Moduli Space of Quasi-Polarized K3 Surfaces of Degree 6 and 8*

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Abstract In this paper, the authors study the moduli space of quasi-polarized complex K3 surfaces of degree 6 and 8 via geometric invariant theory. The general members in such moduli spaces are complete intersections in projective spaces and they have natural GIT constructions for the corresponding moduli spaces and they show that the K3 surfaces with at worst ADE singularities are GIT stable. They give a concrete description of boundary of the compactification of the degree 6 case via the Hilbert-Mumford criterion. They compute the Picard group via Noether-Lefschetz theory and discuss the connection to the Looijenga's compactifications from arithmetic perspective. One of the main ingredients is the study of the projective models of K3 surfaces in terms of Noether-Lefschetz divisors.

Keywords K3 surfaces, GIT, Noether-Lefschetz divisor, Looijenga compactification
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1 Introduction

A primitively quasi-polarized K3 surface (S, L) of degree 2ℓ over $\mathbb{C}$ consists of a complex K3 surfaces, a big and nef line bundle L such that $c_1(L) \in H^2(S, \mathbb{Z})$ is a primitive class and $L^2 = 2\ell$. Let $\mathcal{F}_{2\ell}$ be the moduli space of primitively quasi-polarized complex K3 surfaces of degree 2ℓ. It is well-known that the period map behaves very well on $\mathcal{F}_{2\ell}$. Namely, if we denote by $\mathbb{D}$ the period domain of K3 surfaces and $\Gamma_{2\ell}$ the monodromy group, global Torelli theorem tells us that $\mathcal{F}_{2\ell}$ is isomorphic to $\Gamma_{2\ell}\backslash\mathbb{D}$ via the period map.

Besides the Hodge theoretical construction, there is also explicit algebraic construction of $\mathcal{F}_{2\ell}$ via geometric invariant theory (GIT for short) for low degree K3 surfaces, where such a general K3 surface is a complete intersection in the projective space. For instance, the GIT construction of $\mathcal{F}_2$ and $\mathcal{F}_4$ has been worked out by Shah (see [20–21]). When $2\ell = 6$ or 8, a general element $(S, L) \in \mathcal{F}_{2\ell}$ is a complete intersection of a smooth quadric and a cubic in $\mathbb{P}^4$ or a complete intersection of three quadrics in $\mathbb{P}^5$ respectively.

In this paper, we describe the GIT construction of moduli space of these complete intersects and characterize the image of period map for such complete intersections as a complement of

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certain Noether-Lefschetz (NL for short) divisors in $\Gamma_{2\ell}\backslash\mathbb{D}$. The latter one has a natural arithmetic compactification constructed by Looijenga (see [14]), and we will compare this arithmetic compactification with the natural GIT compactification.

More precisely, for any non-negative integers d, g, the NL-divisor $D_{d,g}^{2\ell} \subset \mathcal{F}_{2\ell} \cong \Gamma_{2\ell}\backslash\mathbb{D}$ is defined to be the locus of quasi-polarized complex K3 surfaces $(S, L) \in \mathcal{F}_{2\ell}$ such that $\mathrm{Pic}(S)$ contains a rank two primitive sublattice of the following form:

$$
\begin{array}{c|c|c}
 & L & \beta \\
\hline
L & 2\ell & d \\
\hline
\beta & d & 2g-2
\end{array}
\tag{1.1}
$$

for some $\beta \in \mathrm{Pic}(S)$. For simplicity of notations, we identify $D_{d,g}^{2\ell}$ as divisors on $\Gamma_{2\ell}\backslash\mathbb{D}$ via period map. One of our main result is in the following.

Theorem 1.1 *For $\ell = 3, 4$, the complete intersections in $\mathbb{P}^{\ell+1}$ of degree 2ℓ with at worst simple singularities (i.e., isolated* ADE *singularities) are* GIT*-stable. Let $\mathcal{M}_{2\ell}$ be the moduli space of such complete intersections with at worst simple singularities. Then the period map extends to $\mathcal{M}_{2\ell}$ and its image in $\Gamma_{2\ell}\backslash\mathbb{D}$ is the complement of $D_{1,1}^{2\ell}$, $D_{2,1}^{2\ell}$ and $D_{3,1}^{2\ell}$.*

Furthermore, the natural GIT *compactification $\overline{\mathcal{M}}_{2\ell}$ is not isomorphic to Looijenga's compactification of the complement $\Gamma_{2\ell}\backslash\mathbb{D} - \bigcup\limits_{d=1}^{3} D_{d,1}^{2\ell}$.*

Remark 1.1 We refer the readers to [6] for the analysis of GIT stability for such complete intersections with semi log canonical singularities.

Secondly, we have classified the boundary components of $\mathcal{M}_6$ in its GIT compactification $\overline{\mathcal{M}}_6$. The main result is as follows.

Theorem 1.2 *The boundary $\overline{\mathcal{M}}_6\backslash\mathcal{M}_6$ consists of 9 irreducible components whose general member X is described as follows:*

α) (**Semitable**) *X has two corank 3 singularities, but not a union of a quadric surface and a quadric cone with multiplicity two.*

$\widetilde{\beta}$) (**Stable**) *X is a union of a smooth quadric surface and a smooth complete intersection of two quadrics.*

γ) (**Semistable**) *X has two simple elliptic singularities of type $\widetilde{E}_8$, whose projective tangent cone intersect X along lines, and not the union of three quadric cones.*

δ) (**Stable**) *X has an isolated $\widetilde{E}_7$ singularity.*

ϵ) (**Stable**) *X has an isolated $\widetilde{E}_8$ singularity, whose projective tangent cone meets X at a point.*

ζ) (**Stable**) *X is singular along a line.*

η) (**Stable**) *X is singular along a conic.*

θ) (**Stable**) *X is singular along a twisted cubic.*

ϕ) (**Stable**) *X is singular along a rational normal curve of degree 4.*

The stratum α is 6-dimensional, $\widetilde{\beta}$ is 7-dimensional (it contains a 2-dimensional semistable loci β), γ and ϕ are 2-dimensional, δ and ζ are 11-dimensional, ϵ is 8-dimensional, η is 7-dimensional, θ is 3-dimensional.

According to the work of [22], the Baily-Borel compactification $(\Gamma_6 \backslash \mathbb{D})^*$ of the Shimura variety $\Gamma_6 \backslash \mathbb{D}$ consists of 10 irreducible components. The extended period map induces a birational map

$$\overline{\mathcal{M}_6} \dashrightarrow (\Gamma_6 \backslash \mathbb{D})^*. \tag{1.2}$$

According to the spirit of Hassett-Keel-Looijenga program raised by [12], it is expected that the map (1.2) can factor through a sequence of elementary birational transformations of Shimura type, i.e., the exceptional loci comes from Shimura subvarieties. This problem will be solved in a forthcoming paper (see [8]).

In [16], Maulik and Pandharipande have conjectured that the Picard group of $\mathcal{F}_{2\ell}$ with $\mathbb{Q}$-coefficients is spanned by the NL-divisors $\{D_{d,g}^{2\ell}\}$ on $\mathcal{F}_{2\ell}$. This conjecture has been verified in [3] via automorphic representation theory and finding a geometric approach remains highly interesting. Here, using the main theorem, we can compute the Picard group of $\mathcal{F}_6$ and $\mathcal{F}_8$ from the GIT construction.

Corollary 1.1 *When $2\ell = 6$ or 8, the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\ell})$ with rational coefficients is spanned by* NL*-divisors $D_{d,1}^{2\ell}, d = 1, 2, 3, 4$. Moreover,*

$$\dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\ell}) = \dim H^2(\Gamma_{2\ell}, \mathbb{Q}) = 4$$

for $2\ell = 6$ or 8.

The first part of this result has been also obtained by O'grady in [18] using a slightly different method. We just point out that the similar approach has been applied to K3 surfaces with Mukai models (i.e., $10 \leq 2\ell \leq 18$ or $2\ell = 22$) in [9].

2 Noether-Lefschetz Divisors for K3 Surfaces

Let us recall the Noether-Lefschetz theory on K3 surfaces.

2.1 Noether-Lefschetz divisors

Let (S, L) be a primitively quasi-polarized K3 surface of degree 2ℓ. The middle cohomology $\Lambda := H^2(S, \mathbb{Z})$ is a unimodular even lattice of signature $(3, 19)$ under the intersection form $\langle, \rangle$. Let $h_{2\ell} = c_1(L)$, then the orthogonal complement $\Lambda_{2\ell} := h_{2\ell}^{\perp} \subset \Lambda$ is an even lattice of signature $(2, 19)$, which has a unique representation:

$$\Lambda_{2\ell} = \mathbb{Z}\omega \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}, \tag{2.1}$$

where $\langle \omega, \omega \rangle = -2\ell$, U is the hyperbolic plane and $E_8(-1)$ is the unimodular, negative definite even lattice of rank 8.

Let $\Lambda_{2\ell}^{\mathbb{C}} = \Lambda_{2\ell} \otimes \mathbb{C}$. The period domain $\mathbb{D}$ associated to $\Lambda_{2\ell}$ can be realized as a connected component of

$$\mathbb{D}^{\pm} := \{v \in \mathbb{P}(\Lambda_{2\ell}^{\mathbb{C}}) \mid \langle v, v \rangle = 0, -\langle v, \overline{v} \rangle > 0\}.$$

The monodromy group

$$\Gamma_{2\ell} = \{g \in \mathrm{Aut}(\Lambda_{2\ell})^+ \mid g \text{ acts trivially on } \Lambda_{2\ell}^{\vee} / \Lambda_{2\ell}\},$$

naturally acts on $\mathbb{D}$, where $\mathrm{Aut}(\Lambda_{2\ell})^+$ is the identity component of $\mathrm{Aut}(\Lambda_{2\ell})$. According to the global Torelli theorem of K3 surfaces, there is an isomorphism

$$\mathcal{F}_{2\ell} \cong \Gamma_{2\ell} \backslash \mathbb{D}$$

via the period map. Then $\mathcal{F}_{2\ell}$ is a locally Hermitian symmetric variety with only quotient singularities, and hence $\mathbb{Q}$-factorial.

