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Derived categories of $K3$ surfaces, O'Grady's filtration, and zero-cycles on holomorphic symplectic varieties

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Abstract

Moduli spaces of stable objects in the derived category of a $K3$ surface provide a large class of holomorphic symplectic varieties. In this paper, we study the interplay between Chern classes of stable objects and zero-cycles on holomorphic symplectic varieties which arise as moduli spaces. First, we show that the second Chern class of any object in the derived category lies in a suitable piece of O'Grady's filtration on the CH_0-group of the $K3$ surface. This solves a conjecture of O'Grady and improves on previous results of Huybrechts, O'Grady, and Voisin. Second, we propose a candidate for the Beauville–Voisin filtration on the CH_0-group of the moduli space of stable objects. We discuss its connection with Voisin's recent proposal via constant cycle subvarieties, and prove a conjecture of hers on the existence of special algebraically coisotropic subvarieties for the moduli space.

Introduction

The purpose of this paper is twofold. On one hand, we study objects in the derived category of a $K3$ surface and their Chern classes. We locate the second Chern classes in the CH_0-group of the $K3$ surface with respect to a filtration introduced by O'Grady, proving and generalizing a conjecture of his. On the other hand, we consider zero-cycles on holomorphic symplectic varieties which arise as moduli spaces in the derived category. We search for a filtration envisioned by Beauville and Voisin on the CH_0-group of the moduli space, suggesting that it should come from the derived category.

Aspects of derived categories, moduli spaces, and algebraic cycles are brought together.

0.1 Zero-cycles on $K3$ surfaces

Let X be a nonsingular projective $K3$ surface. In [BV04], Beauville and Voisin showed that X carries a canonical zero-cycle class of degree 1,

$$[o_X] \in \mathrm{CH}_0(X),$$

where o_X can be taken any point lying on a rational curve in X. It has the remarkable property that all intersections of divisor classes in X, as well as the second Chern class of X, lie in $\mathbb{Z}\cdot[o_X]$.

In [O'Gr13], O'Grady introduced an increasing filtration $S_\bullet(X)$ on $\mathrm{CH}_0(X)$,

$$S_0(X) \subset S_1(X) \subset \cdots \subset S_i(X) \subset \cdots \subset \mathrm{CH}_0(X),$$

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where $S_i(X)$ is the union of $[z]+\mathbb{Z}\cdot[o_X]$ for all effective zero-cycles z of degree i. In particular, we have

$$S_0(X)=\mathbb{Z}\cdot[o_X].$$

An alternative characterization of $S_\bullet(X)$ via effective orbits is given by Voisin in [Voi15].

0.2 Derived categories

Let $D^b(X)$ denote the bounded derived category of coherent sheaves on X. Given an object $\mathcal{E}\in D^b(X)$, we write

$$v(\mathcal{E})\in H^0(X,\mathbb{Z})\oplus H^2(X,\mathbb{Z})\oplus H^4(X,\mathbb{Z})$$

for the Mukai vector of $\mathcal{E}$, and define

$$d(\mathcal{E})=\tfrac{1}{2}\dim\operatorname{Ext}^1(\mathcal{E},\mathcal{E})\in\mathbb{Z}_{\geqslant 0}.$$

An interesting link between the second Chern classes of objects in $D^b(X)$ and the filtration $S_\bullet(X)$ was discovered by Huybrechts and O'Grady. In [Huy10], Huybrechts showed under certain assumptions[1] that if $\mathcal{E}\in D^b(X)$ is a spherical object (and hence $d(\mathcal{E})=0$), then

$$c_2(\mathcal{E})\in\mathbb{Z}\cdot[o_X].$$

Later, O'Grady conjectured[2] in [O'Gr13] that if $\mathcal{E}$ is a Gieseker-stable sheaf with respect to a polarization H on X, then

$$c_2(\mathcal{E})\in S_{d(\mathcal{E})}(X).$$

He verified the conjecture again under certain assumptions on the Picard rank of X and/or on the Mukai vector $v(\mathcal{E})$. Further, in [Voi15], Voisin proved (a generalization of) the conjecture for any simple vector bundle $\mathcal{E}$ on X.

Our first result completes the proof of O'Grady's conjecture and generalizes it to arbitrary objects in $D^b(X)$.

Theorem 0.1. *For any object $\mathcal{E}\in D^b(X)$, we have*

$$c_2(\mathcal{E})\in S_{d(\mathcal{E})}(X).$$

Note that Theorem 0.1 does not involve any stability condition and should be viewed as a statement purely on the derived category $D^b(X)$. However, the proof uses (slope) stability and ultimately relies on Voisin's proof of the vector bundle case.

Theorem 0.1 has an important consequence. Let $\widetilde{S}_\bullet(X)$ be the extension of O'Grady's filtration to the Chow ring $\mathrm{CH}^*(X)$ by the trivial filtration on $\mathrm{CH}^0(X)$ and $\mathrm{CH}^1(X)$. In particular, we have

$$\widetilde{S}_0(X)=R^*(X)$$

which is the Beauville–Voisin ring of X generated by divisor classes. Let

$$\Phi: D^b(X)\xrightarrow{\sim} D^b(X')$$

[1] The assumptions (on the Picard rank of X or on the Mukai vector $v(\mathcal{E})$) were subsequently removed following [Voi15].

[2] The statement in [O'Gr13, Conjecture 0.4] takes a slightly stronger form. However, as is shown in [O'Gr13, Proposition 1.3], it is equivalent to what is stated here.

be a derived equivalence between two nonsingular projective $K3$ surfaces. It induces an isomorphism of (ungraded) Chow groups

$$\Phi^{\mathrm{CH}} : \mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CH}^*(X').$$

We have the following generalization of Huybrechts' result in [Huy10] that Φ^{CH} preserves the Beauville–Voisin ring.[3]

Corollary 0.2. *The isomorphism Φ^{CH} preserves O'Grady's filtration $\widetilde{S}_\bullet$.*

The generality of Theorem 0.1 also suggests a natural increasing filtration on $D^b(X)$,

$$S_0(D^b(X)) \subset S_1(D^b(X)) \subset \cdots \subset S_i(D^b(X)) \subset \cdots \subset D^b(X),$$

where $S_i(D^b(X))$ consists of objects $\mathcal{E} \in D^b(X)$ with $c_2(\mathcal{E}) \in S_i(X)$. By Corollary 0.2, this filtration does not depend on the $K3$ surface X and is 'intrinsic' to the triangulated category $\mathbf{D} = D^b(X)$.

0.3 Moduli spaces of stable objects

Moduli spaces of stable sheaves on X provide a large class of holomorphic symplectic varieties[4] of $K3^{[d]}$-type. The subject has been developed by many people, including Beauville, Mukai, Huybrechts, O'Grady, and Yoshioka; see [Bea83, Huy97, Huy99, Muk84, O'Gr97, Yos01]. More recently, Bridgeland [Bri07, Bri08] and Bayer–Macrì [BM14a, BM14b] obtained all holomorphic symplectic birational models of these moduli spaces by considering moduli spaces of objects in $D^b(X)$ satisfying certain stability conditions.

