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# Rational curves in holomorphic symplectic varieties and Gromov–Witten invariants

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ABSTRACT

We use Gromov–Witten theory to study rational curves in holomorphic symplectic varieties. We present a numerical criterion for the existence of uniruled divisors swept out by rational curves in the primitive curve class of a very general holomorphic symplectic variety of $K3^{[n]}$ type. We also classify all rational curves in the primitive curve class of the Fano variety of lines in a very general cubic 4-fold, and prove the irreducibility of the corresponding moduli space. Our proofs rely on Gromov–Witten calculations by the first author, and in the Fano case on a geometric construction of Voisin. In the Fano case a second proof via classical geometry is sketched.


**Contents**



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## 0. Introduction

### *0.1. Overview*

Rational curves in $K3$ surfaces have been investigated for decades from various angles. In contrast, not much is known about the geometry of rational curves in the higher-dimensional analogs of $K3$ surfaces—holomorphic symplectic varieties.[1] In this paper, we use *Gromov–Witten theory* (intersection theory of the moduli space of stable maps) together with classical methods to study these rational curves.

### *0.2. Rational curves*

Let $(X, H)$ be a very general polarized holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be the primitive curve class. The moduli space $\overline{M}_{0,m}(X, \beta)$ of genus 0 and $m$-pointed stable maps to $X$ in class $\beta$ is pure of expected dimension $2n - 2 + m$; see Proposition 1.1. Consider the decomposition

$$\overline{M}_{0,1}(X, \beta) = M^0 \cup M^1 \cup \cdots \cup M^{n-1} \tag{1}$$

such that the general fibers of the restricted evaluation map

$$\mathrm{ev} : M^i \to \mathrm{ev}(M^i) \subset X$$

are of dimension $i$. The image of $M^0$ under ev is precisely the union of all uniruled divisors swept out by rational curves in class $\beta$. More generally, the image $\mathrm{ev}(M^i)$ is the codimension $i+1$ locus of points on $X$ through which passes an $i$-dimensional family of rational curves in class $\beta$.

In [20, Conjecture 4.3], Mongardi and Pacienza conjectured that for all $i$

$$M^i \neq \emptyset,$$

which would imply the existence of algebraically coisotropic subvarieties in $X$ in the sense of Voisin [26].

[1] A nonsingular projective variety $X$ is holomorphic symplectic if it is simply connected and $H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate holomorphic 2-form.

In Theorems 0.1 and 0.2 below, we provide counterexamples to this conjecture which illustrate "pathologies" of rational curves in higher-dimensional holomorphic symplectic varieties. Two typical examples are as follows.

(i) There exist a very general pair $(X, H)$ of $K3^{[8]}$ type with $M^0 = \emptyset$. In other words, on $(X, H)$ there exists no uniruled divisor swept out by rational curves in the primitive class $\beta$.
(ii) For the Fano variety of lines in a very general cubic 4-fold, we have $M^1 = \emptyset$.

Here a variety is of $K3^{[n]}$ type if it is deformation equivalent to the Hilbert scheme of $n$ points on a $K3$ surface.

*0.3. Uniruled divisors*

On a holomorphic symplectic variety $X$, let

$$(-,-) : H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Q}$$

denote the unique $\mathbb{Q}$-valued extension of the Beauville–Bogomolov form on $H^2(X,\mathbb{Z})$. If $X$ is of $K3^{[n]}$ type and $n \geq 2$, there is an isomorphism of abelian groups

$$r : H_2(X,\mathbb{Z})/H^2(X,\mathbb{Z}) \to \mathbb{Z}/(2n-2)\mathbb{Z},$$

unique up to multiplication by $\pm 1$, such that $r(\alpha) = 1$ for some $\alpha \in H_2(X,\mathbb{Z})$ with $(\alpha,\alpha) = \frac{1}{2-2n}$. Given a class $\beta \in H_2(X,\mathbb{Z})$, we define its residue set by

$$\pm[\beta] = \{\pm r(\beta)\} \subset \mathbb{Z}/(2n-2)\mathbb{Z}.$$

In case $n = 1$, we set $\pm[\beta] = 0$.

The following theorem provides a complete numerical criterion for the existence of uniruled divisors swept out by rational curves in the primitive class of a very general variety of $K3^{[n]}$ type.