The NL-divisor $D_{d,g}^{2\ell}$ can be identified as the quotient of the union of subdomains on $\mathbb{D}$ as following:

$$D_{d,g}^{2\ell} \cong \Gamma_{2\ell} \backslash \bigcup_{M \subset \Lambda} \{z \in \mathbb{D} \mid \langle z, m \rangle = 0, \forall m \in M\}, \tag{2.2}$$

where M runs for all rank two primitive sublattice of $\Lambda_{2\ell}$ of the form (1.1). In the language of Heegner divisors, the right-hand side of (2.2) is called the arithmetic quotient of hyperplane arrangement in $\mathbb{D}$. As known in [18, Proposition 1.3], we have the irreducibility theorem.

Theorem 2.1 *All the* NL*-divisors* $D_{d,g}^{2\ell} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\ell})$ *are irreducible.*

Remark 2.1 The definition of NL-divisors we used here is slightly different from the one in [16]. Maulik and Pandharipande define the NL-divisors without the assumption of primitivity of the sublattice M in (2.2). But the span of these divisors are the same as ours (see [16, §0.2]).

2.2 Dimension formula

Let us denote by $\mathrm{Pic}_{\mathbb{Q}}(\Gamma_{2\ell} \backslash \mathbb{D})^{\mathrm{NL}}$ the subgroup of $\mathrm{Pic}_{\mathbb{Q}}(\Gamma_{2\ell} \backslash \mathbb{D})$ generated by NL-divisors with $\mathbb{Q}$-coefficients. By [5, 16], we know that the dimension $\rho_{2\ell}$ of the span of Heegner divisors on $\Gamma_{2\ell} \backslash \mathbb{D}$ can be explicitly computed by the following formula:

$$\rho_{2\ell} = \frac{31}{24}\ell + \frac{55}{24} - \frac{1}{6\sqrt{6\ell}} \mathrm{Re}(\mathrm{e}^{\frac{5\pi\mathrm{i}}{12}}(G(-1,4\ell) + G(3,4\ell)))$$

$$- \frac{1}{4\sqrt{2\ell}} \mathrm{Re}(G(-1,2\ell)) - \sum_{k=0}^{\ell} \left\{ \frac{k^2}{4\ell} \right\} - \sharp\left\{ k \mid \frac{k^2}{4\ell} \in \mathbb{Z}, 0 \le k \le \ell \right\}, \tag{2.3}$$

where $\{,\}$ denotes the fraction part and $G(a,b)$ is the generalized quadratic Gauss sum:

$$G(a,b) = \sum_{k=0}^{b-1} \mathrm{e}^{2\pi\mathrm{i}\frac{ak^2}{b}}.$$

Denote by $d_{E\mathrm{is}} = \sharp\left\{ k \mid \frac{k^2}{4\ell} \in \mathbb{Z}, 0 \le k \le \ell \right\}$. After applying the summation formula proved by Gauss in 1811 (see [4, §2.2]), one can simply get the following lemma.

Lemma 2.1

$$\rho_{2\ell} = \frac{31\ell + 55}{24} - \frac{1}{4}\alpha_\ell - \frac{1}{6}\beta_\ell - \sum_{k=0}^{\ell} \left\{ \frac{k^2}{4\ell} \right\} - d_{E\mathrm{is}}, \tag{2.4}$$

where

$$\alpha_\ell = \begin{cases} \left(\dfrac{2\ell}{2\ell-1}\right), & \ell \text{ is even}; \\ 0, & \text{otherwise}. \end{cases} \quad, \quad \beta_\ell = \begin{cases} \left(\dfrac{\ell}{4\ell-1}\right) - 1, & \text{if } 3|\ell; \\ \left(\dfrac{\ell}{4\ell-1}\right) + \left(\dfrac{\ell}{3}\right), & \text{otherwise}, \end{cases}$$

and $\left(\frac{a}{b}\right)$ is the Jacobi symbol.

As shown in [16], the span of NL-divisors are the same as the span of non irreducible divisors on $\Gamma_{2\ell}\backslash\mathbb{D}$.

2.3 Projective models of K3 surfaces

Let (S, L) be a smooth K3 surface with a primitive quasi-polarization L of degree 2ℓ. The linear system $|L|$ defines a map ψ_L from S to $\mathbb{P}^{\ell+1}$. The image of ψ_L is called a projective model of S.

In [19], Saint-Donat gives a precise description of all projective models of (S, L) when ψ_L is not a birational morphism.

Proposition 2.1 (see [19]) *Let L be the primitive quasi-polarization of degree 2ℓ on S and let ψ_L be the map defined by $|L|$. Then there are the following possibilities*:

1. *ψ_L is birational to a degree 2ℓ surface in $\mathbb{P}^{\ell+1}$. In particular, ψ_L is a closed embedding when L is ample.*

2. *ψ_L is a generically $2:1$ map and $\psi_L(S)$ is a smooth rational normal scroll of degree ℓ, or a cone over a rational normal curve of degree ℓ.*

3. *$|L|$ has a fixed component D, which is a smooth rational curve. Moreover, $\psi_L(S)$ is a rational normal curve of degree $\ell + 1$ in $\mathbb{P}^{\ell+1}$.*

We call K3 surfaces of type (1), (2), (3) nonhyperelliptic, unigonal, and digonal K3 surfaces accordingly. When $\ell = 2, 3, 4$, the projective model of a general quasi-polarized K3 surface (S, L) is a complete intersection in the projective space $\mathbb{P}^{\ell+1}$.

Remark 2.2 Assume that ψ_L is a birational morphism. Then one can easily see that L is not ample if and only if there exists an exceptional (-2) curve $D \subseteq S$. The morphism ψ_L will factor through a contraction $\pi : S \to \widetilde{S}$, where $\widetilde{S}$ is a singular K3 surface with simple singularities.

Recalling that the NL-divisor $D_{0,0}^{2\ell}$ parametrizes all K3 surfaces (S, L) of degree 2ℓ with exceptional (-2) curves. Therefore, the projective model of a general member in $D_{0,0}^{2\ell}$ is a surface in $\mathbb{P}^{l+1}$ of degree 2ℓ with simple singularities.

In this paper, we mainly consider the case $2\ell = 6$ and 8, where the classification of projective models of S can be read off from the Picard lattice of S.

Lemma 2.2 *Let (S, L) be a smooth quasi-polarized K3 surface of degree 2ℓ ($2\ell = 6$ or 8). Then*

1. *$(S, L) \in D_{1,1}^{2\ell}$ if and only if S is digonal except*

(∗) *$L^2 = 8$ and $L = L' + E + C$, where C is a rational curve, E is an irreducible elliptic curve and L' is irreducible of genus two with $L' \cdot C = E \cdot C = 1$ and $L' \cdot E = 2$. The image $\psi_L(S)$ is contained in a cone over cubic surface in $\mathbb{P}^4$.*

2. *$(S, L) \in D_{2,1}^{2\ell}$ if and only if S is unigonal.*

3. *$(S, L) \in D_{3,1}^{2\ell}$ if and only if S is one of the following*:

• *When $\ell = 3$, S is birational to the complete intersection of a singular quadric and a cubic in $\mathbb{P}^4$ via ψ_L.*

• When $\ell = 4$, S is either birational to a bidegree $(2, 3)$ hypersurface of the Serge variety $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ via ψ_L or in case $(*)$.

Proof The proof of (1) and (2) are straightforward from Proposition 2.1. See also [19, §2, §5] for more detailed discussion.

Now we suppose that a quasi-polarized K3 surface $(S, L) \in D_{3,1}^6$ is neither unigonal nor diagonal. Then ψ_L is a birational map to a complete intersection of a quadric and a cubic. Our first statement of (3) comes from the fact that any quadric threefold containing a plane cubic must be singular. If $(S, L) \in D_{3,1}^8$, the assertion follows from [19, Proposition 7.15 and Example 7.19].

Remark 2.3 We would like to refer the readers to [9–10] for a detailed description of projective models of low degree $(2\ell \leq 22)$ K3 surfaces.

3 Complete Intersection of a Quadric and a Cubic

In this section, we construct the moduli space of the complete intersection of a smooth quadric and a cubic in $\mathbb{P}^4$ via geometric invariant theory.

3.1 Terminology and notations

In the rest of this paper, we will use the following terminology. Let $f(u, v, w)$ be an analytic function in $\mathbb{C}[[u, v, w]]$ whose leading term defines an isolated singularity at the origin. We have the following types of singularities:

• Simple singualrities: Isolated A_n, D_k, E_r singularities.
• Simple elliptic singularities $\widetilde{E}_r$:
$-$ $\widetilde{E}_6$: $f = u^3 + v^3 + w^3 + auvw$,
$-$ $\widetilde{E}_7$: $f = u^2 + v^4 + w^4 + auvw$,
$-$ $\widetilde{E}_8$: $f = u^2 + v^3 + w^6 + auvw$,

We will use the notation $l(x), q(x)$ and $c(x)$ as linear, quadratic and cubic polynomials of $x = (x_0, \cdots, x_n)$.

3.2 Cubic sections on quadric threefolds

Let Q be the smooth quadric threefold in $\mathbb{P}^4$ defined by the equation

$$x_0 x_4 + x_1 x_3 + x_2^2 = 0.$$

Since every nonsingular quadric hypersurface in $\mathbb{P}^4$ is projectively equivalent to Q, a complete intersection of a smooth quadric and a cubic can be identified with an element in $|\mathcal{O}_Q(3)|$.

The automorphism group of Q is the reductive Lie group $SO(Q)(\mathbb{C})$ which is isomorphic to $SO(5)(\mathbb{C})$. Then we can naturally describe the moduli space of the complete intersection of a smooth quadric and a cubic as the GIT quotient of the linear system $|\mathcal{O}_Q(3)| = \mathbb{P}(V)$, where V is a 30-dimensional vector space defined by the exact sequence

$$0 \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) \to V \to 0.$$

Let us take the set of monomials

$$\mathcal{B} := \left\{ x_0^{a_0} x_1^{a_1} \cdots x_4^{a_4} \ \Big| \ \sum_{i=0}^{4} a_i = 3 \text{ and } a_0 a_4 = 0 \right\} \tag{3.1}$$

to be a basis of V. Sometimes, we may change the basis for simpler computations.