Let

$$\mathbf{v} \in H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$$

be a primitive algebraic class with Mukai self-intersection $\mathbf{v}^2 > 0$. In [Bri08], Bridgeland described a connected component $\mathrm{Stab}^\dagger(X)$ of the space of stability conditions on $D^b(X)$, which admits a chamber decomposition depending on $\mathbf{v}$. When $\sigma \in \mathrm{Stab}^\dagger(X)$ is a generic[5] stability condition with respect to $\mathbf{v}$, there is a nonsingular projective moduli space $M_\sigma(\mathbf{v})$ of σ-stable objects $\mathcal{E} \in D^b(X)$ with Mukai vector $v(\mathcal{E}) = \mathbf{v}$. The moduli space $M_\sigma(\mathbf{v})$ only depends on the chamber containing σ. It is of dimension[6]

$$2d(\mathbf{v}) = \mathbf{v}^2 + 2 > 2$$

and is holomorphic symplectic by the pairing

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathbb{C}.$$

When σ is in the chamber corresponding to the large volume limit, the moduli space $M_\sigma(\mathbf{v})$ recovers the moduli space of Gieseker-stable sheaves with respect to a generic polarization H on X.

In the first version of this paper, we proposed the following conjecture relating the second Chern classes of objects in $M_\sigma(\mathbf{v})$ to the corresponding point classes on $M_\sigma(\mathbf{v})$.

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[3] As before, the assumption in [Huy10] on the Picard rank can be removed following [Voi15].

[4] They are also referred to as irreducible holomorphic symplectic varieties, or equivalently, hyper-Kähler varieties following [Bea83]. In this paper, we emphasize the holomorphic symplectic point of view.

[5] The word 'generic' means away from the walls.

[6] By Riemann–Roch and Serre duality, we have $d(\mathbf{v}) = d(\mathcal{E})$ for any $\mathcal{E} \in M_\sigma(\mathbf{v})$.

Conjecture 0.3. *Two objects $\mathcal{E}, \mathcal{E}' \in M_\sigma(\mathbf{v})$ satisfy*

$$[\mathcal{E}] = [\mathcal{E}'] \in \mathrm{CH}_0(M_\sigma(\mathbf{v})) \tag{i}$$

if and only if

$$c_2(\mathcal{E}) = c_2(\mathcal{E}') \in \mathrm{CH}_0(X). \tag{ii}$$

Note that the two conditions above have different flavors. Condition (i) only depends on the triangulated category $\mathbf{D} = D^b(X)$ with a given stability condition, while (ii) requires the underlying $K3$ surface X.

Later, Marian and the third author found a short proof of Conjecture 0.3 in [MZ17]. This renders a number of subsequent statements unconditional.

0.4 Beauville–Voisin filtration for zero-cycles

Our study of zero-cycles on the moduli spaces $M_\sigma(\mathbf{v})$ is motivated by the Beauville–Voisin conjecture for holomorphic symplectic varieties. The conjecture predicts that the Chow ring (with rational coefficients) of a holomorphic symplectic variety admits a multiplicative decomposition; see [Bea07, Voi08, Voi16]. Another way to phrase it is the existence of a new filtration on the Chow ring which is opposite to the conjectural Bloch–Beilinson filtration. Recently, rather than proving consequences of the Beauville–Voisin conjecture, much effort has been put to construct this new filtration, which we shall call the Beauville–Voisin filtration.

In the case of a moduli space $M_\sigma(\mathbf{v})$, our previous discussion suggests a natural candidate for the Beauville–Voisin filtration on the Chow group $\mathrm{CH}_0(M_\sigma(\mathbf{v}))$ of zero-cycles. It is simply given by the restriction of the filtration $S_\bullet(\mathbf{D})$ to $\mathrm{CH}_0(M_\sigma(\mathbf{v}))$. More concretely, we have an increasing filtration

$$S_0\mathrm{CH}_0(M_\sigma(\mathbf{v})) \subset S_1\mathrm{CH}_0(M_\sigma(\mathbf{v})) \subset \cdots \subset S_i\mathrm{CH}_0(M_\sigma(\mathbf{v})) \subset \cdots \subset \mathrm{CH}_0(M_\sigma(\mathbf{v})),$$

where $S_i\mathrm{CH}_0(M_\sigma(\mathbf{v}))$ is the subgroup spanned by $[\mathcal{E}] \in \mathrm{CH}_0(M_\sigma(\mathbf{v}))$ for all $\mathcal{E} \in M_\sigma(\mathbf{v})$ with $c_2(\mathcal{E}) \in S_i(X)$.

An immediate consequence of Theorem 0.1 is

$$S_{d(\mathbf{v})}\mathrm{CH}_0(M_\sigma(\mathbf{v})) = \mathrm{CH}_0(M_\sigma(\mathbf{v})),$$

where $2d(\mathbf{v}) = \mathbf{v}^2 + 2$ is the dimension of $M_\sigma(\mathbf{v})$. Moreover, by an argument in [O'Gr13], the subset

$$\{c_2(\mathcal{E}) : \mathcal{E} \in M_\sigma(\mathbf{v})\} \subset S_{d(\mathbf{v})}(X)$$

equals the full subset of $S_{d(\mathbf{v})}(X)$ of the given degree. In particular, we have

$$S_0\mathrm{CH}_0(M_\sigma(\mathbf{v})) \neq 0.$$

Further, since $S_0(X) = \mathbb{Z} \cdot [o_X]$, Conjecture 0.3 (now proven) implies that

$$S_0\mathrm{CH}_0(M_\sigma(\mathbf{v})) \simeq \mathbb{Z}.$$

In other words, the moduli space $M_\sigma(\mathbf{v})$ carries a canonical zero-cycle class of degree 1, which matches the predictions of the Beauville–Voisin conjecture.

We also show that the filtration $S_\bullet\mathrm{CH}_0$ is independent of birational models or modular interpretations. Hence $S_\bullet\mathrm{CH}_0$ is 'intrinsic' to $M = M_\sigma(\mathbf{v})$ as a moduli space of stable objects in the triangulated category $\mathbf{D} = D^b(X)$.

Proposition 0.4. *For any $(X', \sigma', \mathbf{v}')$ such that $M_{\sigma'}(\mathbf{v}')$ is birational to $M_\sigma(\mathbf{v})$, the canonical isomorphism of Chow groups*

$$\mathrm{CH}_0(M_\sigma(\mathbf{v})) \xrightarrow{\sim} \mathrm{CH}_0(M_{\sigma'}(\mathbf{v}'))$$

preserves the filtration $S_\bullet \mathrm{CH}_0$.

