\mathcal{X}$ (Figure 1), the solid chain of curves—with C_1 at one end, the line $L_{1,N}$ at the other end-forms a teeth of bounded degree of T_J.

Step 5: Attach a very free curve C_i at P_i in $E_{i,0}$, with $\deg_{\widetilde{\mathcal{H}}}(C_i)$ bounded by some d_0, such that C_i meets the exceptional divisor $\mathbb{P}^{n-1}$ of $E_{i,1}$ transversely at P_i. The number d_0 is independent of the points b_i. This can be achieved by [15, Theorem 2].

Step 6: Take a chain of $N+1$ $\mathbb{P}^1$'s as a teeth T_i attached to B at b_i. Map the teeth T_i to the union of $L_{i,N}$, $\mathbb{P}^1$'s, and C_i obtained from the above steps. We have the bound $\deg_{\widetilde{\mathcal{H}}}(T_i) \le d_0$.

Let $p_i \in T_{J_N}$ be points mapped to r_i. Since $\mathcal{N}_{\sigma_{N-1}}\left(-N\left(\sum b_i\right)\right) = \mathcal{N}_{\tilde{\sigma}_{N-1}}$, the calculation of [10, Lemma 26], combined with Lemma 3.3, shows that as long as m is at least

$\max(m_0, C_{N-1,L})$, we can assemble a new comb over $\mathbb{F}_{q^m}$ with a morphism to $\mathcal{X}J_N$, still denoted by T_{J_N} with a little abuse of notations, such that $\mathcal{N}_{T_{J_N}}\left(-\sum p_i\right)$ is globally generated and has no H^1. The new comb T_{J_N} is constructed by attaching r very free immersed (or embedded if the fiber dimension is at least three) rational curves with bounded $\widetilde{\mathcal{H}}$-degree ($\leq d_0$) in fibers outside $b_1, \ldots, b_l$, The $\widetilde{\mathcal{H}}$-degree of T_{J_N} is bounded from above by $d_{N,L} := d_{N-1,L} + (r+L)d_0$.

Now let us consider the space $\mathcal{B}$ parameterizing the following data:

$$\left\{\begin{array}{l} \text{a collection of } L \text{ distinct points } (b_1, \ldots, b_L) \in \underbrace{B \times \cdots \times B}_{L}; \\ \text{a sequence } (\widehat{s}_1|_{\operatorname{Spec}(\widehat{\mathcal{O}}_{B,b_1}/\mathfrak{m}_{B,b_1}^{N+1})}, \ldots, \widehat{s}_L|_{\operatorname{Spec}(\widehat{\mathcal{O}}_{B,b_1}/\mathfrak{m}_{B,b_1}^{N+1})}) \\ \ \text{of } N\text{-jet of formal sections } \widehat{s}_i; \\ \text{iterated blow-up } \mathcal{X}(J_N),\ J_N = (N, (b_1, \ldots, b_L), (\widehat{s}_1, \ldots, \widehat{s}_L)); \\ \text{sequence of points } (r_1, \ldots, r_L) \in (E_{1,N}, \ldots, E_{L,N}), \\ \text{where } r_i \text{ is the intersection of strict transform of } \widehat{s}_i \text{ and } E_{i,N}; \end{array}\right\}$$

Let $\mathcal{M}$ be a family of sections over $\mathcal{B}$, with the fiber $\mathcal{M}_{\mathfrak{b}}$ over each point $\mathfrak{b} \in \mathcal{B}$ parameterizing the collection of all sections $\sigma : B \to \mathcal{X}(J_N)$ passing through $(r_1, \ldots, r_L)$ such that $\beta(J_N) \circ \sigma$ is in the smooth locus of π and σ has $\widetilde{\mathcal{H}}$-degree at most $d_{N,L}$ together with an immersion f to $\mathcal{X}(J_N)$, where J_N is the jet data parameterized by the point $\mathfrak{b}$. Each moduli space $\mathcal{M}_{\mathfrak{b}}$ has finitely many components. Consider the closure $\overline{\mathcal{M}}_{\mathfrak{b}}$ of $\mathcal{M}_{\mathfrak{b}}$ in the moduli space of stable maps. Let $\overline{M}_{\mathfrak{b}}$ denote the coarse moduli space of $\overline{\mathcal{M}}_{\mathfrak{b}}$. Since a section has no automorphisms, $\mathcal{M}$ is an open subset of $\overline{M}$.