3.3 Numerical criterion

Now we classify stability of the points in $\mathbb{P}(V)$ under the action of $SO(Q)(\mathbb{C})$ by applying the Hilbert-Mumford numerical criterion (see [17]).

As is customary, a one parameter subgroup (1-PS) of $SO(Q)(\mathbb{C})$ can be diagonalized as

$$\lambda_{u,v} : t \in \mathbb{C}^* \to \operatorname{diag}(t^u, t^v, 1, t^{-v}, t^{-u})$$

for some $u, v \in \mathbb{Z}$. We call such $\lambda_{u,v} : \mathbb{C}^* \to SO(Q)(\mathbb{C})$ a normalized 1-PS of $SO(Q)(\mathbb{C})$ if $u \geq v \geq 0$.

Let $\lambda_{u,v}$ be a normalized 1-PS of $SO(Q)(\mathbb{C})$. Then the weight of a monomial $x_0^{a_0} x_1^{a_1} \cdots x_4^{a_4} \in \mathcal{B}$ with respect to $\lambda_{u,v}$ is

$$(a_0 - a_4)u + (a_1 - a_3)v. \tag{3.2}$$

If we denote by $M_{\leq 0}(\lambda_{u,v})$ (resp. $M_{<0}(\lambda_{u,v})$) the set of monomials of degree 3 which have non-positive (resp. negative) weight with respect to $\lambda_{u,v}$, one can easily compute the maximal subsets $M_{\leq 0}(\lambda_{u,v})$ (resp. $M_{<0}(\lambda_{u,v})$), as listed in Table 1 (resp. Table 2).

Table 1 Maximal subsets $M_{\leq 0}(\lambda)$

Cases	(u,v)	Maximal monomials
(N1)	(1,0)	$x_1^{a_1} x_2^{a_2} x_3^{a_3}, \sum a_i = 3$
(N2)	(1,1)	$x_0 x_2 x_3, x_2^3$
(N3)	(2,1)	$x_0 x_3^2, x_1^2 x_4, x_1 x_2 x_3, x_2^3$

Table 2 Maximal subsets $M_{<0}(\lambda)$

Cases	(u,v)	Maximal monomials
(U1)	(1,1)	$x_0 x_3^2, x_2^2 x_3$
(U2)	(3,1)	$x_1^2 x_4, x_1 x_3^2, x_2^2 x_3$

According to the Hilbert-Mumford criterion, an element $f(x_0, \cdots, x_4) \in \mathbb{P}(V)$ is not properly stable (resp. unstable) if and only if the weight of all monomial in f is non-positive (resp. negative) for some 1-PS. Thus we obtain the following lemma.

Lemma 3.1 *Let X be the surface defined by an element in $\mathbb{P}(V)$. Then X is not properly stable if and only if $X = Q \cap Y$ for some cubic hypersurface $Y \subseteq \mathbb{P}^4$ defined by a cubic polynomial in one of following cases*:

- $c(x_1, x_2, x_3, x_4)$;
- $x_0 x_3 l(x_2, x_3) + x_1 x_2 l_1(x_3, x_4) + x_1 q(x_3, x_4) + c(x_2, x_3, x_4)$;
- $x_0 x_3^2 + x_1 x_3 l_1(x_2, x_3) + x_1 x_4 l_2(x_1, x_2, x_3, x_4) + c(x_2, x_3, x_4)$.

For $f \in \mathbb{P}(V)$ not properly stable, using the destabilizing 1-PS λ, the limit $\lim_{t \to 0} f_t = f_0$ exists and it is invariant with respect to λ. The invariant part of polynomials of type (N1)–(N3) are the followings:

(α) $c(x_1, x_2, x_3) = 0$;

(β) $\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 + \lambda_3 x_0 x_2 x_3 + \lambda_4 x_1 x_2 x_4 = 0$, $\lambda_i \in \mathbb{C}$;

(γ) $\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 + \lambda_3 x_0 x_3^2 + \lambda_4 x_1^2 x_4 = 0$, $\lambda_i \in \mathbb{C}$.

Similarly, we get the following lemma.

Lemma 3.2 *With the notation above, X is not semistable if and only if $X = Q \cap Y$ for some cubic hypersurface Y defined by one of the following equations*:

- $x_0 x_3^2 + x_1 q(x_3, x_4) + c(x_2, x_3, x_4)$, *and $c(x_2, x_3, x_4)$ has no x_2^3 term*;
- $x_4 q_1(x_1, x_2, x_3, x_4) + x_3 q_2(x_2, x_3) + \lambda x_1 x_3^2$.

3.4 Geometric interpretation of stability

We use the terminology of the corank of the hypersurface singularities as in [1, 12].

Definition 3.1 *Let $0 \in \mathbb{C}^n$ be a hypersurface singularity given by the equation $f(z_1, \cdots, z_n) = 0$. The corank of 0 is n minus the rank of the Hessian of $f(z_1, \cdots, z_n)$ at 0.*

Theorem 3.1 *A complete intersection $X = Q \cap Y$ is not properly stable if and only if X satisfies one of the following conditions*:

(i) *X has a hypersurface singularity of corank 3.*

(ii) *X is singular along a line L and there exists a plane P such that $P \cap Q = 2L$ and P is contained in the projective tangent cone $\mathbb{P}(CT_p(X))$ for any point $p \in L$.*

(iii) *X has a singularity p which deforms to a singularity of $\widetilde{E}_8$ class, and the restriction of the projective cone $\mathbb{P}(CT_p(X))$ to X contains a line L passing through p with multiplicity at least 6.*

Proof As a consequence of Lemma 3.1, it suffices to find the geometric characterizations of the complete intersections of type (N1)–(N3). Here we do it case by case.

(i) If X is of type (N1), then X can be considered as the intersection of Q and a cubic cone Y with the vertex $p_0 = [1, 0, 0, 0, 0] \in Q$. It is easy to see that p_0 is a corank of 3 singularity of X.

Conversely, we write the equation of Y as

$$x_0 q(x_0, x_1, x_2, x_3) + c(x_1, x_2, x_3, x_4) = 0.$$

If we choose the affine coordinate

$$y_i := \frac{x_i}{x_0}, \tag{3.3}$$

then the affine equation near p_0 is

$$q(1, y_1, y_2, y_3) + c(y_1, y_2, y_3, -y_2^2 - y_1 y_3) = 0 \tag{3.4}$$

in $\mathbb{C}^3$. It has a corank 3 singularity at the origin if and only if the quadric q is 0.

(ii) If X is of type (N2), then the equation of Y is given by

$$x_0 x_3 l(x_2, x_3) + x_1 x_2 \ell_1(x_3, x_4) + x_1 q(x_3, x_4) + c(x_2, x_3, x_4),$$

and therefore X is singular along the line $L : x_2 = x_3 = x_4 = 0$.

Moreover, for any point $p = [z_0, z_1, 0, 0, 0] \in L$, the projective tangent cone $\mathbb{P}(CT_p(X))$ at p is defined as

$$z_0 x_4 + z_1 x_3 = z_0 x_3 l(x_2, x_3) + z_1 (x_2 \ell_1(x_3, x_4) + q(x_2, x_3)) = 0, \tag{3.5}$$

which contains the plane $P : x_3 = x_4 = 0$ for each $p \in L$ and $P \cap Q = 2L$.

Conversely, since the intersection of P and Q is a double line L, we may certainly assume that the plane P is defined by

$$x_3 = x_4 = 0$$

after some coordinate transform persevering the quadric form Q. Then the line $L = P \cap Q$ is given by $x_2 = x_3 = x_4 = 0$.

Because X is singular along L, the equation of Y can be written as:

$$x_0 q_1(x_2, x_3) + x_1 q_2(x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0. \tag{3.6}$$

Then the projective tangent cone

$$\mathbb{P}(CT_p(X)) = \{z_0 x_4 + z_1 x_3 = z_0 q_1(x_2, x_3, x_4) + z_1 q_2(x_2, x_3, x_4) = 0\}$$

contains the plane P for each point $p = [z_0, z_1, 0, 0, 0] \in L$ only if the quadrics q_i have no x_2^2 term.

(iii) For X of type (N3), a similar discussion is as follows: If Y is defined by

$$x_0 x_3^2 + x_1 x_3 l_1(x_2, x_3) + x_1 x_4 l_2(x_1, x_2, x_3) + c(x_2, x_3, x_4) = 0, \tag{3.7}$$

then $X = Q \cap Y$ is singular at p_0. After choosing the affine coordinates as (3.3), the affine equation near p_0 is

$$y_3^2 + y_1 y_3^2 f(y_1, y_2, y_3) + y_1 y_2^2 \ell(y_1, y_2) + a y_1 y_2 y_3 + g(y_2, y_3) = 0 \tag{3.8}$$

for some polynomials ℓ, f, g with ℓ linear, $\deg(f) \geq 1, \deg(g) \geq 3$. Therefore, p_0 is a hypersurface singularity of corank 2 and its projective tangent cone is a double plane $2P : x_3^2 = x_4 = 0$. The remaining part is straightforward.

Conversely, we take p_0 to be the isolated singular point which deforms to a singularity of $\widetilde{E}_8$ class. As it has corank at least 2, the equation of Y can be written as

$$x_0 q_1(x_1, \cdots, x_3) + x_1 q_2(x_1, \cdots, x_4) + c(x_2, x_3, x_4) = 0.$$

Then the quadric $q_1(x_1, x_2, x_3)$ is of the form $l(x_1, x_2, x_3)^2$ for some linear polynomial l because p_0 is singular of corank at least 2.