In [Voi16], Voisin proposed a filtration on $\mathrm{CH}_0(M)$ for any holomorphic symplectic variety M of dimension $2d$. Given a (closed) point $x \in M$, consider the orbit of x under rational equivalence

$$O_x = \{x' \in M : [x] = [x'] \in \mathrm{CH}_0(M)\}.$$

It is a countable union of constant cycle subvarieties.[7] We write $\dim O_x$ for the maximal dimension of these subvarieties. There is an increasing filtration

$$\begin{aligned} S_0^V \mathrm{CH}_0(M) \subset S_1^V \mathrm{CH}_0(M) \subset \cdots \subset S_i^V \mathrm{CH}_0(M) \\ \subset \cdots \subset S_d^V \mathrm{CH}_0(M) = \mathrm{CH}_0(M), \end{aligned}$$

where $S_i^V \mathrm{CH}_0(M)$ is the subgroup spanned by $[x] \in \mathrm{CH}_0(M)$ for all $x \in M$ with $\dim O_x \geqslant d - i$. Many questions around the filtration $S_\bullet^V \mathrm{CH}_0(M)$ remain open, among which the existence of algebraically coisotropic subvarieties

$$\begin{array}{ccc} Z_i & \hookrightarrow & M \\ {\scriptstyle q}\,\downarrow & & \\ B_i & & \end{array}$$

where Z_i is a subvariety of codimension i and the general fibers of q are constant cycle subvarieties (in M) of dimension i.

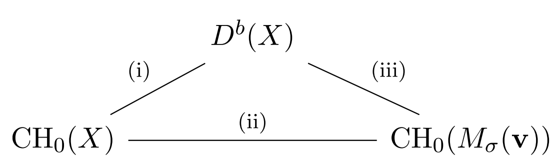
The following result constructs such algebraically coisotropic varieties and connects the filtrations $S_\bullet \mathrm{CH}_0(M)$ and $S_\bullet^V \mathrm{CH}_0(M)$ in case $M = M_\sigma(\mathbf{v})$. In particular, this verifies [Voi16, Conjecture 0.4] when the holomorphic symplectic variety arises as a moduli space of stable objects in $D^b(X)$.[8]

Theorem 0.5. *For $0 \leqslant i \leqslant d = d(\mathbf{v})$, the following hold:*

(i) *there exists an algebraically coisotropic subvariety $Z_i \dashrightarrow B_i$ of codimension i with constant cycle fibers;*
(ii) *we have $S_i \mathrm{CH}_0(M) \subset S_i^V \mathrm{CH}_0(M)$.*

0.5 Summary

We summarize the main themes of this paper by the following diagram.



[7] A constant cycle subvariety is a subvariety whose points all share the same class in the CH_0-group of the ambient variety; see [Huy14, Voi16].

[8] It was shown in [Voi16, Theorem 1.3] that [Voi16, Conjecture 0.4] is equivalent to the existence of algebraically coisotropic subvarieties with constant cycle fibers.

(i) Theorem 0.1, i.e., O'Grady's conjecture, provides a sheaf/cycle correspondence and lifts O'Grady's filtration $S_\bullet(X)$ to $D^b(X)$.
(ii) Conjecture 0.3, now proven in [MZ17], relates point classes on $M_\sigma(\mathbf{v})$ to zero-cycles classes on X.
(iii) The lifted filtration $S_\bullet(D^b(X))$ in turn provides a natural candidate for the Beauville–Voisin filtration on $\mathrm{CH}_0(M_\sigma(\mathbf{v}))$, with many of the required properties.

In a sequel [SY18] to this paper, we extend the picture above to more general $K3$ categories, especially Kuznetsov's noncommutative $K3$ category associated to a nonsingular cubic 4-fold [Kuz10].

0.6 Conventions

Throughout, we work over the complex numbers $\mathbb{C}$. All varieties are assumed to be (quasi-)projective, and $K3$ surfaces are nonsingular and projective. Equivalences of triangulated categories are $\mathbb{C}$-linear.

1. Chern classes and O'Grady's filtration

In this section, we prove Theorem 0.1 and Corollary 0.2.

1.1 Preliminaries

We first list a few useful facts. Let X be a $K3$ surface.

Lemma 1.1 [O'Gr13, Corollary 1.7]. *Let $\alpha, \alpha' \in \mathrm{CH}_0(X)$.*

(i) *If $\alpha \in S_i(X)$ and $\alpha' \in S_{i'}(X)$, then $\alpha + \alpha' \in S_{i+i'}(X)$.*
(ii) *If $\alpha \in S_i(X)$, then $m\alpha \in S_i(X)$ for any $m \in \mathbb{Z}$.*

Corollary 1.2. *Let*

$$\mathcal{F} \to \mathcal{E} \to \mathcal{G} \to \mathcal{F}[1]$$

be a distinguished triangle in $D^b(X)$. If two of $c_2(\mathcal{E}), c_2(\mathcal{F}), c_2(\mathcal{G})$ lie in $S_i(X)$ and $S_{i'}(X)$ respectively, then the third lies in $S_{i+i'}(X)$.

Proof. By the distinguished triangle, we have

$$c_2(\mathcal{E}) = c_2(\mathcal{F}) + c_2(\mathcal{G}) + D,$$

where D is spanned by intersections of divisor classes. Hence $D \in S_0(X)$ by [BV04] and the statement follows immediately from Lemma 1.1. □

We will need the following generalization of a lemma of Mukai [Muk87, Corollary 2.8].

Lemma 1.3 [BB17, Lemma 2.5]. *Let*

$$\mathcal{F} \to \mathcal{E} \to \mathcal{G} \to \mathcal{F}[1]$$

be a distinguished triangle in $D^b(X)$. If $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$, then there is an inequality

$$d(\mathcal{F}) + d(\mathcal{G}) \leqslant d(\mathcal{E}).$$

The following is a direct consequence of Corollary 1.2 and Lemma 1.3.

Proposition 1.4. *Let*

$$\mathcal{F} \to \mathcal{E} \to \mathcal{G} \to \mathcal{F}[1]$$

be a distinguished triangle in $D^b(X)$ *satisfying* $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$. *If*

$$c_2(\mathcal{F}) \in S_{d(\mathcal{F})}(X) \quad \text{and} \quad c_2(\mathcal{G}) \in S_{d(\mathcal{G})}(X),$$

then

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

We also recall the theorems of Huybrechts and Voisin which serve as the starting point of our proof.

Theorem 1.5 ([Huy10, Theorem 1] and [Voi15, Corollary 1.10]). *If* $\mathcal{E} \in D^b(X)$ *is spherical, i.e.,* $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) = H^*(\mathbb{S}^2, \mathbb{C})$, *then*

$$c_2(\mathcal{E}) \in S_0(X).$$

Theorem 1.6 [Voi15, Theorem 1.9]. *If* $\mathcal{E}$ *is a simple vector bundle on* X, *then*

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

1.2 Slope-stable sheaves

From now on, we fix a polarization H on X. The following proposition proves Theorem 0.1 for μ-stable sheaves.