For any $m \geq \max(m_0, C_{N-1,L})$, and any $\mathbb{F}_{q^m}$-point $\mathfrak{b} \in \mathcal{B}$, our construction of comb T_{J_N} gives a smooth and non-stacky $\mathbb{F}_{q^m}$-point of $\overline{\mathcal{M}}_{\mathfrak{b}}$, which has a smoothing so that a general member of the smoothing lies in $\mathcal{M}$. This indicates that there is a geometrically irreducible component of $\overline{\mathcal{M}}_{\mathfrak{b}}$ defined over $\mathbb{F}_{q^m}$. Furthermore, this component contains an open subset that is a geometrically irreducible component of $\mathcal{M}_{\mathfrak{b}}$ defined over $\mathbb{F}_{q^m}$.

Now we are ready to apply the Lang–Weil estimate and the strategy is the same as Lemma 3.3. Namely, as $\mathfrak{b} \in \mathcal{B}$ varies, $\overline{M}_{\mathfrak{b}}$ forms a bounded family. There is a universal bound for the dimension, degree, and $\deg\left(\partial \overline{M}_{\mathfrak{b}}\right)$ of each fiber. Then Lang–Weil estimate indicates that there exists $C_{N,L} > 0$ such that when $m \geq C_{N,L}$, there is an $\mathbb{F}_{q^m}$-point of $\mathcal{M}_{\mathfrak{b}}$.

Finally, we take the family $\mathcal{M}_{N,L}$ of sections of $\mathcal{X}^{\mathrm{sm}}$ to be the family of sections whose degree is at most $d_{N,L}$. ■

In the following, we show that we can go back to a smaller field using the special geometry of cubic hypersurfaces. The first is a result on "lifting" formal sections.

Lemma 3.5. Assume that q is at least eleven. Given a finite sequence $(b_1, \ldots, b_k)$ of distinct closed points of B, such that each fiber $\mathcal{X}_{b_i}$ is a smooth cubic hypersurface, and a sequence $(\widehat{s}_1, \ldots, \widehat{s}_k)$ of formal power series sections of π over Spec $\widehat{\mathcal{O}}_{B,b_i}$, there is a sequence $(\widehat{s}_1{}', \ldots, \widehat{s}_k{}')$ of formal power series sections of π over Spec $\widehat{\mathcal{O}}_{B,b_i} \otimes \mathbb{F}_{q^2}$ but not over Spec $\widehat{\mathcal{O}}_{B,b_i}$, and a sequence $(\widehat{L}_1, \ldots, \widehat{L}_k)$ of lines defined over Spec $\widehat{\mathcal{O}}_{B,b_i}$ such that $\widehat{L}_i \cap \mathcal{X} = \{\widehat{s}_i, \widehat{s}_i', \widehat{s}_i''\}$, where $\widehat{s}_i''$ is the conjugate of $\widehat{s}_i'$ under $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. □

Proof. Let x_i be the intersection of $\widehat{s}_i$ with $\mathcal{X}_{b_i}$.

By Lemma 9.4 [13], given any smooth cubic hypersurface $X \subset \mathbb{P}^n, n \geq 2$ over $\mathbb{F}_q, q \geq 11$, and any $\mathbb{F}_q$-point $x_i \in X(\mathbb{F}_q)$, there is a point $y_i \in X(\mathbb{F}_{q^2}) - X(\mathbb{F}_q)$ and a line L_i defined over $\mathbb{F}_q$ such that $L_i \cap X = \{x_i, y_i, y_i^\sigma\}$, where y_i^σ is the conjugate point of y_i under $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$.