After we make a coordinate change preserving Q and p_0, the defining equation of Y has two possibilities:

1. $x_0x_2^2 + x_1q(x_1, x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0,$
2. $x_0x_3^2 + x_1q(x_1, x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0.$

The projective tangent cone at $\mathbb{P}(CT_{p_0}(X))$ is a double plane

$$2P : x_4 = x_2^2 = 0 \quad \text{or} \quad x_4 = x_3^2 = 0.$$

The line L contained in the restriction of $2P$ to X has to be defined by $x_2 = x_3 = x_4 = 0$. It follows that the first case can not happen since $P \cap X$ contains L with multiplicity at least 3.

In the second case, the multiplicity condition further implies that the quadric $q(x_1, x_2, x_3, x_4)$ does not have x_1^2, x_1x_2, x_2^2 terms. To see that there is no x_1x_3 term, note that the affine local equation $\left(y_i = \frac{x_i}{x_0}\right)$ near p_0 can be written as

$$y_3^2 + by_1^2y_3 + y_1y_3^2g(y_1, y_2, y_3) + y_1y_2^2\ell(y_1, y_2) + ay_1y_2y_3 + c(y_2, y_3) = 0,$$

where $a, b \in \mathbb{C}$, g, c are polynomials with $\deg(c) \geq 3$. Since p_0 deforms to $\widetilde{E}_8$, we know that a has to be 0.

Theorem 3.2 *A complete intersection $X = Q \cap Y$ is unstable if and only if X satisfies one of the following conditions*:

(i′) *X is singular along a line L satisfying the condition*: *There exists a plane P such that $\mathbb{P}(CT_p(X)) = 2P$ for any point $p \in L$*;

(ii′) *there exists a plane P whose restriction to X is a line L with multiplicity 6 and X has a corank 3 singularity p on L. Moreover, the projective tangent cone $\mathbb{P}(CT_p(X))$ at p is the union of the plane P and a quadric surface and they meet at L with multiplicity two.*

Proof We check the complete intersections of type (U1)–(U2) case by case.

(i′) To simplify the proof, we choose another monomial basis of V as below:

$$\mathcal{B}' := \left\{ x_0^{a_0} \cdots x_4^{a_4} \mid \sum_{i=0}^{4} a_i = 3, \ a_2 \leq 1 \right\}. \tag{3.9}$$

Then the polynomial of type (U1) has the form

$$x_0q_0(x_3, x_4) + x_1q_1(x_3, x_4) + x_2q_2(x_3, x_4) + c(x_3, x_4) = 0. \tag{3.10}$$

At this time, X is singular along the line $L : x_2 = x_3 = x_4 = 0$ and satisfies the condition described in (i′).

On the other hand, the line L on Q can be written as

$$L : x_2 = x_3 = x_4 = 0$$

for a suitable change of coordinates preserving Q. Then the equation of Y has the form

$$\sum_{i=0}^{1} x_iq_i(x_2, x_3, x_4) + x_2q_2(x_3, x_4) + c(x_3, x_4) = 0,$$

where q_i does not contain x_2^2 term.

Moreover, for any point $p = [z_0, z_1, 0, 0, 0] \in L$, the projective tangent cone $\mathbb{P}(CT_p(X))$ is given by

$$z_0 x_3 + z_1 x_4 = z_0 q_0(x_2, x_3, x_4) + z_1 q_1(x_2, x_3, x_4) = 0.$$

They have a common plane P with multiplicity 2 if and only if P is defined by $x_3 = x_4 = 0$ and $q_i(x_2, x_3, x_4)$ does not contain the $x_2 x_3, x_2 x_4$ terms.

(ii') When Y has the equation

$$x_4 q_1(x_1, \cdots, x_4) + x_3 q_2(x_2, x_3) + \lambda x_1 x_3^2 = 0,$$

one observe that X contains the line $L : x_2 = x_3 = x_4 = 0$ which is contained in the plane $P := x_3 = x_4 = 0$. It is easy to see that P intersect with X is the line L with multiplicity 6. Moreover, X is singular at $p_0 = [1, 0, 0, 0, 0]$ and the projective cone at p_0 is given by

$$\{x_4 = x_3 q_2(x_2, x_3) + \lambda x_1 x_3^2 = 0\},$$

which is the union of the plane $X_1 : x_3 = x_4 = 0$ and the quadratic surface $X_2 : x_4 = q_2(x_2, x_3) + \lambda x_1 x_3 = 0$ satisfying the desired conditions. The proof of the converse is quite similar as the previous cases and we omit the details here.

Corollary 3.1 *A complete intersection $X = Q \cap Y$ is semistable (resp. stable) if X has at worst isolated singularities (resp. simple singularities).*

Proof By Theorem 3.2, the singular locus of X is at least one dimensional if it is unstable. Then X has to be semistable if it has at worst isolated singularities.

Next, from Theorem 3.1, we know that if X is not properly stable, then either X is singular along a curve or it contains at least an isolated simple elliptic singularity. It follows that X with simple singularities is stable.

Now it makes sense to talk about the moduli space $\mathcal{M}_6$ of complete intersections of a smooth quadric and a cubic with simple singularities. Let $\mathcal{U}_6$ be the open subset of $\mathbb{P}(V)^s$ parameterizing such complete intersections in $\mathbb{P}^4$. Then we have $\mathcal{M}_6 = \mathcal{U}_6 / SO(5)(\mathbb{C})$.

Theorem 3.3 *There is an open immersion $\mathcal{P}_6 : \mathcal{M}_6 \to \mathcal{F}_6$ via the period map and the image of $\mathcal{P}_6$ in $\mathcal{F}_6$ is the complement of three NL-divisors $D_{1,1}^6, D_{2,1}^6$ and $D_{3,1}^6$. The Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_6)$ is spanned by $\{D_{d,1}^6, \ 1 \le d \le 4\}$.*

Proof For the first statement, one only need the fact that the complete intersections with simple singularities correspond to degree 6 quasi-polarized K3 surfaces containing a (-2) curve. Therefore, we obtain an open immersion $\mathcal{P}_6 : \mathcal{M}_6 \to \mathcal{F}_6$ from Torelli theorem. By Lemma 2.2, we know that the boundary divisors of the image $\mathcal{P}_6(\mathcal{F}_6)$ is the union of $D_{1,1}^6, D_{2,1}^6$ and $D_{3,1}^6$.

Next, the moduli space $\mathcal{M}_6$ is isomorphic to the quotient $\mathcal{U}_6 / SO(5)(\mathbb{C})$. Observing that $\mathrm{Pic}(\mathcal{U}_6) \cong \mathrm{Pic}(\mathbb{P}(V))$ has rank one since the boundary of $\mathcal{U}_6$ in $\mathbb{P}(V)$ has codimension at least two, we claim that the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_6)$ is at most one. Denote by $\mathrm{Pic}(\mathcal{U}_6)_{SO(5)(\mathbb{C})}$ the set of $SO(5)(\mathbb{C})$-linearized line bundles on $\mathcal{U}_6$. There is an injection

$$\mathrm{Pic}(\mathcal{U}_6 / SO(5)(\mathbb{C})) \hookrightarrow \mathrm{Pic}(\mathcal{U}_6)_{SO(5)(\mathbb{C})}$$

by [11, Proposition 4.2] for the reductive group $SO(5)(\mathbb{C})$. Our claim then follows from the fact that the forgetful map $\mathrm{Pic}(\mathcal{U}_6)_{SO(5)(\mathbb{C})} \to \mathrm{Pic}(\mathcal{U}_6)$ is an injection. Actually, one can easily see that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_6)$ is spanned by the descent of the tautological line bundle $\mathcal{O}_{\mathcal{U}_6}(1)$ on $\mathcal{U}_6$ to the quotient $\mathcal{U}_6/SO(5)(\mathbb{C})$, and we denote it by $\mathcal{O}_{\mathcal{M}_6}(1)$.

Since the complement of $\mathcal{P}_6(\mathcal{M}_6)$ in $\mathcal{F}_6$ is the union of three irreducible divisors and $\dim_{\mathbb{Q}}(\mathrm{Pic}(\mathcal{F}_6)) \geq 4$, it follows that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_6)$ is spanned by NL-divisors $\{D_{d,1}^6, 1 \leq d \leq 4\}$ by the dimension consideration.

Remark 3.1 There is another natural GIT construction of moduli space of complete intersections in projective spaces, see [2, 14]. There exists a projective bundle $\pi : \mathbb{P}E \to \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))) \cong \mathbb{P}^{14}$ parameterizing all complete intersections of a quadric and a cubic in $\mathbb{P}^5$. Then one can consider the GIT quotient

$$\mathbb{P}(E) /\!\!/_{H_t} SL_5(\mathbb{C})$$

for the line bundle $H_t = \pi^* \mathcal{O}_{\mathbb{P}^{14}}(1) + t\mathcal{O}_{\mathbb{P}E}(1)$. We want to point out that $\mathbb{P}(E) /\!\!/_{H_t} SL_5(\mathbb{C})$ is isomorphic to our GIT quotient $\mathbb{P}(V) /\!\!/ SO(5)(\mathbb{C})$ when $t < \frac{1}{6}$. This will be discussed in the upcoming paper [8].

3.5 Minimal orbits

In this section, we give a description of the semistable boundary components of the GIT compactification. It consists of strictly semistable points with minimal orbits. From Subsection 3.2, it suffices to discuss the points of type (α)–(γ). As in [12], our approach is to use Luna's criterion.

Lemma 3.3 (Luna's criterion (see [15])) *Let G be a reductive group acting on an affine variety V. If H is a reductive subgroup of G and $x \in V$ is stabilized by H, then the orbit $G \cdot x$ is closed if and only if $C_G(H) \cdot x$ is closed.*

To start with, we first observe that Type (α), (β) and (γ) have a common specialization, which we denote by Type (ξ):

$$\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 = 0.$$

Lemma 3.4 *If X is of Type (ξ), it is strictly semistable with closed orbits.*

Proof The stabilizer of Type (ξ) contains a 1-PS:

$$H = \{\mathrm{diag}(t^2, t, 1, t^{-1}, t^{-2}) \mid t \in \mathbb{C}^*\}$$

of distinct weights. So the center

$$C_G(H) = \{\mathrm{diag}(a_0, a_1, 1, a_1^{-1}, a_0^{-1})\} \subset SO(Q)(\mathbb{C})$$

is a maximal torus. It acts on $V^H = \langle x_0 x_3^2, x_1^2 x_4, x_1 x_2 x_3, x_2^3 \rangle \subset V$. It is straightforward to see any element of Type (ξ) is semistable with closed orbit in V^H under the action. Then the statement follows from Luna's criterion.