**Theorem 0.1.** *Let $X$ be a holomorphic symplectic variety of $K3^{[n]}$ type, and let $\beta \in H_2(X,\mathbb{Z})$ be a primitive curve class. If*

$$(\beta,\beta) = -2 + \sum_{i=1}^{n-1} 2d_i - \frac{1}{2n-2}\left(\sum_{i=1}^{n-1} r_i\right)^2,$$

$$\pm[\beta] = \pm\left[\sum_{i=1}^{n-1} r_i\right]$$

*for some* $d_i, r_i \in \mathbb{Z}$ *satisfying* $2d_i - \frac{r_i^2}{2} \geq 0$*, then there exists a uniruled divisor on* $X$ *swept out by rational curves in class* $\beta$*. The converse holds if* $\beta$ *is irreducible.*

For a very general pair $(X, \beta)$ with $X$ of $K3^{[n]}$ type and $\beta$ the primitive curve class, Theorem 0.1 implies that

(i) $M^0 \neq \emptyset$ when $n \leq 7$, and
(ii) for every $n \geq 8$, there exists $(X, \beta)$ such that $M^0 = \emptyset$.

The first instance of case (ii) is given by a very general pair $(X, \beta)$ of $K3^{[8]}$ type with $(\beta, \beta) = \frac{3}{14}$ and $\pm[\beta] = \pm[5]$.[2]

*0.4. Fano varieties of lines*

Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic 4-fold. By Beauville and Donagi [4], the Fano variety of lines in $Y$

$$F = \{l \in \mathrm{Gr}(2,6) : l \subset Y\}$$

is a holomorphic symplectic 4-fold. These varieties form a 20-dimensional family of polarized holomorphic symplectic varieties of $K3^{[2]}$ type.

In [25], Voisin constructed a rational self-map

$$\varphi : F \dashrightarrow F \tag{2}$$

sending a general line $l$ to its residual line with respect to the unique plane $\mathbb{P}^2 \subset \mathbb{P}^5$ tangent to $Y$ along $l$. When $Y$ is very general, the exceptional divisor associated to the resolution of $\varphi$

$$\begin{array}{ccc} D = \mathbb{P}(\mathcal{N}_{S/F}) & \xrightarrow{\phi} & F \\ \downarrow{\scriptstyle p} & & \\ S & & \end{array} \tag{3}$$

is a $\mathbb{P}^1$-bundle over a nonsingular surface $S \subset F$; see Amerik [1]. The image of each fiber

$$\phi(p^{-1}(s)) \subset F, \quad s \in S$$

is a rational curve lying in the primitive curve class in $H_2(F, \mathbb{Z})$.

The following theorem shows that every rational curve in the primitive curve class is of this form in a unique way.

[2] Such a pair $(X, \beta)$ can be obtained by deforming $(\mathrm{Hilb}^8(\mathsf{S}), \beta')$, where $\mathsf{S}$ is a $K3$ surface of genus 2 with polarization $\mathsf{H}$ and $\beta' = \mathsf{H} + 5\mathsf{A}$ with $\mathsf{A}$ the exceptional curve class; see Section 2 for the notation.

**Theorem 0.2.** *Let $F$ be the Fano variety of lines in a very general cubic 4-fold. Then for every rational curve $C \subset F$ in the primitive curve class, there is a unique $s \in S$ such that $C = \phi(p^{-1}(s))$.*

We also show that $S$ is connected and calculate its first Chern class; see Corollary 3.3. In particular, the moduli space of rational curves in the primitive curve class of a very general $F$ is irreducible. This implies $M^1 = \emptyset$ in the decomposition (1) and the following.

**Corollary 0.3.** *For a very general $F$, there is a unique irreducible uniruled divisor swept out by rational curves in the primitive curve class.*

The moduli space of rational curves in the primitive curve class of a very general $K3$ surface always has more than one irreducible component. Corollary 0.3 indicates a difference between rational curves in $K3$ surfaces and in higher-dimensional holomorphic symplectic varieties.

### *0.5. Idea of proofs*

We briefly explain how Gromov–Witten theory [10] controls rational curves in the primitive class $\beta$ of a very general polarized holomorphic symplectic variety $(X, H)$ of $K3^{[n]}$ type.