So there are lines $\widehat{L}_i$ defined over Spec $\widehat{\mathcal{O}}_{B,b_i}$ such that $\widehat{L}_i \times \mathrm{Spec}\,\kappa(b_i) = L_i$ and such that $\widehat{s}_i \in \widehat{L}_i \cap \mathcal{X}$. Take $\widehat{s}_i'$ and $\widehat{s}_i''$ to be the intersection of $\widehat{L}_i$ with $\mathcal{X}$. ■

Now we prove the key reduction lemma.

Lemma 3.6. Let X be a smooth cubic hypersurface defined over $\mathbb{F}_{q^k}(B)$. Assume that q is at least eleven. If weak approximation at order N holds for places of good reduction over $\mathbb{F}_{q^{2k}}(B)$, then weak approximation at order N holds for places of good reduction over $\mathbb{F}_{q^k}(B)$. □

Proof. By Lemma 3.5, given a finite sequence $(b_1, \ldots, b_l)$ of distinct closed points of B such that the fiber $\mathcal{X}_{b_i}$ is smooth, and a sequence $(\widehat{s}_1, \ldots, \widehat{s}_l)$ of formal power series sections of π over Spec $\widehat{\mathcal{O}}_{B,b_i}$, there is a sequence $(\widehat{s}_1', \ldots, \widehat{s}_l')$ of formal power series sections of π over Spec $\widehat{\mathcal{O}}_{B,b_i} \otimes \mathbb{F}_{q^{2k}}$ but not over Spec $\widehat{\mathcal{O}}_{B,b_i}$, and a sequence $(\widehat{L}_1, \ldots, \widehat{L}_l)$ of lines defined over Spec $\widehat{\mathcal{O}}_{B,b_i}$ such that $\widehat{L}_i \cap \mathcal{X} = \{\widehat{s}_i, \widehat{s}_i', \widehat{s}_i''\}$, where $\widehat{s}_i''$ is the conjugate of $\widehat{s}_i'$ under $\mathrm{Gal}(\mathbb{F}_{q^{2k}}/\mathbb{F}_{q^k})$.

By assumption, there is a section s' defined over $\mathbb{F}_{q^{2k}}$ such that

$$s' \cong \widehat{s}_i' \otimes \mathbb{F}_{q^{2k}} \mod \mathfrak{m}_{B,b_i}^{N+1}.$$

By the choice of $\widehat{s}_i'$, s' is not defined over $\mathbb{F}_{q^k}$. Let s'' be the conjugate section of s'. Then it satisfies

$$s'' \cong \widehat{s}_i'' \otimes \mathbb{F}_{q^{2k}} \mod \mathfrak{m}_{B,b_i}^{N+1}.$$

Take the family of line $\mathcal{L}$ spanned by s' and s''. The family $\mathcal{L}$ is necessarily defined over $\mathbb{F}_{q^k}$. Let s be the third section in the intersection of $\mathcal{L}$ and $\mathcal{X}$. Then this section s meets the approximation requirement by construction. ■

We now finish the proof of Theorem 1.1. For any cubic hypersurface of dimension at least two defined over $\mathbb{F}_q(B)$, there is a model $\pi : \mathcal{X} \to B$ over B such that for each closed point $b \in B$, if the corresponding place is a place of good reduction, then the fiber of $\mathcal{X}$ over b is a smooth cubic hypersurface of dimension at least two. By using Néron desingularization [20, Corollary 2.4], we may assume that for the model $\pi : \mathcal{X} \to B$, there is a section $B \to \mathcal{X}^{\mathrm{sm}}$ in the smooth locus of $\mathcal{X}$. Any smooth cubic hypersurface of dimension at least two is separably rationally connected. Thus the set $S \subset B$ in Lemma 3.4 is just the set of points over which the fibers are singular. Given the collection of closed points $b_1, \ldots, b_l$ and the order $N \geq 0$ we want to approximate, there is a number M such that weak approximation at order N holds over these points over $\mathbb{F}_{q^{2m}}$ for any $m \geq M$ by the Lemma 3.4. Hence, weak approximation at order N for $q \geq 11$ at these points of good reduction follows from Lemma 3.6.