Proposition 3.1 *Let X be a surface of Type (α). Then it has two corank 3 singularities. Moreover, we have*

1. X is unstable if it is union of a quadric surface and a quadric cone with multiplicity two.

2. The orbit of X is not closed if X is singular along two lines. It degenerates to type (ξ). Otherwise, X is semistable with closed orbit.

Proof The stabilizer of Type (α) contains a 1-PS:

$$H_1 = \{\text{diag}(t, 1, 1, 1, t^{-1}) \mid t \in \mathbb{C}^*\}.$$

The center $C_G(H_1) \cong SO(Q_1)(\mathbb{C}) \times SO(Q_2)(\mathbb{C})$, where $Q_1 = x_0 x_4$ and $Q_2 = x_1 x_3 + x_2^2$. The group $SO(Q_1)(\mathbb{C}) \cong SO(2; \mathbb{C})$ acts linearly on variable x_0, x_4, while $SO(Q_2)(\mathbb{C}) \cong SO(3)(\mathbb{C})$ acts linearly on the variables x_1, x_2 and x_3.

The action of $C_G(H_1)$ on $V^{H_1} = \left\langle x_1^{d_1} x_2^{d_2} x_3^{d_3}, \sum\limits_{k=1}^{3} d_k = 3 \right\rangle \subset V$ is equivalent to the action of $SO(Q_2)(\mathbb{C})$ on the set of cubic polynomials in three variables x_1, x_2, x_3 preserving the quadratic form Q_2. By Luna's criterion, we can reduce our problem to a simpler GIT question $V^{H_1} /\!\!/ SO(3)(\mathbb{C})$. Any 1-PS $\lambda : \mathbb{C}^* \to SO(Q_2)(\mathbb{C})$ of $SO(Q_2)(\mathbb{C})$ can be diagonalized in the form

$$\lambda(t) = \text{diag}(t^a, 1, t^{-a}). \tag{3.11}$$

The weight of a monomial $x_1^{d_1} x_2^{d_2} x_3^{d_3}$ with respect to (3.11) is $a(d_1 - d_3)$. Then our assertion follows easily from the Hilbert-Mumford criterion.

The remaining cases can be shown in a similar way. Here we omit the proof.

Proposition 3.2 *Let X be a surface of type (β). Then it is a union of a quadric surface and a complete intersection of two quadrics. Moreover, we have*

(i) X is unstable if X consists of two quadric cones and a quadric surface intersecting at a line.

(ii) The orbit of X is not closed if its equation can be written as $\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 + \lambda_3 x_1 x_2 x_4$ up to a coordinate transform preserving Q. It degenerates to type (ξ).
Otherwise, X is semistable with closed orbit.

Proposition 3.3 *A general member X of type (γ) has two simple elliptic singularities of type $\widetilde{E}_8$. Moreover, we have*

(i) X is unstable if X consists of three quadric cones.

(ii) The orbit of X is not closed if its equation has the form $\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 + \lambda_3 x_1^2 x_4$ up to a coordinate change preserving Q.
Otherwise, X is semistable with closed orbit.

4 Stable Singular Complete Intersection

In this section, we will discuss the stable loci of singular complete intersections of a smooth quadric and a cubic hypersurface. As a result, we prove Theorem 1.2.

4.1 Stable complete intersection with isolated singularity

Let $X = Q \cap Y$ be a complete intersection of the smooth quadric Q and a cubic threefold Y. A first observation is in the following.

Proposition 4.1 *If $X = Q \cap Y$ has only isolated singularity, then X is stable if and only if the non-ADE singularities can only be one of the following situations*:

i) $\widetilde{E}_7$ *type*,

ii) $\widetilde{E}_8$ *type and its projective tangent cone* $\mathbb{P}(CT_p)$ *meets X at a point, the general equation of Y is of the form.*

Up to a coordinate change, the general equation of Y for $Y \cap Q$ with an $\widetilde{E}_7$ singularity is

$$(\delta) : x_0 x_2^2 + x_2 q(x_1, x_2, x_3, x_4) + x_4 q(x_1, x_3, x_4) = 0.$$

Similarly, the general equation of Y for $Y \cap Q$ with an $\widetilde{E}_8$ singularity of type ii) *is*

$$(\epsilon) : x_0 x_3^2 + x_2 x_3 \ell(x_1, x_3, x_4) + c(x_1, x_3, x_4) = 0,$$

where ℓ is linear and c is a cubic polynomial in x_1, x_3, x_4.

Proof Suppose X has only at worst isolated singularities of type i) or ii). This means that X does not have a corank 3 singularity or $\widetilde{E}_8$ singularity whose projective tangent cone meets X along a line. By Theorem 3.1, we know that X is stable. Conversely, suppose X is stable and it has a non-ADE isolated singularity at $p = [1, 0, 0, 0, 0]$. If p is not of type i) or ii), by Theorem 3.1 (i), p has to be a $\widetilde{E}_8$ type singularity. Let us analysis the local equation of p. As in the proof in Theorem 3.1, up to a change of coordinates preserving the quadric Q, the defining equation of Y has two possibilities:

1. $x_0 x_2^2 + x_1^2 \ell(x_1, x_2, x_3, x_4) + x_1 q(x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0$,
2. $x_0 x_3^2 + x_1^2 \ell(x_1, x_2, x_3, x_4) + x_1 q(x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0$.

In the first case, the affine local equation near p can be written as

$$x_2^2 + \sum_{6 \geq d \geq 3} f_d(x_1, x_2, x_3) = 0,$$

where f_d is a homogenous polynomial of degree d. Note that there is no term $x_1^3 x_2, x_3^3 x_2$ and $x_1 x_3^4, x_1^4 x_3$ in the fourth and fifth jet. One can easily see that p can not be an $\widetilde{E}_8$ type singularity.

In the second case, we know that the projective tangent cone $\mathbb{P}(CT_p(X))$ meets X along either a line $L : x_2 = x_3 = x_4 = 0$ or the point p (with multiplicity). If the intersection is a line, X can not be stable by the proof in Theorem 3.1. The only possibility is that the intersection is a point. In this situation, the third jet contains the term x_1^3 and the weights on variables x_1, x_2, x_3 are $(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. The equation of Y is of the form

$$x_0 x_3^2 + x_2 x_3 \ell(x_1, x_3, x_4) + c(x_1, x_3, x_4) = 0,$$

where ℓ is linear and c is a cubic in x_1, x_3, x_4.

At the end, let us give the general equations for X with an $\widetilde{E}_7$ singularity. Without loss of generality, we assume the singularity is at $p = [1, 0, 0, 0, 0]$. As it is corank 2, the defining equation of Y can be written as

$$x_0 x_2^2 + x_2 q(x_1, x_2, x_3, x_4) + c(x_1, x_3, x_4) = 0$$

or

$$x_0 x_3^2 + x_3 q'(x_1, x_2, x_3, x_4) + c'(x_1, x_2, x_4) = 0.$$

The weights on (x_1, x_2, x_3) are either $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ or $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. In either case, the direct computation shows that all monomials in $c(x_1, x_3, x_4)$ and $c'(x_1, x_2, x_4)$ must have x_4 term. The assertions follows.

From the proof, we can see that if X has only isolated singularities of $\widetilde{E}_8$ type, then X will be stable if the projective tangent cone $\mathbb{P}(CT_p(X))$ of the singularity meets X at a point and X is strictly semistable if $\mathbb{P}(CT_p(X))$ meets X along a line.

4.2 Stable loci of complete intersection with non-isolated singularity

Let us consider the non-normal case. With the notations as above, we denote by $\mathrm{Sing}(X)$ the singular loci of X. Then we have the following theorem.

Theorem 4.1 *Let X be a complete intersection of a smooth quadric Q and a cubic hypersurface Y with non-isolated singularities. Then one of the following holds*:
i) $\mathrm{Sing}(X)$ *contains a line. The general equations of such X are of the form*

$$(\zeta) : x_0 x_4 + x_1 x_3 + x_2^2 = x_3 q_1(x_0, x_1, x_2) + x_4 q_2(x_1, x_2) + c(x_0, x_1, x_2) = 0,$$

where q_1, q_2 are quadrics and c is a cubic.
ii) $\mathrm{Sing}(X)$ *contains a conic. The general equations of such X are of the form*

$$(\eta) : x_0 x_4 + x_1 x_3 + x_2^2 = x_0^2 \ell_1 + x_0 x_4 \ell_2 + x_4^2 \ell_3 = 0,$$

where ℓ_i are linear polynomials in $x_0, \cdots, x_4$.
iii) $\mathrm{Sing}(X)$ *contains a twisted cubic. The general equations of X are of the form*

$$Q : \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \end{vmatrix} + x_4 \ell(x_0, x_1, x_2, x_4) = 0,$$

$$(\theta) : \\ Y : \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ b_1 x_4 & b_2 x_4 + a_1 x_1 + a_2 x_0 & b_3 x_4 + a_1 x_2 + a_2 x_1 + a_3 x_0 \end{vmatrix}$$
$$+ x_4 x_0 \ell(x_1, x_2, x_3, x_4) + x_4^2 \ell' = 0,$$

where $a_i, b_i \in \mathbb{C}, \ell'$ is a linear polynomial in $x_0, \cdots, x_4$ and ℓ represents a linear polynomial in four variables.
iv) $\mathrm{Sing}(X)$ *contains an elliptic curve of degree four. The general equations of such X are of the form*

$$(\widetilde{\beta}) : x_0 x_4 + x_1 x_3 + x_2^2 = \ell q = 0,$$

where ℓ is a linear polynomial and q is a quadric polynomial.
v) $\mathrm{Sing}(X)$ *contains a rational normal curve of degree four. The general equations of such X are of the form*

$$(\phi) : x_0 x_4 + x_1 x_3 - 2x_2^2 = \sum_{i=2}^{6} \ell_i \Delta_i = 0,$$

where ℓ_i are linear polynomials in (4.6) and Δ_i are quadric polynomials defined in (4.4). Moreover, the general members of each type is stable.