Proposition 1.7. *If* $\mathcal{E}$ *is torsion-free and* μ*-stable on* (X, H), *then*

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

Proof. The double dual $\mathcal{E}^{\vee\vee}$ of $\mathcal{E}$ is locally free. There is a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{E}^{\vee\vee} \to \mathcal{Q} \to 0, \tag{1}$$

where $\mathcal{Q}$ is a 0-dimensional sheaf whose support is of length l. A direct calculation yields $d(\mathcal{Q}) \geqslant l$ and

$$c_2(\mathcal{Q}) \in S_l(X) \subset S_{d(\mathcal{Q})}(X).$$

Now since $\mathcal{E}$ is μ-stable, the double dual $\mathcal{E}^{\vee\vee}$ is also μ-stable and hence simple. Applying Theorem 1.6, we find

$$c_2(\mathcal{E}^{\vee\vee}) \in S_{d(\mathcal{E}^{\vee\vee})}(X).$$

Consider (1) as a distinguished triangle

$$\mathcal{Q}[-1] \to \mathcal{E} \to \mathcal{E}^{\vee\vee} \to \mathcal{Q}.$$

Since $\mathcal{Q}$ is 0-dimensional and $\mathcal{E}^{\vee\vee}$ is locally free, we have

$$\mathrm{Hom}(\mathcal{Q}[-1], \mathcal{E}^{\vee\vee}) = 0.$$

Applying Proposition 1.4, we conclude that $c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X)$. □

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We continue to treat sheaves which can be obtained as iterated extensions of μ-stable sheaves.

Proposition 1.8. *Let $\mathcal{F}$ be torsion-free and μ-stable on (X, H). If $\mathcal{E}$ is an iterated extension of $\mathcal{F}$, then*

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

Proof. Suppose $\mathcal{E}$ is an iterated extension of m copies of $\mathcal{F}$. Then we have

$$c_2(\mathcal{E}) = c_2(\mathcal{F}^{\oplus m}) = mc_2(\mathcal{F}) + D, \tag{2}$$

where D is spanned by intersections of divisor classes and hence lies in $S_0(X)$. Combining Lemma 1.1 and Proposition 1.7, we find

$$c_2(\mathcal{E}) \in S_{d(\mathcal{F})}(X). \tag{3}$$

If $\mathcal{F}$ is spherical, i.e., $v(\mathcal{F})^2 = -2$, then $c_2(\mathcal{F}) \in S_0(X)$ by Theorem 1.5. By (2), we see that $c_2(\mathcal{E}) \in S_0(X)$, and hence the statement holds.

We may focus on the case $v(\mathcal{F})^2 \geqslant 0$. Then we have

$$\begin{aligned} 2d(\mathcal{E}) &= v(\mathcal{E})^2 + 2\dim \operatorname{Hom}(\mathcal{E}, \mathcal{E}) \\ &= m^2 v(\mathcal{F})^2 + 2\dim \operatorname{Hom}(\mathcal{E}, \mathcal{E}) \\ &\geqslant v(\mathcal{F})^2 + 2 \\ &= 2d(\mathcal{F}), \end{aligned}$$

where we use that $\mathcal{F}$ is simple in the last equality. In this case the proposition follows from (3). □

1.3 Torsion-free sheaves

The next step is to prove Theorem 0.1 for arbitrary torsion-free sheaves.

The following proposition provides a nice splitting of a μ-semistable vector bundle.

Proposition 1.9. *Let $\mathcal{E}$ be a μ-semistable vector bundle on (X, H). There exists a short exact sequence of sheaves*

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{G} \to 0 \tag{4}$$

with the following properties:

(i) *the sheaf $\mathcal{M}$ is an iterated extension of a μ-stable vector bundle $\mathcal{F}$;*
(ii) *the quotient sheaf $\mathcal{G}$ is torsion-free;*
(iii) *we have* $\operatorname{Hom}(\mathcal{M}, \mathcal{G}) = 0$.

Proof. We only need to consider the case when $\mathcal{E}$ is not μ-stable. First, we can always find a μ-stable sub-vector bundle $\mathcal{F} \subset \mathcal{E}$ with $\mu(\mathcal{F}) = \mu(\mathcal{E})$.

The construction goes as follows. Let $\mathcal{F}_0$ be any μ-stable subsheaf of $\mathcal{E}$ with $\mu(\mathcal{F}_0) = \mu(\mathcal{E})$. The double dual $\mathcal{F} = \mathcal{F}_0^{\vee\vee}$ is both μ-stable (of the same slope) and locally free, which admits a nontrivial map

$$i : \mathcal{F} = \mathcal{F}_0^{\vee\vee} \to \mathcal{E}^{\vee\vee} = \mathcal{E}.$$

The map i is injective according to the stability condition. Hence we obtain a short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G}_0 \to 0. \tag{5}$$

Claim. *The quotient sheaf $\mathcal{G}_0$ is torsion-free and μ-semistable.*

Proof of the Claim. The stability condition ensures that the torsion part of $\mathcal{G}_0$ is at most 0-dimensional. Now assume that there is a short exact sequence of sheaves

$$0 \to \mathcal{T} \to \mathcal{G}_0 \to \mathcal{G}_0^F \to 0$$

with $\mathcal{T}$ a nontrivial 0-dimensional sheaf and $\mathcal{G}_0^F$ torsion-free. We have a surjective map $\mathcal{E} \to \mathcal{G}_0^F$ given by $\mathcal{E} \to \mathcal{G}_0 \to \mathcal{G}_0^F$ with kernel $\mathcal{F}'$. It follows that $\mathcal{F}'$ is a nontrivial extension of the 0-dimensional sheaf $\mathcal{T}$ by the vector bundle $\mathcal{F}$, which is a contradiction. This shows that $\mathcal{G}_0$ is torsion-free.

Since $\mu(\mathcal{G}_0) = \mu(\mathcal{F})$, the μ-semistability follows from a standard argument by considering quotients of $\mathcal{G}_0$ and comparing slopes. □

If $\mathrm{Hom}(\mathcal{F}, \mathcal{G}_0) = 0$, then we are done by setting $\mathcal{M} = \mathcal{F}$ and $\mathcal{G} = \mathcal{G}_0$, and (5) gives the desired exact sequence. Otherwise, there exists a nontrivial map

$$i_1 : \mathcal{F} \to \mathcal{G}_0,$$

which must be injective according to the stability condition. We define $\mathcal{G}_1$ to be the quotient $\mathcal{G}_0/\mathcal{F}$. The same argument as in the Claim implies that $\mathcal{G}_1$ is torsion-free and μ-semistable. Hence we obtain a short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{G}_1 \to 0,$$

where $\mathcal{F}_1$ is a self-extension of $\mathcal{F}$.