4 Proof of Theorem 1.2

In this section, we will discuss weak approximation for del Pezzo surfaces of degree 4 at places of good reduction whose residue fields have at least thirteen elements (assuming the existence of rational points) by applying ideas similar to the case of cubic hypersurfaces (cf. Section 3).

By Lemma 3.4, the set of rational points of a del Pezzo surface of degree at least four (in fact any smooth projective separably rationally connected variety) defined over a global function field is either Zariski dense or empty. The classical geometry of degree 4 del Pezzo surfaces naturally suggests that one may blow up a point not contained in the sixteen lines and get a smooth cubic surface with a line, and thus one may reduce the weak approximation problem to the case of cubic surfaces. However, in our case we cannot use this strategy, at least not in this simple-minded way. The reason is the following: for a place ν of good reduction, we do not know after the blow-up it is a place of good reduction if there is rational point over the function for the cubic surface under our definition (cf. Section 1). More precisely, it is possible that the central fiber of the family only becomes a weak Fano surface and the linear system of the anti-canonical bundle contracts some (-2)-curves. This could happen if all the $\mathbb{F}_q$-rational points of the fiber are contained in one of the sixteen lines.

Hence, instead of using the reduction to cubic surfaces as the above, we will prove the following proposition, then weak approximation for del Pezzo surfaces of degree 4 at places of good reductions follows by the same argument as in the proof of Theorem 1.1 in Section 3.

Throughout the rest of this section, let X be a smooth del Pezzo surface of degree 4 defined over a finite field $\mathbb{F}_q$, and let $X \hookrightarrow \mathbb{P}^4$ be the embedding given by the anti-canonical bundle of X.

Proposition 4.1. If $q \geq 13$, then given a smooth $\mathbb{F}_q$-rational point x of X, there exists a plane Π defined over $\mathbb{F}_q$ such that the intersection of Π with X consists of x and three other points, each of which is defined over $\mathbb{F}_{q^3}$. □

We need the following lemmas.

Lemma 4.2. If $q \geq 7$, then given any smooth $\mathbb{F}_q$-rational point x of X, there exists a hyperplane $H \subset \mathbb{P}^4$ containing x defined over $\mathbb{F}_q$ such that the intersection of H with X is a geometrically integral genus 1 curve which is smooth at x. □

Proof. The intersection $H \cap X$ fails to be geometrically integral if and only if H contains a line or a conic in X. In the case where H contains a conic not passing through x, there is a "residual" conic (or a chain of two lines) passing through the point x.

The geometry of del Pezzo surfaces of degree 4 provides us with the following information.

- Given a line contained in X, if the point x does not lie in the line, then there are $q+1$ hyperplanes containing the point x and the line.
- If the point x lies in a line, then there are q^2+q+1 hyperplanes containing x and the line.
- Since there are sixteen lines in X and a point in X can lie in at most three lines, we have at most $3(q^2+q+1)+13(q+1)$ $\mathbb{F}_q$-hyperplanes containing a line and the point x.
- Since there are at most five conics containing x, and any hyperplane containing a conic also contains the plane spanned by the conic, we have at most $5(q+1)$ hyperplanes containing such a conic.
- There are $q+1$ tangent hyperplanes at x.

There are q^3+q^2+q+1 hyperplanes defined over $\mathbb{F}_q$ containing x. As long as $q \geq 7$, we can find a hyperplane over $\mathbb{F}_q$ with desired properties. ■

Lemma 4.3. Let $E \subset \mathbb{P}^2$ be a geometrically integral degree 3 curve over a finite field $\mathbb{F}_q$. If $q \geq 13$, then given any smooth $\mathbb{F}_q$-rational point x of X, there exists a line L defined over $\mathbb{F}_q$ such that the intersection of L with E consists three Galois conjugate points, all of which are defined over $\mathbb{F}_{q^3}$. □

Proof. We first consider the case E is smooth.

Denote by n (respectively, m) the number of $\mathbb{F}_q$ (respectively, $\mathbb{F}_{q^2}$) points of E. There are three types of lines we want to avoid.