Proof Let $C \subseteq \text{Sing}(X)$ be an irreducible curve. If $X = X_1 \cup X_2$ is reducible, then $\deg(X_i) = 2$ or 4 as X_i is contained in a smooth quadric threefold Q. The only possibility is $X = Q \cap Y$ with Y a union of a $\mathbb{P}^3$ and a quadric threefold. This is exactly type $(\widetilde{\beta})$.

If X is irreducible, taking a general hyperplane H, then $H \cap X$ is an irreducible curve singular along $H \cap C$. Note that the arithmetic genus of H is at most 4, $H \cap C$ has at most 4 points. It follows that the degree of C is at most 4. Hence C can be a line, a conic, a plane cubic, a twisted cubic or a rational normal curve of degree 4. If C is a plane cubic, then C is contained in the intersection $\mathbb{P}^2 \cap Q$, which is a conic. This is clearly impossible. Let us now describe their equations of X case by case.

i) Take the quadric threefold $Q : x_0x_4 + x_1x_3 + x_2^2 = 0$ and we can assume X is singular along the line $C : x_0 = x_1 = x_2 = 0$. The equation of Y is of the form

$$f = x_3^2\ell_1 + x_3x_4\ell_2 + x_4^2\ell_3 + x_3q_1 + x_4q_2 + c = 0, \tag{4.1}$$

where ℓ_i are linear, q_i are quadric and c is a cubic polynomial in x_0, x_1 and x_2. Then the Jacobian of X on the line C given by

$$\begin{pmatrix} x_4 & x_3 & 0 & 0 & 0 \\ \dfrac{\partial f}{\partial x_0} & \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & 0 & 0 \end{pmatrix}$$

has rank one. The only possibility is that all $\ell_i = 0$. This gives the equation (ξ).

ii) Take the quadric as above and we assume that X is singular along a smooth conic $C : x_0 = x_4 = x_1x_3 + x_2^2 = 0$. The equation of a cubic hypersurface Y containing C is of the form

$$x_0q_1 + x_4q_2 + (x_1x_3 + x_2^2)\ell(x_1, x_2, x_3) = 0 \tag{4.2}$$

for some quadric polynomials q_1 and q_2. Similarly as i), one can compute that the equation of Y is of type (η).

iii) Take Q as above. If C is an elliptic curve of degree 4, the span of C is a three dimension linear subspace, denoted by H. Note that $H \cap X$ can not be a curve as C is contained in $H \cap X$ with multiplicity at least 2. This means $H \cap X$ is a surface and thus X is reducible. Such X is of type $(\widetilde{\beta})$.

iv) If C is a twisted cubic, the defining equations of C can be written as

$$x_4 = 0, \quad x_1x_3 - x_2^2 = 0, \quad x_1x_2 - x_0x_3 = 0, \quad x_0x_2 - x_1^2 = 0.$$

Then the equations of a complete intersection X containing C can be written as

$$\begin{aligned} Q : \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \end{vmatrix} + x_4\ell = 0, \\[2em] Y : \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ \ell_1 & \ell_2 & \ell_3 \end{vmatrix} + x_4q(x_0, x_1, x_2, x_3) + x_4^2\ell' = 0 \end{aligned} \tag{4.3}$$

for some linear polynomials ℓ_i, ℓ, ℓ' and a quadric polynomial q. If X is singular along C, then via computing the Jacobian of (4.3) , we get the equation (θ).

vi) Without loss of generality, we can assume that the curve C is defined by equations

$$
\begin{aligned}
\Delta_1 &= x_1x_3 - x_2^2, \quad \Delta_2 = x_1x_2 - x_0x_3, \quad \Delta_3 = x_0x_2 - x_1^2, \\
\Delta_4 &= x_2x_4 - x_3^2, \quad \Delta_5 = x_2x_3 - x_1x_4, \quad \Delta_6 = x_1x_3 - x_0x_4,
\end{aligned}
\tag{4.4}
$$

and Q is defined by the equation $2\Delta_1 - \Delta_6 = 0$. As Y contains C, we may assume that the equation of Y is given by

$$
\sum_2^6 \ell_i\Delta_i = 0
\tag{4.5}
$$

for some linear polynomial ℓ_i. As X is singular along C, the Jacobian matrix of X along C is

$$
\begin{pmatrix}
x_4 & x_3 & -4x_2 & x_1 & x_0 \\
\sum_i \ell_i\frac{\partial\Delta_i}{\partial x_0} & \cdots & \cdots & \cdots & \sum_i \ell_i\frac{\partial\Delta_i}{\partial x_4}
\end{pmatrix}.
$$

Then via a computation, one can get the linear functions ℓ_i are of the form

$$
\begin{aligned}
\ell_4 &= \sum_{i=0}^2 a_ix_i, \quad \ell_5 = \sum_{i=0}^2 b_ix_i + a_2x_3, \quad \ell_2 = \sum_{i=1}^3 c_ix_i + (b_2 - a_1)x_4, \\
\ell_3 &= c_1x_2 + (c_2 - b_0)x_3 + (c_3 + a_0 - b_1)x_4, \\
\ell_6 &= \frac{1}{2}(c_1x_0 + (c_2 - 2b_0)x_1 + (c_3 - 2b_1 + 3a_0)x_2 + (2a_1 - b_2)x_3 + a_2x_4).
\end{aligned}
\tag{4.6}
$$

There are nine parameters a_i, b_i for $i = 0, 1, 2$ and c_j for $j = 1, 2, 3$. We left the details to readers.

Finally, the assertion of stability follows directly from Theorem 3.1.

Proof of Theorem 1.2 It basically follows from the combination of Propositions 3.1–3.3, 4.1 and Theorem 4.1. The strata α, β, γ are strictly semistable, which are described in Propositions 3.1, 3.2 and 3.3 respectively. The dimension of these components can be computed via Luna's slice theorem as below.

With the notations as in Propositions 3.1–3.3, we have

$$
\begin{aligned}
\dim \alpha &= \dim \mathbb{P}(V^{H_1}) /\!/ SO(3)(\mathbb{C}) = 6, \\
\dim \beta &= \dim \mathbb{P}(V^{H_2}) /\!/ \mathbb{C}^* = 2, \\
\dim \gamma &= \dim(\mathbb{P}(V^{H_3}) /\!/ \mathbb{C}^* = 2,
\end{aligned}
\tag{4.7}
$$

where $V^{H_2} = \langle x_2^3, x_1x_2x_3, x_0x_2x_3, x_1x_2x_4 \rangle$ and $V^{H_3} = \langle x_2^3, x_1x_2x_3, x_0x_3^2, x_1^2x_4 \rangle$ parameterizing the equations of type (β) and (γ) respectively. For stable components, we can also compute the dimension as follows:

1. For ζ, note that the NL-divisor $D_{1,0}^6$ parametrizing X containing a line has dimension 18. The general member in $D_{1,0}^6$ is of the form (4.1). Thus one can see that ζ has codimension 7 in $D_{1,0}^6$ and it follows $\dim \zeta = 11$.

2. Similarly, the general equations of elements in $D^6_{2,0}, D^6_{3,0}$ and $D^6_{4,0}$ are given in (4.2), (4.3) and (4.5) respectively. Then one can directly see that η has codimension 7 in $D^6_{2,0}$, while θ has codimension 15 in $D^6_{3,0}$ and ϕ has codimension 16 in $D^6_{4,0}$.

3. Let us consider $\widetilde{\beta}$ consisting of the union of a $\mathbb{P}^2$ and a complete intersection of two quadrics meeting along a degree 4 curve in $\mathbb{P}^3$. The elements in $\widetilde{\beta}$ are parameterized by the product of two projective spaces $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$, where $V_1 = H^0(Q, \mathcal{O}_Q(1))$ and $V_2 = H^0(Q, \mathcal{O}_Q(2))$. Hence its dimension is

$$\dim \mathbb{P}(V_1) + \dim \mathbb{P}(V_2) - \dim SO(5)(\mathbb{C}) = 7.$$

4. For δ, it can be viewed as the quotient space $\mathbb{P}(V)/G_1$, where V is the vector space spanned by monomials in the equation (δ) and G_1 is the subgroup of $SO(5)(\mathbb{C})$ fixing the singular point p_0 and the hyperplane $x_2 = 0$. As $\dim V = 17$ and $\dim G_1 = 5$, we get $\dim \delta = 11$.

5. Similar as above, ϵ is the quotient space $\mathbb{P}(V')/G_2$ with $\dim V' = 14$ and $\dim G_2 = 5$. It follows that $\dim \epsilon = 8$.

5 Complete Intersection of Three Quadrics in $\mathbb{P}^5$

Let $W = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ be the space of global sections of $\mathcal{O}_{\mathbb{P}^5}(2)$. Since every complete intersection X is determined by a net of quadrics Q_1, Q_2, Q_3, the complete intersection of three quadrics are parametrized by the Grassmannian $Gr(3, W)$. The moduli space of complete intersections can be constructed as the GIT quotient $Gr(3, W)^{ss} /\!\!/ SL_6(\mathbb{C})$. In this situation, the complete GIT strata is very complicated. For example, see [7] for the GIT stability of a net of quadrics in $\mathbb{P}^4$. However, we are satisfied with the following result.

Theorem 5.1 *Let X be a complete intersection of three quadrics in $\mathbb{P}^5$. If X has at worst simple singularities, then X is GIT stable.*

5.1 Set up

We first make some notations. Given a net of quadrics $\{Q_1, Q_2, Q_3\}$, the Plücker coordinates of $\{Q_1, Q_2, Q_3\}$ in $\mathbb{P}(\bigwedge^3 W)$ can be represented by

$$\{x_{i_1} x_{j_1} \wedge x_{i_2} x_{j_2} \wedge x_{i_3} x_{j_3}\}$$

for three distinct pairs (i_k, j_k).