If $\mathrm{Hom}(\mathcal{F}_1, \mathcal{G}_1) \neq 0$, we can continue this process until we reach the desired exact sequence (4). □

Remark 1.10. One may expect a similar splitting for any μ-semistable sheaf via the Jordan–Hölder filtration (for slope stability). However, the difficulty is that there exist nontrivial morphisms between nonisomorphic μ-stable sheaves with the same slope. For example, there is the inclusion

$$\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$$

with $\mathcal{I}_Z$ the ideal sheaf of a 0-dimensional subscheme $Z \subset X$. Here we use a μ-stable locally free factor $\mathcal{F}$ to avoid this trouble.

Proposition 1.11. *If $\mathcal{E}$ is a torsion-free sheaf on X, then*

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

Proof. We proceed by induction on the rank of $\mathcal{E}$. If $\mathrm{rank}(\mathcal{E}) = 1$, then $\mathcal{E}$ is μ-stable, and Proposition 1.7 gives the base case of the induction.

Now assume that $\mathcal{E}$ is torsion-free of rank $r > 0$. If $\mathcal{E}$ is not μ-semistable, then by the Harder–Narasimhan filtration (for slope stability), we have a short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0.$$

Here $\mathcal{F}$ and $\mathcal{G}$ are nonzero and torsion-free, and the slope of every μ-stable factor of $\mathcal{F}$ is greater than the slope of any μ-stable factor of $\mathcal{G}$. In particular, we have $\mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$, $\mathrm{rank}(\mathcal{G}) < \mathrm{rank}(\mathcal{E})$, and $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$. The induction hypothesis yields

$$c_2(\mathcal{F}) \in S_{d(\mathcal{F})}(X) \quad \text{and} \quad c_2(\mathcal{G}) \in S_{d(\mathcal{G})}(X).$$

Applying Proposition 1.4, we find $c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X)$.

It remains to treat the case when $\mathcal{E}$ is μ-semistable. By the same argument as in Proposition 1.7, it suffices to prove Theorem 0.1 for $\mathcal{E}^{\vee\vee}$, which is a μ-semistable locally free sheaf satisfying $\mathrm{rank}(\mathcal{E}) = \mathrm{rank}(\mathcal{E}^{\vee\vee})$.

Hence we may assume $\mathcal{E}$ to be μ-semistable and locally free. We apply Proposition 1.9 to $\mathcal{E}$. Either $\mathcal{E}$ is an iterated extension of some μ-stable sheaf $\mathcal{F}$, or the extension (4) is nontrivial. In the first case, the statement of the proposition holds by Proposition 1.8. In the second case, the induction hypothesis and Proposition 1.4 complete the proof. □

1.4 Torsion sheaves

Theorem 0.1 for torsion sheaves is essentially proven in [O'Gr13]. We begin by recalling the following criterion of O'Grady.

Lemma 1.12 [O'Gr13, Claim 0.2]. *Let C be an irreducible nonsingular curve of genus g, and let $f : C \to X$ be a nonconstant map. Then*

$$f_*\mathrm{CH}_0(C) \subset S_g(X).$$

Let $\mathcal{E}$ be a pure 1-dimensional torsion sheaf on X with Mukai vector

$$v(\mathcal{E}) = (0, l, s) \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}).$$

By Lemma 1.12, we have at worst

$$c_2(\mathcal{E}) \in S_g(X),$$

where $g = \frac{1}{2}l^2 + 1$ is the arithmetic genus of the support curve of $\mathcal{E}$.

On the other hand, we find

$$d(\mathcal{E}) = \tfrac{1}{2}v(\mathcal{E})^2 + \dim \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \geqslant \tfrac{1}{2}l^2 + 1 = g.$$

Hence for any pure 1-dimensional sheaf $\mathcal{E}$, we have $c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X)$.

Now we can prove Theorem 0.1 for arbitrary sheaves.

Proposition 1.13. *If $\mathcal{E}$ is a coherent sheaf on X, then*

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

Proof. Given a torsion sheaf $\mathcal{T}$, there is a short exact sequence of sheaves

$$0 \to \mathcal{T}_0 \to \mathcal{T} \to \mathcal{T}_1 \to 0,$$

where $\mathcal{T}_0$ is 0-dimensional and $\mathcal{T}_1$ is pure and 1-dimensional. Clearly

$$\mathrm{Hom}(\mathcal{T}_0, \mathcal{T}_1) = 0.$$

By the discussion above, we have

$$c_2(\mathcal{T}_0) \in S_{d(\mathcal{T}_0)}(X) \quad \text{and} \quad c_2(\mathcal{T}_1) \in S_{d(\mathcal{T}_1)}(X).$$

Applying Proposition 1.4, we find $c_2(\mathcal{T}) \in S_{d(\mathcal{T})}(X)$ which proves the statement for torsion sheaves.

Let $\mathcal{E}$ be an arbitrary sheaf. There is a short exact sequence of sheaves

$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{T}$ torsion and $\mathcal{F}$ torsion-free. In particular, we have $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$. Since the statement of the proposition holds for both $\mathcal{T}$ and $\mathcal{F}$, we conclude by Proposition 1.4 that $c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X)$. □

1.5 Proof of Theorem 0.1 and Corollary 0.2

Given a bounded complex $\mathcal{E} \in D^b(X)$, we define its length by

$$\ell(\mathcal{E}) = \max\{|i-j| : h^i(\mathcal{E}) \neq 0, h^j(\mathcal{E}) \neq 0\}.$$

Clearly $\ell(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a (shifted) sheaf.

Proof of Theorem 0.1. We proceed by induction on $\ell(\mathcal{E})$. Proposition 1.13 provides the base case of the induction.

Now consider a bounded complex $\mathcal{E} \in D^b(X)$. Let m be the largest integer such that $h^m(\mathcal{E}) \neq 0$. There is a standard distinguished triangle

$$\mathcal{F} \to \mathcal{E} \to \mathcal{G} \to \mathcal{F}[1].$$

Here $\mathcal{G}$ is the shifted sheaf $h^m(\mathcal{E})[-m]$ and $\mathcal{F} \in D^b(X)$ is the truncated complex $\tau^{\leqslant m-1}\mathcal{E}$ which satisfies

$$\ell(\mathcal{F}) < \ell(\mathcal{E}).$$

By the induction hypothesis, we have

$$c_2(\mathcal{F}) \in S_{d(\mathcal{F})}(X) \quad \text{and} \quad c_2(\mathcal{G}) \in S_{d(\mathcal{G})}(X).$$

Since $\mathcal{F}$ is concentrated in degrees $< m$ and $\mathcal{G}$ in degree m, we have $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$. Applying Proposition 1.4, we find $c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X)$. The proof of Theorem 0.1 is complete. □

Let X and X' be two $K3$ surfaces. Suppose there is a derived equivalence

$$\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$$

with Fourier–Mukai kernel $\mathcal{F} \in D^b(X \times X')$. The induced isomorphism of (ungraded) Chow groups

$$\Phi^{\mathrm{CH}} : \mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CH}^*(X')$$

is given by the correspondence[9]

$$v^{\mathrm{CH}}(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_{X \times X'}} \in \mathrm{CH}^*(X \times X').$$

Recall the following theorem of Huybrechts and Voisin.