- There are n tangent lines to E.
- There are at most $\frac{m-n}{2}$ lines which intersect E at one $\mathbb{F}_q$ point and two $\mathbb{F}_{q^2}$ (but not $\mathbb{F}_q$) points.
- There are at most $\frac{1}{3}\binom{n}{2}$ lines which intersect E at three $\mathbb{F}_q$ points.

Thus there are at most $\frac{n(n+2)}{6} + \frac{m}{2}$ lines to avoid. By the Hasse–Weil estimate, $n \leq 1 + q + 2\sqrt{q}, m \leq 1 + q^2 + 2q$. So when $q \geq 13$, there is at least one line which is not of the three types listed above.

If E is singular, then it is either a nodal or a cupidal cubic. In either case there is a unique singular point. Assume the number of $\mathbb{F}_q$ (respectively, $\mathbb{F}_{q^2}$) points of E is $1 + n$ (respectively, $1 + m$). Then $n = q - 1, q$ or $q + 1$, $m = q^2$ or $q^2 - 1$.

There are $q+1$ lines containing the unique singular point, n lines tangent to E at an $\mathbb{F}_q$ point, $\frac{m-n}{2}$ lines which intersect E at one $\mathbb{F}_q$ point and two $\mathbb{F}_{q^2}$ (but not $\mathbb{F}_q$) points, at most $\frac{1}{3}\binom{n}{2}$ lines which intersect E at three $\mathbb{F}_q$ points. By a simple computation, for any q we can find the desired line L. ■

Proposition 4.1 now follows easily.

Proof of Proposition 4.1. We use the hyperplane $H \subset \mathbb{P}^4$ given by Lemma 4.2, and take the intersection $H \cap X$ to get a genus one curve in $H \cong \mathbb{P}^3$. Projecting the curve from x yields a degree 3 curve in $\mathbb{P}^2 \subset H \subset \mathbb{P}^4$. Take the plane in H spanned by x and the line given by Lemma 4.3. This is the plane we want. ■

5 Proof of Theorem 1.3

The general idea of the proof is first to approximate the ν-adic points by a rational point p in a degree 2 field extension K' of K, then produce a rational curve over the original field K containing all the Galois conjugate points of p. Since weak approximation holds for $\mathbb{P}^1_K$, we prove weak approximation over the original field K. This strategy depends closely on the existence of rational curves connecting two general rational points over a global field, which we discuss below.

The following lemma is essentially [18, Lemma 1.3]. The only difference is that in our situation it is crucial to have a single rational curve, as opposed to the situation in *loc. cit.*, where the author studies R-equivalence. The proof is adapted from the proof in *loc. cit.*.

Lemma 5.1. Let K be a purely imaginary number field or the function field of a smooth irreducible curve B defined over a finite field. Let V be a singular cubic hypersurface of dimension at least four with an ordinary double point P which is a K-rational point. Then given any other K-rational point R, there is a K-morphism $f : \mathbb{P}^1_K \to V$ such that $f(0) = P, f(\infty) = R$. □

Proof. We may assume that the line $\overline{PR}$ spanned by the two points P and R is not contained in V, otherwise there is nothing to prove.

Without loss of generality, we can assume P is the point $[1, 0, \ldots, 0] \in \mathbb{P}^n_{[X_0,\ldots,X_n]}$. Then we can write the equation of the cubic hypersurface V as $X_0 Q(X_1, \ldots, X_n) + C(X_1, \ldots, X_n) = 0$, where Q is a quadratic form and C is a cubic form. Denote the coordinate of R by $[r_0, r_1, \ldots, r_n]$. Since the line $\overline{PR}$ is not in V, $Q(r_1, \ldots, r_n) \neq 0$. In particular at least one of $r_1, \ldots, r_n$ is non-zero.