Let $\lambda : \mathbb{C}^* \to SL_6(\mathbb{C})$ be a normalized one-parameter subgroup, i.e., $\lambda(t) = \text{diag}(t^{a_0}, t^{a_1}, \cdots, t^{a_5})$ satisfying $a_0 \geq a_1 \geq \cdots \geq a_5$ and $\sum\limits_{i=0}^{5} a_i = 0$. We denote by

$$w_\lambda(x_i x_j) := a_i + a_j$$

the weight of the monomial $x_i x_j$ with respect to λ. The weight of a Plücker coordinate $x_{i_1} x_{j_1} \wedge x_{i_2} x_{j_2} \wedge x_{i_3} x_{j_3}$ with respect to λ is simply $\sum\limits_{k=1}^{3} w_\lambda(x_{i_k} x_{j_k})$.

5.2 Numerical criterion for nets

By the Hilbert-Mumford numerical criterion, a net of quadrics $\{Q_1, Q_2, Q_3\}$ is not properly stable if and only if for a suitable choice of coordinates, there exists a normalized 1-PS λ : $t \to \text{diag}(t^{a_0}, t^{a_1}, \cdots, t^{a_5})$ such that the weight of all Plücker coordinates of $\{Q_1, Q_2, Q_3\}$ with respect to λ is not positive. We say that $\{Q_1, Q_2, Q_3\}$ is not properly stable with respect to λ.

Given a normalized 1-PS $\lambda : \mathbb{C}^* \to SL_6(\mathbb{C})$, we can define two complete orders on quadratic monomials:

1. " $>$ " : $x_0^2 > x_0 x_1 > \cdots > x_0 x_5 > x_1^2 > x_1 x_2 > \cdots > x_4 x_5 > x_5^2$.
2. " $>_\lambda$ " : $x_i x_j >_\lambda x_k x_l$ if either $w_\lambda(x_i x_j) > w_\lambda(x_k x_l)$ or $w_\lambda(x_i x_j) = w_\lambda(x_k x_l)$ for a given normalized 1-PS λ and $x_i x_j > x_k x_l$.

Since the 1-PS $\lambda : \mathbb{C}^* \to SL_6(\mathbb{C})$ is normalized, $x_i x_j >_\lambda x_k x_l$ implies $\max\{i, j\} > \min\{k, l\}$.

We denote by m_i the leading term of Q_i with respect to the order " $>_\lambda$ " and we say that a monomial $x_k x_l \notin Q_i$ if the quadratic polynomial Q_i does not contain $x_k x_l$ term. Moreover, we can always set

$$m_1 >_\lambda m_2 >_\lambda m_3, \tag{5.1}$$

up to replacing Q_1, Q_2, Q_3 with a linear combination of the three polynomials. Then the term $m_1 \wedge m_2 \wedge m_3$ appears in the Plücker coordinates of $Q_1 \wedge Q_2 \wedge Q_3$ and has the largest weight with respect to λ. Hence the net $\{Q_1, Q_2, Q_3\}$ is not properly stable with respect to λ if and only if $w_\lambda(m_1 \wedge m_2 \wedge m_3) \leq 0$.

Lemma 5.1 *With the notation above, let X be the complete intersection $Q_1 \cap Q_2 \cap Q_3$. Then X has a singularity with multiplicity greater than two if one of the following conditions does not hold:*

(1) $m_1 \geq_\lambda x_0 x_4$;
(2) $m_2 \geq_\lambda x_1 x_5$ *if* $m_1 = x_0^2$, *and* $m_2 \geq_\lambda x_0 x_5$ *otherwise*;
(3) $m_3 \geq_\lambda x_3^2$ *if* $m_1 <_\lambda x_0 x_3$.

Moreover, X is singular along a curve if one of the following conditions does not hold:

(1′) $m_1 \geq_\lambda x_1^2$ *if* $m_3 <_\lambda x_1 x_5$; *or* $m_1 \geq_\lambda \max\{x_1 x_3, x_2^2\}$ *if* $m_2 <_\lambda x_1 x_4$;
(2′) $m_2 \geq_\lambda x_2^2$ *if* $m_3 <_\lambda x_2 x_5$; $m_2 \geq_\lambda \max\{x_1 x_4, x_3^2\}$ *if* $m_1 <_\lambda x_1^2$; *and* $m_2 \geq_\lambda \max\{x_2 x_4, x_3^2\}$ *otherwise*;
(3′) $m_3 \geq_\lambda \max\{x_3 x_5, x_4^2\}$.

Proof Let p_0 be the point $[1, 0, 0, 0, 0, 0]$ in $\mathbb{P}^5$. For (1) and (2), if either $m_1 <_\lambda x_0 x_4$ or $m_2 <_\lambda x_0 x_5$ and $m_1 <_\lambda x_0^2$, the surface X contains the point p_0 and two quadrics Q_2, Q_3 are both singular at p_0. It follows that multiplicity of p_0 is greater than 2.

If $m_1 = x_0^2$ and $m_2 <_\lambda x_1 x_5$, then X is singular along the two points

$$\{Q_1 = x_2 = x_3 = x_4 = x_5 = 0\}$$

with multiplicity greater than 2. Similarly, one can easily check our assertion for (3).

For (1′), (2′) and (3′), we will only list the singular locus of X and leave the proof to readers:

- X is singular along the line $L : x_2 = x_3 = x_4 = x_5 = 0$ if condition (1′) is invalid.

• X is either reducible or singular along L or $C_1 : x_3 = x_4 = x_5 = Q_1 = 0$ if condition $(2')$ is invalid.

• X is either reducible or singular along the curve $C_2 : x_4 = x_5 = Q_1 = Q_2 = 0$ if condition $(3')$ is invalid.

As before, we need to know the maximal set $M_{\leq 0}(\lambda)$ of triples of distinct quadratic monomials $\{q_1, q_2, q_3\}$, whose sum of their weights with respect to λ is non-positive. Instead of looking at all maximal subsets, we are interested in the maximal subset $\overline{M}_{\leq 0}(\lambda)$ which contains a triple $\{m_1, m_2, m_3\}$ satisfying the conditions (1)–(3) and $(1')$–$(3')$ in Lemma 5.1. It is not difficult to compute that there are four such maximal subset. See Table 3 below.

Table 3 Maximal set $\overline{M}_{\leq 0}(\lambda)$

Cases	$\lambda = (a_0, \cdots, a_5)$	Maximal triples $\{q_1, q_2, q_3\}$		
		q_1	q_2	q_3
$(N1')$	$(2, 1, 0, 0, -1, -2)$	x_0x_2, x_1^2	x_0x_5, x_1x_4, x_2^2	x_2x_5, x_4^2
$(N2')$	$(3, 1, 1, -1, -1, -3)$	x_0x_3, x_1^2	x_0x_5, x_1x_3	x_1x_5, x_3^2
$(N3')$	$(4, 1, 1, -2, -2, -2)$	x_0x_3, x_1^2	x_0x_3, x_1^2	x_3^2
$(N4')$	$(5, 3, 1, -1, -3, -5)$	x_0x_4, x_1x_3, x_2^2	x_0x_5, x_1x_4, x_2x_3	x_1x_5, x_2x_4, x_3^2

The lemma below gives a geometric description of X of type $(N1')$–$(N4')$.

Lemma 5.2 *Let X be a general element of type* $(N1')$–$(N4')$. *Then X has an isolated simple elliptic singularity.*

Proof Obviously, X is singular at $p_0 = [1, 0, 0, 0, 0, 0]$. Moreover, p_0 is an isolated hypersurface singularity when X is general. To show that it is simple elliptic, let us compute the analytic type of p_0 case by case.

If X is a general element of type $(N1')$, then the equations of Q_i can be written as

$$Q_1 : x_0x_2 + q(x_1, \cdots, x_5) = 0,$$
$$Q_2 : x_0x_5 + x_1x_4 + q'(x_2, x_3, x_4, x_5) = 0,$$
$$Q_3 : x_4^2 + x_5 l(x_2, x_3, x_4, x_5) = 0$$

up to a linear change of the coordinates. Let us take the local coordinates near p_0:

$$y_i = \frac{x_i}{x_0}. \tag{5.2}$$

From the first two quadratic equations, one can get

$$y_2 = f_1(y_1, y_3, y_4),$$
$$y_5 = y_1y_4 + by_3^2 + b'y_3 f_1(y_1, y_2, y_4) + f_2(y_1, y_3, y_4)$$

for some formal power series $f_1 \in \mathbb{C}[[y_1, y_3, y_4]]_{\geq 2}, f_2 \in \mathbb{C}[[y_1, y_3, y_4]]_{\geq 4}$ and some constants $b, b' \in \mathbb{C}$. Therefore, the local equation of p_0 is

$$y_4^2 + \alpha_1 y_3^3 + \alpha_2 y_3^2 y_1^2 + \alpha_3 y_3 y_1^4 + \alpha_4 y_1^6 + (\geq \text{ higher order terms}) = 0 \tag{5.3}$$

for some complex number α_i. According to Subsection 3.1, the singularity p_0 is simple elliptic of type $\widetilde{E}_8$.

If X is a general element of type (N2′), we write the equations as

$$Q_1 : x_0 x_3 + q(x_1, \cdots, x_5) = 0,$$
$$Q_2 : x_0 x_5 + x_1 x_3 + x_2 x_4 = 0,$$
$$Q_3 : q'(x_3, x_4, x_5) + x_5 l(x_1, x_2) = 0.$$

Still, we take the affine coordinate (5.2) near p_0 and then we have

$$y_3 = f(y_1, y_2, y_4), \quad y_5 = -y_1 f(y_1, y_2, y_4) - y_2 y_4$$

for some $f \in \mathbb{C}[[y_1, y_2, y_4]]_{\geq 2}$. Thus the local equation around p_0 is

$$\alpha y_4^2 + g(y_1, y_4) + y_4 g'(y_1, y_2, y_4) = 0, \tag{5.4}$$

where $g \in \mathbb{C}[[y_1, y_2]]_{\geq 4}$, $g' \in \mathbb{C}[[y_1, y_2, y_4]]_{\geq 2}$ and $\alpha \in \mathbb{C}$ is a constant. Hence p_0 is simple elliptic of type $\widetilde{E}_7$ by Subsection 3.1.