Theorem 1.14 ([Huy10, Theorem 2] and [Voi15, Corollary 1.10]). *The isomorphism Φ^{CH} preserves the Beauville–Voisin ring.*

Proof of Corollary 0.2. Since Φ is a derived equivalence, we only need to prove that

$$\Phi^{\mathrm{CH}}(\widetilde{S}_i(X)) \subset \widetilde{S}_i(X').$$

Since Φ^{CH} preserves the Beauville–Voisin ring by Theorem 1.14, it suffices to show that for any effective zero-cycle $z = x_1 + \cdots + x_i$, we have

$$\Phi^{\mathrm{CH}}([z]) \in \widetilde{S}_i(X'). \tag{6}$$

[9] Here the square root is taken with respect to the canonical classes $[o_X] \in \mathrm{CH}_0(X)$ and $[o_{X'}] \in \mathrm{CH}_0(X')$; see [Huy10].

Further, we may assume $x_1, \dots, x_i$ distinct, since multiplicities result in $[z] \in S_{i'}(X)$ for some $i' < i$.

Let $\mathcal{E}$ be the direct sum of skyscraper sheaves

$$\mathbb{C}_{x_1} \oplus \cdots \oplus \mathbb{C}_{x_i}.$$

Then $c_2(\mathcal{E}) = [z]$ and $d(\Phi(\mathcal{E})) = d(\mathcal{E}) = i$. Applying Theorem 0.1, we find

$$c_2(\Phi(\mathcal{E})) \in S_i(X'),$$

which implies (6). □

2. Zero-cycles on moduli spaces of stable objects

In this section, we discuss the Beauville–Voisin conjecture in the case of moduli spaces of stable objects. We prove Proposition 0.4 and Theorem 0.5.

2.1 Independence of modular interpretations

The proof of Proposition 0.4 uses Bayer and Macrì's work [BM14a, BM14b] on the birational transforms of moduli spaces of stable objects.

Let X be a $K3$ surface. Recall that given a primitive Mukai vector[10] $\mathbf{v}$ with $\mathbf{v}^2 > 0$, and a generic stability condition $\sigma \in \mathrm{Stab}^{\dagger}(X)$ with respect to $\mathbf{v}$, there is a moduli space $M_\sigma(\mathbf{v})$ of σ-stable objects in $D^b(X)$.

Bayer and Macrì realized all holomorphic symplectic birational models of $M_\sigma(\mathbf{v})$ as other moduli spaces of stable objects. Their following theorem describes the procedure concretely.

Theorem 2.1 [BM14a, Corollary 1.3]. *With the notation above, let $(X', \sigma', \mathbf{v}')$ be another triple. The moduli spaces $M_\sigma(\mathbf{v})$ and $M_{\sigma'}(\mathbf{v}')$ are birational if and only if there exists a derived (anti-)equivalence*

$$\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$$

which sends $\mathbf{v}$ to $\mathbf{v}'$ and induces an isomorphism

$$\Sigma : U \xrightarrow{\sim} U'$$

between two nonempty open subsets $U \subset M_\sigma(\mathbf{v})$ and $U' \subset M_{\sigma'}(\mathbf{v}')$.[11]

It is well known that the CH_0-group is invariant under birational transforms of nonsingular projective varieties.[12] The statement can be made slightly more precise.

Lemma 2.2. *Let $f : V \dashrightarrow V'$ be a birational map between nonsingular projective varieties, and let*

$$f_* : \mathrm{CH}_0(V) \xrightarrow{\sim} \mathrm{CH}_0(V')$$

be the induced isomorphism of Chow groups. Then for any point $x \in V$, there exists a point $x' \in V'$ such that

$$[x'] = f_*([x]) \in \mathrm{CH}_0(V').$$

[10] In this section we always assume $\mathbf{v}^2 > 0$, so that the moduli space $M_\sigma(\mathbf{v})$ is of dimension >2. This assumption is crucial in Theorem 2.1. See § 3.1 for a discussion of the dimension 2 case.

[11] Here Σ sends an object $\mathcal{E} \in U$ to $\Phi(\mathcal{E}) \in U'$.

[12] In [Rie14], Rieß proved that birational holomorphic symplectic varieties have isomorphic Chow rings.

Proof. Consider a resolution

$$\begin{array}{ccc} & \widetilde{V} & \\ {}^{p}\swarrow & & \searrow^{q} \\ V & \overset{f}{\dashrightarrow} & V' \end{array}$$

with $\widetilde{V}$ nonsingular and projective. Then f_* is realized as q_*p^*. By weak factorization, both p and q can be taken a sequence of blow-ups and blow-downs with nonsingular centers. We are reduced to the case of a blow-up, for which the statement is obvious. □

Let $(X, \sigma, \mathbf{v})$ and $(X', \sigma', \mathbf{v}')$ be such that $M_\sigma(\mathbf{v})$ and $M_{\sigma'}(\mathbf{v}')$ are birational. By Theorem 2.1, a derived (anti-)equivalence

$$\Phi : D^b(X) \xrightarrow{\sim} D^b(X')$$

induces a birational map

$$\Sigma : M_\sigma(\mathbf{v}) \dashrightarrow M_{\sigma'}(\mathbf{v}'),$$

which identifies two nonempty open subsets $U \subset M_\sigma(\mathbf{v})$ and $U' \subset M_{\sigma'}(\mathbf{v}')$. By further composing with $R\mathcal{H}om(-, \mathcal{O}_X)$, we may assume that Φ is a derived equivalence.

Let $\mathcal{E}$ be an object in $M_\sigma(\mathbf{v})$. By Lemma 2.2, there exists an object $\mathcal{F}$ in $M_{\sigma'}(\mathbf{v}')$ such that

$$[\mathcal{F}] = \Sigma_*([\mathcal{E}]) \in \mathrm{CH}_0(M_{\sigma'}(\mathbf{v}')). \tag{7}$$

LEMMA 2.3. *With the notation above, for any pair of objects $\mathcal{E} \in M_\sigma(\mathbf{v})$ and $\mathcal{F} \in M_{\sigma'}(\mathbf{v}')$ satisfying (7), we have*[13]

$$v^{\mathrm{CH}}(\mathcal{F}) = \Phi^{\mathrm{CH}}(v^{\mathrm{CH}}(\mathcal{E})) \in \mathrm{CH}^*(X').$$

Proof. Since any class in $\mathrm{CH}_0(M_\sigma(\mathbf{v}))$ is supported on U, we may write

$$[\mathcal{E}] = \sum_j a_j [\mathcal{E}_j] \in \mathrm{CH}_0(M_\sigma(\mathbf{v}))$$

for some $\mathcal{E}_j \in U$. Using the (quasi-)universal family on $M_\sigma(\mathbf{v}) \times X$, we have

$$v^{\mathrm{CH}}(\mathcal{E}) = \sum_j a_j v^{\mathrm{CH}}(\mathcal{E}_j) \in \mathrm{CH}^*(X). \tag{8}$$

On the other hand, it is clear from the definition that

$$[\mathcal{F}] = \sum_j a_j [\Sigma(\mathcal{E}_j)] = \sum_j a_j [\Phi(\mathcal{E}_j)] \in \mathrm{CH}_0(M_{\sigma'}(\mathbf{v}')).$$

Again using the (quasi-)universal family on $M_{\sigma'}(\mathbf{v}') \times X'$, we have

$$v^{\mathrm{CH}}(\mathcal{F}) = \sum_j a_j v^{\mathrm{CH}}(\Phi(\mathcal{E}_j)) \in \mathrm{CH}^*(X'). \tag{9}$$

[13] Recall that $v^{\mathrm{CH}}(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_X} \in \mathrm{CH}^*(X)$ for $\mathcal{E} \in D^b(X)$.