Since the evaluation map ev is generically finite on the component $M^0$ but contracts positive dimensional fibers on all other components in the decomposition (1), the (non)emptiness of $M^0$ is detected by the pushforward

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)] \in H^2(X,\mathbb{Q}). \tag{4}$$

For the Fano variety of lines $X = F$, a key observation is that the emptiness of $M^1$ can be further detected by the Gromov–Witten correspondence

$$\mathrm{ev}_{12*}[\overline{M}_{0,2}(X,\beta)] \in H^{4n}(X \times X,\mathbb{Q}). \tag{5}$$

The class (5) has contributions from all of the components in (1), and contains strictly more information than the 1-pointed class (4).

Since $\overline{M}_{0,m}(X,\beta)$ is pure of the expected dimension, its fundamental class coincides with the (reduced) virtual fundamental class [5,17],

$$[\overline{M}_{0,m}(X,\beta)] = [\overline{M}_{0,m}(X,\beta)]^{\mathrm{vir}}.$$

Hence the classes (4) and (5) are determined by the Gromov–Witten invariants of $X$. By deformation invariance, the Gromov–Witten invariants can be calculated on a special model given by the Hilbert scheme of points of an elliptic $K3$ surface; see [22] and Section 2.

Our proofs of Theorems 0.1 and 0.2 are intersection-theoretic. In Appendix A, we also sketch an alternative proof of Theorem 0.2 using a series of classification results in classical projective geometry.[3]

### *0.6. Conventions*

We work over the complex numbers. A statement holds for a *very general* polarized projective variety $(X, H)$ if it holds away from a countable union of proper Zariski-closed subsets in the corresponding component of the moduli space.

## 1. Moduli spaces of stable maps

We discuss properties of the moduli spaces of stable maps to holomorphic symplectic varieties, and introduce tools from Gromov–Witten theory.

### *1.1. Dimensions*

Let $X$ be a holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be an *irreducible* curve class. We show that the moduli space $\overline{M}_{0,1}(X, \beta)$ of genus 0 pointed stable maps to $X$ in class $\beta$ is pure of the expected dimension.

Let $M$ be an irreducible component of $\overline{M}_{0,1}(X, \beta)$. We know *a priori*

$$\dim M \geq \int_\beta c_1(X) + \dim X - 1 = 2n - 1.$$

Consider the restriction of the evaluation map to $M$,

$$\mathrm{ev} : M \to Z = \mathrm{ev}(M) \subset X. \tag{6}$$

**Proposition 1.1.** *If a general fiber of (6) is of dimension $r - 1$, then*

(i) $\dim Z = 2n - r$, *so that* $\dim M = 2n - 1$*;*
(ii) $r \leq n$*;*
(iii) *a general fiber of the MRC fibration*[4] $Z \dashrightarrow B$ *is of dimension* $r$*.*

**Proof.** Since the curve class $\beta$ is irreducible, the family of rational curves $M \to T \subset \overline{M}_{0,0}(X, \beta)$ viewed as in $X$ is unsplit in the sense of [15, IV, Definition 2.1]. Given a

[3] The proof in Appendix A was found only after a first version of this article appeared online. While Theorem 0.2 can be proven classically, the quantitative information obtained from Gromov–Witten theory was essential for us to find the statement.

[4] We refer to [11] for the definition and properties of the maximal rationally connected (MRC) fibration.

general point $x \in Z$, let $T_x \subset T$ be the Zariski-closed subset parametrizing maps passing through $x$. Consider the universal family $\mathcal{C}_x \to T_x$ and the restricted evaluation map

$$\mathrm{ev} : \mathcal{C}_x \to V_x = \mathrm{ev}(\mathcal{C}_x) \subset Z.$$

By [15, IV, Proposition 2.5], we have

$$\dim T = \dim Z + \dim V_x - 2.$$

Hence $\dim V_x = \dim M - \dim Z + 1 = r$. In other words, rational curves through a general point of $Z$ cover a Zariski-closed subset of dimension $r$.

A general fiber of the MRC fibration $Z \dashrightarrow B$ is thus of dimension $\geq r$. By an argument of Mumford (see [26, Lemma 1.1]), this implies $\dim Z \leq 2n - r$ and $r \leq n$. On the other hand, since $\dim M \geq 2n - 1$, we have

$$\dim Z = \dim M - (r - 1) \geq 2n - r.$$

Hence there is equality $\dim Z = 2n - r$, and the dimension of a general fiber of $Z \dashrightarrow B$ is exactly $r$. $\square$

Proposition 1.1 shows that $\overline{M}_{0,1}(X, \beta)$ is pure of the expected dimension $2n - 1$ and justifies the decomposition (1). Similar arguments have also appeared in [2, Theorem 4.4] and [3, Proposition 4.10].