Since the point P is an ordinary double point, the equation $Q(X_1, \ldots, X_n) = 0$ defines a smooth quadric hypersurface of dimension at least three in $\mathbb{P}^{n-1}$. By the assumption on the field K, the set of rational points of this smooth quadric hypersurface is Zariski dense. In particular, there is a K-point $S = [s_1, \ldots, s_n]$ in $\mathbb{P}^{n-1}$ such that $Q(s_1, \ldots, s_n) = 0$ and $C(s_1, \ldots, s_n)$ does not vanish.

Now consider the K-rational map

$$\phi : \mathbb{P}^{n-1} \dashrightarrow V$$

$$[X_1, \ldots, X_n] \mapsto [-C(X_1, \ldots, X_n), X_1 Q(X_1, \ldots, X_n), \ldots, X_n Q(X_1, \ldots, X_n)].$$

This gives a birational isomorphism between $\mathbb{P}^{n-1}$ and V. Furthermore, the birational map ϕ is a morphism near the points S and $[r_1, \ldots, r_n]$ by the above discussion. In particular, $\phi(S) = P, \phi([r_1, \ldots, r_n]) = R$. Then ϕ maps the line in $\mathbb{P}^{n-1}$ connecting S and $[r_1, \ldots, r_n]$ to a rational curve connecting P and R in V. ■

The following lemma is essentially proved in [18, Proposition 1.4]. Our version has stronger hypothesis and stronger result.

Lemma 5.2. Let K be either a purely imaginary number field or a global function field and let X be a smooth cubic hypersurface of dimension $n \geq 10$ defined over K. Let x and y be two K-rational points such that the line L spanned by x and y intersects X at three distinct points x, y, z. Denote by H_x, H_y, H_z, respectively, the projective tangent hyperplane at x, y, z. Assume the following conditions.

(1) The family of lines passing through x is geometrically irreducible and has dimension $n-3$.
(2) The intersection $H_z \cap X$ is a singular cubic hypersurface with only one ordinary double point z.
(3) The intersection $H_x \cap H_z \cap X$ is a smooth cubic hypersurface of dimension $n-2$.

Then there is a K-morphism $f : \mathbb{P}^1_K \to X$ such that $f(0) = x, f(\infty) = y$. □

Proof. By assumption neither x nor y is contained in H_z.

Let $t_x : X \dashrightarrow X$ be the birational involution defined by sending a general point p to the point q such that x, p, q are colinear. For a point p in $H_x \cap X$, not equal to the point x, such that the line spanned by p and x is not contained in X, the birational map t_x is defined near p and $t_x(p) = x$.

The intersection $H_x \cap H_z \cap X$ is a divisor in $H_x \cap X$. By the assumption ((1)), the locus of points swept out by the family of lines through x,

$$\mathrm{Line}(x) = \{w | w \neq x, \text{ and the line spanned by } x \text{ and } w \text{ is contained in } X\} \cup \{x\}$$

is an irreducible divisor and contains the point x by definition. Since H_z does not contain x, the intersection $H_x \cap H_z \cap X$ is not completely contained in $\mathrm{Line}(x)$. By the assumption on the field K, there is a K-rational point u in the intersection $H_x \cap H_z \cap X$. If K is a global function field, this follows from the fact that K is a C_2-field in the sense of Lang. In the case of number fields, this is proved in [1]. By [12, Theorem 1], the intersection $H_x \cap H_z \cap X$ is then K-unirational and thus has a Zariski dense set of rational points. In particular, there is a K-rational point u such that the birational involution t_x is defined around u. Lemma 5.1 shows that there is a K-rational curve in $H_z \cap X$, thus also in X, containing the two points z and u. Then we can take f to the restriction of t_x to this curve. ■

Recall the statement of Theorem 1.3.

Theorem 1.3 *Let X be a smooth cubic hypersurface in $\mathbb{P}^n$, $n \geq 11$ defined over a global field K, which is either a purely imaginary field or $\mathbb{F}_q(B)$, the function field of a curve B defined over a finite field $\mathbb{F}_q$ of characteristic not equal to* 2, 3, 5. *Then X satisfies weak approximation.*

Proof of Theorem 1.3. Under the assumptions, X is K-unirational and has a Zariski dense set of K-rational points, as explained in the proof of Lemma 5.2. Choose x to be a general K-point such that

- The family of lines through x is geometrically irreducible and has dimension $n-4$.
- Denote by H_x the projective tangent hyperplane at x. Then the projective tangent cone of $H_x \cap X$ has an ordinary double point at x.