Combining (8) and (9) and using the commutative diagram

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi} & D^b(X') \\ {\scriptstyle v^{\mathrm{CH}}}\downarrow & & \downarrow{\scriptstyle v^{\mathrm{CH}}} \\ \mathrm{CH}^*(X) & \xrightarrow{\Phi^{\mathrm{CH}}} & \mathrm{CH}^*(X') \end{array}$$

we find

$$v^{\mathrm{CH}}(\mathcal{F}) = \Phi^{\mathrm{CH}}\bigg(\sum_j a_j v^{\mathrm{CH}}(\mathcal{E}_j)\bigg) = \Phi^{\mathrm{CH}}(v^{\mathrm{CH}}(\mathcal{E})) \in \mathrm{CH}^*(X'). \qquad \square$$

Proof of Proposition 0.4. Let $\mathcal{E}$ be an object in $M_\sigma(\mathbf{v})$ such that

$$c_2(\mathcal{E}) \in S_i(X).$$

Equivalently, we have

$$v^{\mathrm{CH}}(\mathcal{E}) \in \widetilde{S}_i(X).$$

By Lemma 2.2, there exist an object $\mathcal{F}$ in $M_{\sigma'}(\mathbf{v}')$ satisfying

$$[\mathcal{F}] = \Sigma_*([\mathcal{E}]) \in \mathrm{CH}_0(M_{\sigma'}(\mathbf{v}')).$$

Applying Lemma 2.3, we find

$$v^{\mathrm{CH}}(\mathcal{F}) = \Phi^{\mathrm{CH}}(v^{\mathrm{CH}}(\mathcal{E})) \in \mathrm{CH}^*(X').$$

Since Φ^{CH} preserves the filtration $\widetilde{S}_\bullet$ by Corollary 0.2, we conclude that

$$v^{\mathrm{CH}}(\mathcal{F}) \in \widetilde{S}_i(X'),$$

or equivalently,

$$c_2(\mathcal{F}) \in S_i(X').$$

The proposition then follows from the definition of $S_\bullet\mathrm{CH}_0$. $\square$

2.2 The Beauville–Voisin filtration

As stated in Theorem 0.5, we compare two proposed filtrations on the CH_0-group of a moduli space of stable objects.

Let X be a $K3$ surface, let $M = M_\sigma(\mathbf{v})$ be a moduli space of stable objects in $D^b(X)$ of dimension $2d = 2d(\mathbf{v})$, and let $X^{[d]}$ be the Hilbert scheme of d points on X. Consider the incidence variety

$$R = \{(\mathcal{E}, \xi) \in M \times X^{[d]} : c_2(\mathcal{E}) = [\mathrm{Supp}(\xi)] + c[o_X] \in \mathrm{CH}_0(X)\},$$

where $\mathrm{Supp}(\xi)$ is the support of ξ and $c \in \mathbb{Z}$ is a constant determined by the Mukai vector $\mathbf{v}$. This incidence variety has already appeared in [O'Gr13, Voi15].

A standard argument using Hilbert schemes shows that R is a countable union of Zariski-closed subsets of $M \times X^{[d]}$. Let

$$p_M : R \to M \quad \text{and} \quad p_{X^{[d]}} : R \to X^{[d]}$$

denote the two projections. By (the now proven) Conjecture 0.3 for $X^{[d]}$ or an explicit calculation, all points on the same fiber of p_M have the same class in $\mathrm{CH}_0(X^{[d]})$. Similarly, by Conjecture 0.3 for M, all points on the same fiber of $p_{X^{[d]}}$ have the same class in $\mathrm{CH}_0(M)$.

An important consequence of Theorem 0.1 is that p_M is dominant. Then, by the argument in [O'Gr13, Proposition 1.3] (see also [Voi15, Corollary 3.4]), we also know that $p_{X^{[d]}}$ is dominant. More precisely, there exists a component $R_0 \subset R$ which dominates both M and $X^{[d]}$. Note that M and $X^{[d]}$ have the same dimension.

Further, up to taking hyperplane sections,[14] we may assume that R_0 is generically finite over both M and $X^{[d]}$. To summarize, we have a diagram

$$\begin{array}{ccc} & R_0 & \\ {\scriptstyle p_M}\swarrow & & \searrow{\scriptstyle p_{X^{[d]}}} \\ U \subset M & & X^{[d]} \supset V \end{array} \tag{10}$$

where $U \subset M$ and $V \subset X^{[d]}$ are nonempty open subsets over which p_M and $p_{X^{[d]}}$ are finite.

We recall two density results on X and $X^{[d]}$.

LEMMA 2.4 ([Voi15, Lemma 2.3]; see also [Huy14, Lemma 6.3]). *The union of constant cycle curves in X is Zariski-dense.*

LEMMA 2.5 ([Mac04, Theorem 1.2]; see also [Voi15, Lemma 3.5]). *For any point $\xi \in X^{[d]}$, its orbit under rational equivalence $O_\xi \subset X^{[d]}$ is Zariski-dense.*

Proof of Theorem 0.5. Given $d-i$ constant cycle curves in X labeled as $C_{i+1}, C_{i+2}, \ldots, C_d$, we consider the rational map

$$\phi : X^{[i]} \times C_{i+1} \times C_{i+2} \times \cdots \times C_d \dashrightarrow X^{[d]}$$

which (generically) sums up the points on the factors. By Lemma 2.4, the union of $\mathrm{Im}(\phi)$ for all choices of constant cycles curves is Zariski-dense in $X^{[d]}$. In particular, there exists such ϕ whose image meets $V \subset X^{[d]}$.

Let $\phi' : Z \dashrightarrow R_0$ denote the pull-back of ϕ via $p_{X^{[d]}}$.[15] We have the following diagram.

$$\begin{array}{ccc} Z & \overset{\phi'}{\dashrightarrow} & R_0 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p_{X^{[d]}}} \\ X^{[i]} \times C_{i+1} \times C_{i+2} \times \cdots \times C_d & \overset{\phi}{\dashrightarrow} & X^{[d]} \end{array}$$

Again by Lemma 2.4, we may assume that $\phi'(Z)$ meets $p_M^{-1}(U) \subset R_0$.