### *1.2. Gromov–Witten theory*

Let $X$ be a holomorphic symplectic variety of dimension $2n$, and let $\beta \in H_2(X, \mathbb{Z})$ be an arbitrary curve class. By Li–Tian [17] and Behrend–Fantechi [5], the moduli space of stable maps $\overline{M}_{0,m}(X, \beta)$ carries a (reduced[5]) virtual fundamental class

$$[\overline{M}_{0,m}(X, \beta)]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(\overline{M}_{0,m}(X, \beta), \mathbb{Q}).$$

It has the following basic properties.

(a) *Virtual dimension.* The virtual fundamental class is of dimension

$$\mathrm{vdim} = 2n - 2 + m. \tag{7}$$

(b) *Expected dimension.* If $\overline{M}_{0,m}(X, \beta)$ is pure of the expected dimension (7), then the virtual and the ordinary fundamental classes agree:

[5] Since $X$ is holomorphic symplectic, the (standard) virtual fundamental class on the moduli space vanishes. The theory is nontrivial only after reduction; see [19, Section 2.2] and [22, Section 0.2]. The virtual fundamental class is always assumed to be reduced in this paper.

$$[\overline{M}_{0,m}(X,\beta)]^{\mathrm{vir}} = [\overline{M}_{0,m}(X,\beta)].$$

(c) *Deformation invariance.* Let $\pi : \mathcal{X} \to B$ be a family of holomorphic symplectic varieties, and let $\beta \in H^0(B, R\pi_*^{4n-2}\mathbb{Z})$ be a class which restricts to a curve class in $H_2(X_b, \mathbb{Z})$ on each fiber.[6] Then there exists a class on the moduli space of relative stable maps

$$[\overline{M}_{0,m}(\mathcal{X}/B,\beta)]^{\mathrm{vir}} \in H_{2(\mathrm{vdim}+\dim B)}(\overline{M}_{0,m}(\mathcal{X}/B,\beta),\mathbb{Q})$$

such that for every fiber $X_b \hookrightarrow \mathcal{X}$, the inclusion $\iota_b : b \hookrightarrow B$ induces

$$\iota_b^![\overline{M}_{0,m}(\mathcal{X}/B,\beta)]^{\mathrm{vir}} = [\overline{M}_{0,m}(X_b,\beta)]^{\mathrm{vir}}.$$

Here $\iota_b^!$ is the refined Gysin pullback. In particular, intersection numbers of $[\overline{M}_{0,m}(X,\beta)]^{\mathrm{vir}}$ against cohomology classes pulled back from $X$ via the evaluation maps

$$\mathrm{ev}_i : \overline{M}_{0,m}(X,\beta) \to X, \quad (f, x_1, \ldots, x_m) \mapsto f(x_i)$$

are invariant under deformations of $(X,\beta)$ which keep $\beta$ of Hodge type.

### 1.3. Gromov–Witten correspondence

Let $X, \beta$ be as in Section 1.1. The evaluation maps from the 2-pointed moduli space

$$\begin{array}{ccc} & \overline{M}_{0,2}(X,\beta) & \\ {}^{\mathrm{ev}_1}\swarrow & & \searrow^{\mathrm{ev}_2} \\ X & & X \end{array}$$

induce an action on cohomology:

$$\mathsf{GW}_\beta : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q}), \quad \gamma \mapsto \mathrm{ev}_{2*}(\mathrm{ev}_1^*\gamma \cap [\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}). \tag{8}$$

We call (8) the *Gromov–Witten correspondence.*

We introduce a factorization of (8) as follows. Consider the diagram

$$\begin{array}{ccc} \overline{M}_{0,1}(X,\beta) & \xrightarrow{\mathrm{ev}} & X \\ \downarrow p & & \\ \overline{M}_{0,0}(X,\beta) & & \end{array} \tag{9}$$

[6] We have suppressed an application of Poincaré duality here. Same with the definition of $\mathsf{GW}_\beta$ and $\Phi_2$ in Section 1.3 below.