These two properties can be achieved since the set of points in X satisfying these properties is non-empty and Zariski dense [26, Lemma 5.4, 5.7].

Denote by x_ν the K_ν-rational point induced by x for each place ν. Given finitely many places $\nu_1, \ldots, \nu_k$ and any ν_i-adic points y_{ν_i} which we want to approximate, let M_i be the line spanned by x_{ν_i} and y_{ν_i} for each i. Without loss of generality, we may assume that the line M_i intersect the cubic hypersurface at a third point z_{ν_i} different from x_{ν_i} and y_{ν_i}. We may also assume that the points z_{ν_i} are general points having the following properties:

- The family of lines through z_{ν_i} is geometrically irreducible and has dimension $n-4$.
- Let $H_{z_{\nu_i}}$ be the tangent hyperplane at z_{ν_i}. Then $H_{z_{\nu_i}} \cap X$ is a singular cubic hypersurface with an ordinary double point z_{ν_i}.
- The intersection $H_{x_{\nu_i}} \cap H_{z_{\nu_i}} \cap X$ is a smooth cubic hypersurface of dimension $n-2$.

For any pre-specified order of weak approximation, there is a line M defined over K which approximates all the M_i's to that order and contains the point x. This follows from weak approximation for the space of lines containing x, which holds as this space is isomorphic to $\mathbb{P}^d_K$ for some d.

If the line M intersect X at two other K-rational points y, z, then by Lemma 5.2, there is a rational curve containing them (note that our assumptions on z_{ν_i} imply the assumptions of the lemma). In particular, the base change of this rational curve to K_{ν_i} contains K_{ν_i} points which approximate the points y_{ν_i} to order N. Then the theorem follows since we know weak approximation for $\mathbb{P}^1_K$ and thus can find a rational point in $\mathbb{P}^1_K$ which approximates the ν_i-adic points y_{ν_i}.

Now assume the line M intersect X at two points y, z which are defined over a separable degree 2 field extension L/K. By construction, the irreducible (over K) degree 2 zero cycle $y + z$ is sufficiently close to the degree 2 zero cycle $y_{\nu_i} + z_{\nu_i}$ in the ν_i-adic topology for each ν_i.

In the following we will show that there is a rational curve defined over K which contains the degree 2 zero cycle $y + z$. Once we know this, the theorem follows for the same reason as above.

Consider the Weil restriction of scalars $\mathrm{Res}_{L/K} X_L$ and the K-rational dominant map $\mathrm{Res}_{L/K} X_L \dashrightarrow X$. After base change to L, the Weil restriction is isomorphic to $X_L \times X_L$ and the rational map maps a general pair of points (u, v) to the point w such that u, v, w are colinear (c.f. [19, Section 15, Proposition 15.1], [13, Example 3.8, Exercise 3.11], and a detailed discussion of the former two in [25, Construction 2.1]).

Recall that by the universal property of Weil restrictions, a K-morphism $f : \mathbb{P}^1_K \to \mathrm{Res}_{L/K} X_L$ is the same as an L-morphism $g : \mathbb{P}^1_L \to X_L$; and an L-point of $\mathrm{Res}_{L/K} X_L$ is the same as two L-points of X_L.

There are two L-rational points u, v of $\mathrm{Res}_{L/K} X_L$, which are mapped to z and y, respectively, that is the points corresponding to the pairs of L-rational points (x, y) and (x, z). Moreover, these two points are conjugate to each other by the Galois group $\mathrm{Gal}(L/K)$.

So it suffices to show that there is a rational curve over K which contains one of the two Galois conjugate L-rational points u, v of $\mathrm{Res}_{L/K} X_L$. This is equivalent to showing that there is a rational curve defined over L containing x and y (or a rational curve defined over L containing x and z), which follows from Lemma 5.2. ■

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