Let $q : Z \dashrightarrow X^{[i]}$ denote the composition of p' and the projection to $X^{[i]}$. For a general point $\xi \in X^{[i]}$, consider the fiber $Z_\xi \subset Z$. By construction, the image

$$p_M(\phi'(Z_\xi)) \subset M$$

consists of objects in M with constant second Chern class. By (the now proven) Conjecture 0.3, this gives a constant cycle subvariety in M. The dimension of $p_M(\phi'(Z_\xi))$ is $d-i$ since p_M and $p_{X^{[d]}}$ are finite over U and V.

[14] In fact, this step is unnecessary: by [Voi16, Theorem 1.3], the orbit of a very general point on M under rational equivalence is discrete.

[15] If Z is not irreducible, we may take an irreducible component of Z.

We have shown that the image $p_M(\phi'(Z))$ is generically covered by constant cycle subvarieties of dimension $d-i$. We conclude by [Voi16, Theorem 0.7] that $p_M(\phi'(Z))$ is algebraically coisotropic of codimension $d-i$ with constant cycle fibers. This proves part (i) of the theorem.

For part (ii), let $\mathcal{E}$ be an object in M such that $c_2(\mathcal{E}) \in S_i(X)$. By definition, there exists a point $\xi_0 \in X^{[i]}$ satisfying

$$c_2(\mathcal{E}) = [\mathrm{Supp}(\xi_0)] + (d-i)[o_X] + c[o_X] \in \mathrm{CH}_0(X).$$

Applying Lemma 2.5 to $\xi_0 \in X^{[i]}$, we may further assume that $p_M(\phi'(Z_{\xi_0}))$ is well defined and is of dimension $d-i$.

By construction, the subvariety $p_M(\phi'(Z_{\xi_0}))$ consists of objects in M whose second Chern class equals $c_2(\mathcal{E})$. By Conjecture 0.3, it is a subvariety of dimension $d-i$ in the orbit $O_{\mathcal{E}} \subset M$. We conclude that $[\mathcal{E}] \in S_i^V \mathrm{CH}_0(M)$, which proves part (ii) of the theorem. □

Remark 2.6. Our proof relies on the Zariski density of subvarieties of maximal dimension in an orbit of $X^{[d]}$. If one could prove such density for M, then an argument using [Voi15, Theorem 2.1] would yield the other inclusion

$$S_i^V \mathrm{CH}_0(M) \subset S_i \mathrm{CH}_0(M).$$

3. Further questions

3.1 The dimension 2 case

In § 2, we focused on the Beauville–Voisin filtration for moduli spaces of dimension $2d(\mathbf{v}) = \mathbf{v}^2 + 2 > 2$. We discuss here the case $\mathbf{v}^2 = 0$.

When $\mathbf{v} \in H^*(X, \mathbb{Z})$ is a primitive Mukai vector satisfying $\mathbf{v}^2 = 0$, and σ is a generic stability condition, the corresponding moduli space $M = M_\sigma(\mathbf{v})$ is a $K3$ surface. Although the Beauville–Voisin filtration on $\mathrm{CH}_0(M)$ is clear by [BV04], its compatibility with the filtration on $\mathbf{D} = D^b(X)$ is not obvious.

If M is a fine moduli space, then the universal family induces a derived equivalence

$$D^b(M) \xrightarrow{\sim} D^b(X).$$

Theorem 1.14 shows that the corresponding isomorphism of Chow groups

$$\mathrm{CH}^*(M) \xrightarrow{\sim} \mathrm{CH}^*(X)$$

preserves the Beauville–Voisin ring. In particular, the canonical class $[o_M] \in \mathrm{CH}_0(M)$ is represented by any object $\mathcal{E} \in M$ with $c_2(\mathcal{E}) \in \mathbb{Z}\cdot[o_X]$. The Beauville–Voisin filtration $S_\bullet \mathrm{CH}_0(M)$ indeed comes from the restriction of the filtration $S_\bullet(\mathbf{D})$ on the derived category.

If M is not a fine moduli space, then $D^b(X)$ is equivalent to a derived category of twisted sheaves on M,

$$D^b(M, \alpha) \xrightarrow{\sim} D^b(X).$$

Recently, Huybrechts showed in [Huy19, Corollary 2.2] that the universal twisted family induces an isomorphism of Chow groups

$$\mathrm{CH}^*(M)_\mathbb{Q} \xrightarrow{\sim} \mathrm{CH}^*(X)_\mathbb{Q}. \tag{11}$$

In this case, we also expect (11) to preserve the Beauville–Voisin ring. More generally, we ask the following question.[16]

[16] By Lemma 1.1(ii), each $\widetilde{S}_i(X)$ has the structure of a cone. Hence it makes sense to extend $\widetilde{S}_\bullet(X)$ to rational coefficients.

Question 3.1. Does the isomorphism (11) preserve O'Grady's filtration $\widetilde{S}_\bullet$?

One may also ask the same question for arbitrary pairs of twisted $K3$ surfaces which are derived equivalent.

3.2 More on the Beauville–Voisin filtration

Let M be a moduli space of stable objects in $D^b(X)$ as in § 2. Recall the filtration $S_\bullet\mathrm{CH}_0(M)$, where $S_i\mathrm{CH}_0(M)$ is the subgroup spanned by the classes of $\mathcal{E} \in M$ satisfying $c_2(\mathcal{E}) \in S_i(X)$. The following question asks for more precision.

Question 3.2. For an object $\mathcal{E} \in M$, is it true that

$$[\mathcal{E}] \in S_i\mathrm{CH}_0(M)$$

if and only if

$$c_2(\mathcal{E}) \in S_i(X)?$$

By (the proof of) Proposition 0.4, the answer to Question 3.2 is independent of birational models or modular interpretations.

Question 3.2 for the Hilbert schemes of points on X alone has an interesting interpretation. Let $\gamma \in \mathrm{CH}_0(X)$ be a zero-cycle class of degree 0. We may assume

$$\gamma = [\mathrm{Supp}(\xi)] - d[o_X]$$

for some $\xi \in X^{[d]}$ with d sufficiently large.

By an explicit calculation via the motivic decomposition of $X^{[d]}$, we have

$$[\xi] \in S_i\mathrm{CH}_0(X^{[d]})$$

if and only if

$$\gamma^{\times(i+1)} = 0 \in \mathrm{CH}_0(X^{i+1}).$$

A positive answer to Question 3.2 for $X^{[d]}$ is then equivalent to the statement that

$$\gamma \in S_i(X)$$

if and only if

$$\gamma^{\times(i+1)} = 0 \in \mathrm{CH}_0(X^{i+1}).$$

The latter is a new characterization of O'Grady's filtration $S_\bullet(X)$ proposed by Voisin.

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