with $p$ the forgetful map (which is flat). We define morphisms

$$\Phi_1 : H^i(X, \mathbb{Q}) \to H_{4n-2-i}(\overline{M}_{0,0}(X, \beta), \mathbb{Q}), \quad \gamma \mapsto p_*(\mathrm{ev}^*\gamma \cap [\overline{M}_{0,1}(X, \beta)]^{\mathrm{vir}}),$$
$$\Phi_2 = \mathrm{ev}_* p^* : H_{4n-2-i}(\overline{M}_{0,0}(X, \beta), \mathbb{Q}) \to H^i(X, \mathbb{Q}).$$

Since $\beta$ is irreducible, there is a Cartesian diagram of forgetful maps

$$\begin{array}{ccccc} & & \overline{M}_{0,2}(X,\beta) & & \\ & \swarrow & & \searrow & \\ \overline{M}_{0,1}(X,\beta) & & & & \overline{M}_{0,1}(X,\beta) \\ & \searrow & & \swarrow & \\ & & \overline{M}_{0,0}(X,\beta). & & \end{array}$$

Hence the Gromov–Witten correspondence (8) factors as

$$\mathsf{GW}_\beta = \Phi_2 \circ \Phi_1 : H^i(X, \mathbb{Q}) \to H^i(X, \mathbb{Q}). \tag{10}$$

*1.4. Hodge classes*

Now let $(X, H)$ be a very general polarized holomorphic symplectic 4-fold of $K3^{[2]}$ type. It is shown in [23, Section 3] that the Hodge classes in $H^4(X, \mathbb{Q})$ are spanned by $H^2$ and $c_2(X)$.

A surface $\Sigma \subset X$ is *Lagrangian* if the holomorphic 2-form $\sigma$ on $X$ restricts to zero on $\Sigma$. The class of any Lagrangian surface is a positive multiple of

$$v_X = 5H^2 - \frac{1}{6}(H, H)c_2(X) \in H^4(X, \mathbb{Q}), \tag{11}$$

where $(-, -)$ is the Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$.[7]

**Proposition 1.2.** *If $(X, H)$ is very general of $K3^{[2]}$ type and $\beta \in H_2(X, \mathbb{Z})$ is the primitive curve class, then for any Hodge class $\alpha \in H^4(X, \mathbb{Q})$, the class*

$$\mathsf{GW}_\beta(\alpha) \in H^4(X, \mathbb{Q})$$

*is proportional to $v_X$.*

[7] This follows from a direct calculation of the constraint $[\Sigma] \cdot \sigma = 0 \in H^6(X, \mathbb{Q})$. The class $v_X$ was first calculated by Markman.

**Proof.** We use the factorization (10). For any Hodge class $\alpha \in H^4(X, \mathbb{Q})$, the class

$$\Phi_1(\alpha) \in H_2(\overline{M}_{0,0}(X, \beta), \mathbb{Q})$$

is represented by curves. Hence $\mathsf{GW}_\beta(\alpha)$ can be expressed as a linear combination of classes of the form

$$[\mathrm{ev}(p^{-1}(C))] \in H^4(X, \mathbb{Q})$$

with $C \subset \overline{M}_{0,0}(X, \beta)$ a curve.

Moreover, we have

$$\mathrm{ev}^*\sigma = p^*\sigma'$$

for some holomorphic 2-form $\sigma'$ on $\overline{M}_{0,0}(X, \beta)$. Hence any surface of the form $\mathrm{ev}(p^{-1}(C))$ is Lagrangian, and the proposition follows. □

Proposition 1.2 implies that the class $v_X$ in (11) is an eigenvector of the Gromov–Witten correspondence

$$\mathsf{GW}_\beta : H^4(X, \mathbb{Q}) \to H^4(X, \mathbb{Q}).$$

An explicit formula for $\mathsf{GW}_\beta$ was calculated in [22] and is recalled in Section 2.5.

## 2. Gromov–Witten calculations

In this Section, we prove Theorem 0.1 using formulas for the 1-pointed Gromov–Witten class in the $K3^{[n]}$ case based on [22]. We also present formulas for the Gromov–Witten correspondence in the $K3^{[2]}$ case, which will be used in Section 3.

### *2.1. Quasi-Jacobi forms*

Jacobi forms are holomorphic functions in variables[8] $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ with modular properties; see [9] for an introduction. Here we will consider Jacobi forms as formal power series in the variables

$$q = e^{2\pi i\tau}, \quad y = -e^{2\pi iz}$$

expanded in the region $|q| < |y| < 1$.

[8] Let $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ denote the upper half-plane.