One can similarly prove that X has a simple elliptic singularity p_0 of type $\widetilde{E}_7$ when it is general of type (N3′), and of type $\widetilde{E}_8$ when it is general of type (N4′).

5.3 Image of the period map

Let $\mathcal{U}_8 \subset Gr(3, W)$ be the open subset consisting of all complete intersections with at worst simple singularities. By Lemma 5.2, we know that $\mathcal{U}_8$ is contained in the stable locus of $Gr(3, W)$. This proves Theorem 5.1. Moreover, similarly as Theorem 3.3, we can get the following result.

Theorem 5.2 *Let* $\mathcal{M}_8 = \mathcal{U}_8 /\!\!/ SL_6(\mathbb{C})$ *be the moduli space of the complete intersection of three quadrics in* $\mathbb{P}^5$ *with simplest simple singularities. Then*

(i) *the boundary of* $\mathcal{M}_8$ *in* $\overline{\mathcal{M}}_8$ *has codimension* ≥ 2;

(ii) *there is an open immersion* $\mathcal{P}_8 : \mathcal{M}_8 \to \mathcal{F}_8$ *as the extended period map and the comple-ment of* $\mathcal{P}_8(\mathcal{M}_8)$ *in* $\mathcal{F}_8$ *is the union of three* NL-*divisors* $D_{1,1}^8, D_{2,1}^8$ *and* $D_{3,1}^8$. *The Picard group* $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_8)$ *is spanned by* $\{D_{d,1}^8, \ 1 \leq d \leq 4\}$.

Proof For (i), let $\Delta \subseteq Gr(3, W)$ be the discriminant divisor which parameterizes singular complete intersections. Then Δ is $SL_6(\mathbb{C})$-invariant and irreducible (see [9]). Consider the GIT quotient $\Delta /\!\!/ SL_6(\mathbb{C})$. By Theorem 5.1, the general members in Δ is stable, so the boundary $\overline{\mathcal{M}}_8 \backslash \mathcal{M}_8$ lies in the boundary of $\Delta /\!\!/ SL_6(\mathbb{C})$ as a proper closed subset. It follows that $\overline{\mathcal{M}}_8 \backslash \mathcal{M}_8$ has codimension two in $\overline{\mathcal{M}}_8$.

For (ii), this follows from the same arguments as in Theorem 3.3.

6 Arithmetic Compactification of Locally Hermitian Symmetric Varieties

Baily and Borel compactify the arithmetic quotient $\Gamma_{2\ell} \backslash \mathbb{D}$ to a normal projective variety $\overline{\Gamma_{2\ell} \backslash \mathbb{D}}^{bb}$ by adding finitely many modular curves and singletons, which correspond to the classes

of $\mathbb{Q}$-isotropic subspaces of $\Lambda_{2\ell}^{\mathbb{C}}$ of dimensions 2 and 1. In [14], Looijgenga gives an arithmetic compactification of the completment of hyperplane arrangements in $\Gamma_{2\ell}\backslash\mathbb{D}$ in the spirit of Satake-Baily-Borel theory. In our situation, the hyperplane arrangement which we are interested in will be the union of three NL-divisors $D_{d,1}^{2\ell}$ for $d = 1, 2, 3$.

6.1 A review of Looijenga's work

Let $\mathfrak{E}$ be a collection of elements in Λ. The orthogonal complement of $\beta \in \mathfrak{E}$ and $h_{2\ell}$ in Λ defines a hyperplane $H_\beta \subseteq \mathbb{P}(\Lambda_{2\ell}^{\mathbb{C}})$. Set $\mathbb{D}_{H_\beta} = \mathbb{D} \cap H_\beta$ to be the hyperplane arrangement and define

$$\mathbb{D}_{\mathfrak{E}}^\circ = \mathbb{D} - \bigcup_{\beta \in \mathfrak{E}} \mathbb{D}_{H_\beta}$$

to be the complement of all subdomains obtained from $\mathfrak{E}$. The quotient $\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ$ is the complement of Heenger divisors.

Looijenga constructed the compactification $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ}$ from the strata of decomposition of rational cones. It can also be viewed as the natural blowdown of certain minimal normal blowup of the Baily-Borel compactification $\overline{\Gamma_{2\ell}\backslash\mathbb{D}}^{bb}$. The structure of the birational map is explicitly provided that how the hyperplanes H_β intersect inside the period domain $\mathbb{D}$. To make it precise, we fix our temporary notation as follows:

• $\mathrm{PO}(\mathfrak{E})$: The collection of subspaces $M \subseteq \Lambda_{2\ell}^{\mathbb{C}}$ which are intersection of the hyperplane arrangements from $\mathfrak{E}$. Denote by

$$\pi_M : \mathbb{P}(\Lambda_{2\ell}^{\mathbb{C}}) - \mathbb{P}(M) \to \mathbb{P}(\Lambda_{2\ell}^{\mathbb{C}}/M)$$

the natural projection. The projection also defines a natural subdomain $\pi_M \mathbb{D}_{\mathfrak{E}}^\circ \subseteq \mathbb{D}_{\mathfrak{E}}^\circ$ (see [14, Section 7]).

• $\mathrm{I}(\mathfrak{E})$: The collection of the common intersection of $I^\perp$ and hyperplane arrangements from $\mathfrak{E}$ containing I, where I is a $\mathbb{Q}$-isotropic subspace of $\Lambda_{2\ell}^{\mathbb{C}}$.

The compactification $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ}$ can be interpreted as below: We define

$$\widehat{\mathbb{D}}_{\mathfrak{E}} = \mathbb{D}_{\mathfrak{E}}^\circ \bigcup \coprod_{M \in \mathrm{PO}(\mathfrak{E})} \pi_M \mathbb{D}_{\mathfrak{E}}^\circ \bigcup \coprod_{V \in \mathrm{I}(\mathfrak{E})} \pi_V \mathbb{D}_{\mathfrak{E}}^\circ, \tag{6.1}$$

then the compactification $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ}$ is isomorphic to the quotient $\Gamma_{2\ell}\backslash\widehat{\mathbb{D}}$, and boundary thus decomposes into finitely many strata. A consequent of this description is that if the r-th self intersection of hyperplane arrangements $(\Gamma_{2\ell}\backslash\bigcup_\beta \mathbb{D}_{H_\beta})^{(r)} \neq \emptyset$, then

$$\dim(\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ} - \Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ) \geq r + 1. \tag{6.2}$$

When the codimesion of $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ} - \Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ$ is greater than one, there is an explicit description of $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ}$ in terms of the algebra of automorphic forms: Let $\mathbb{L}$ be natural automorphic line bundle on $\mathbb{D}$ and $\mathbb{L}^\circ$ be the restriction of $\mathbb{L}$ to $\mathbb{D}_{\mathfrak{E}}^\circ$, then

$$\widehat{\Gamma_{2\ell}\backslash\mathbb{D}_{\mathfrak{E}}^\circ} \cong \mathrm{Proj} \bigoplus_{k \in \mathbb{Z}} H^0(\mathbb{D}_{\mathfrak{E}}^\circ, (\mathbb{L}^\circ)^{\otimes k})^{\Gamma_{2\ell}} \tag{6.3}$$

by [14, Corollary 7.5].

6.2 Application to moduli problem via GIT

Let us discuss the possible geometric interpretation of $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{2\ell}}$. For many known geometric examples, such as Enrique surface, K3 surfaces of degree 2 and cubic fourfolds, the natural GIT compactification is precisely Looijenga's compactification (see [13–14]). So it is interesting to investigate the relations between the compactifiations from GIT and arithmetic for K3 surfaces. For K3 surface with Mukai models, Laza has first found that the two compactifications do not necessarily coincide. This fails for quartic surfaces in $\mathbb{P}^3$. We can show that this actually happens quite often.

In our case, let $\mathfrak{E}_{2\ell}$ be the collection of elements $\beta \in \Lambda$ satisfying $\beta^2 = 0$ and $\beta \cdot h_{2\ell} = 1, 2$ or 3. Then $\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}$ is the complementent of three NL-diviosrs $D^{2\ell}_{d,1}$ for $d = 1, 2, 3$. The following lemma gives a rough description of the dimension of the boundary strata of $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}}$.

Lemma 6.1 *When* $2\ell = 6$ *and* 8, *the boundary of* $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}} - \Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}$ *has codimension* 1.

Proof To understand the dimension of boundary, it suffices to consider the intersection of hyperplane arrangements from $\mathfrak{E}_{2\ell}$. Let M be an even lattice of signature $(1, 17)$ spanned by $h_{2\ell}$ and elements $e_1, e_2, \cdots, e_{17}$ satisfying that $e_i^2 = 0, e_i e_j = 1$ and $h_{2\ell} e_i = 3$ for $i \neq j$. It is easy to check that this lattice has signature $(1, 17)$ and thus can be embedded into Λ. Then the lattice M can represent the intersection of 17 hyperplane arrangements from $\mathfrak{E}_{2\ell}$. This proves the assertion by (6.2).

Corollary 6.1 *For* $2\ell = 6$ *and* 8, *the* GIT *quoitent* $\overline{\mathcal{M}}_{2\ell}$ *is not isomorphic to Looijenga's compactification.*

Proof Since $\mathcal{M}_{2\ell}$ is isomorphic to $\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}$, this is obtained by comparing the dimension of the boundary of $\overline{\mathcal{M}}_{2\ell} - \mathcal{M}_{2\ell}$ and $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}} - \Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}$.

In general, we believe that natural GIT compactifications of K3 surfaces with Mukai models constructed in [9] will not be the same as Looijenga's compactification. This can be achieved by a similar method.

Another interesting problem is to study the birational maps between $\widehat{\Gamma_{2\ell}\backslash\mathbb{D}^\circ_{\mathfrak{E}_{2\ell}}}$ and $\mathcal{M}_{2\ell}$. In a sequel to this paper, the authors together with Greer and Laza will study the birational geometry of $\mathcal{F}_6$ via the variation of GIT and Looijenga's arithmetic approach.

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