Recall the Jacobi theta function

$$\Theta(q,y) = (y^{1/2} + y^{-1/2}) \prod_{m\geq 1} \frac{(1+yq^m)(1+y^{-1}q^m)}{(1-q^m)^2}$$

and the Weierstraß elliptic function

$$\wp(q,y) = \frac{1}{12} - \frac{y}{(1+y)^2} + \sum_{m\geq 1}\sum_{d|m} d((-y)^d - 2 + (-y)^{-d})q^m.$$

Define Jacobi forms $\phi_{k,1}$ of *weight* $k$ and *index* 1 by

$$\phi_{-2,1}(q,y) = \Theta(q,y)^2, \quad \phi_{0,1}(q,y) = 12\Theta(q,y)^2\wp(q,y).$$

We also require the weight $k$ and index 0 Eisenstein series

$$E_k(q) = 1 - \frac{2k}{B_k}\sum_{m\geq 1}\sum_{d|m} d^{k-1}q^m, \quad k = 2,4,6,$$

where the $B_k$ are the Bernoulli numbers, and the modular discriminant

$$\Delta(q) = \frac{E_4^3 - E_6^2}{1728} = q\prod_{m\geq 1}(1-q^m)^{24}.$$

We define the ring of quasi-Jacobi forms of even weight as the free polynomial algebra

$$\mathcal{J} = \mathbb{Q}[E_2, E_4, E_6, \phi_{-2,1}, \phi_{0,1}].$$

The weight/index assignments to the generators induce a bigrading

$$\mathcal{J} = \bigoplus_{k\in\mathbb{Z}}\bigoplus_{m\geq 0} \mathcal{J}_{k,m}$$

by weight $k$ and index $m$.

**Lemma 2.1** *([9, Theorem 2.2]). Let $\phi \in \mathcal{J}_{*,m}$ be a quasi-Jacobi form of index $m \geq 1$. For all $d,r\in\mathbb{Z}$, the coefficient $[\phi]_{q^d y^r}$ only depends on $2d - \frac{r^2}{2m}$ and the set $\{\pm[r]\}$, where $[r]\in\mathbb{Z}/2m\mathbb{Z}$ is the residue of $r$.*

By Lemma 2.1, we may denote the $q^d y^r$-coefficient of $\phi$ by

$$\phi\left[2d - \frac{r^2}{2m}, \pm[r]\right] = [\phi]_{q^d y^r}. \tag{12}$$

If $\phi$ is of index 0, we set $\phi[2d, 0] = [\phi]_{q^d}$. Lemma 2.1 remains valid if we replace $\phi$ by $f(q)\phi$ for any Laurent series $f(q)$, and we keep the notation as in (12) for the coefficients.

We will mainly focus on the quasi-Jacobi form

$$\phi = \left(-\wp + \frac{1}{12}E_2\right)\Theta^2. \tag{13}$$

The following are some positivity results.

**Lemma 2.2.** *Let $\phi$ be as in (13). Then $\phi[D] \geq 0$ for all $D$ and*

$$\phi[D] > 0 \Longleftrightarrow D = 2n - \frac{r^2}{2} \geq 0 \text{ for some } n, r \in \mathbb{Z}.$$

**Proof.** By the Jacobi triple product, we have $\Theta = \vartheta_1/\eta^3$ where

$$\vartheta_1(q, y) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} y^n q^{\frac{1}{2}n^2}, \quad \eta(q) = q^{\frac{1}{24}} \prod_{n \geq 1}(1 - q^n).$$

If we write $\Theta = \sum_{n,r} c(n,r) q^n y^r$, we therefore get

$$c(n, r) > 0 \Longleftrightarrow \left(r \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \text{ and } 2n \geq r^2 - \frac{1}{4}\right)$$

and $c(n, r) = 0$ otherwise. By the explicit expressions for the action of differential operators on quasi-Jacobi forms in [22, Appendix B], we have the identity

$$\phi = \Theta^2 D_y^2 \log \Theta = D_y^2(\Theta)\Theta - D_y(\Theta)^2.$$

Hence

$$\begin{aligned} \left[\phi\right]_{q^n y^k} &= \sum_{\substack{n=n_1+n_2 \\ k=k_1+k_2}} c(n_1, k_1)c(n_2, k_2)(k_1^2 - k_1 k_2) \\ &= \frac{1}{2} \sum_{\substack{n=n_1+n_2 \\ k=k_1+k_2}} c(n_1, k_1)c(n_2, k_2)(k_1 - k_2)^2 \geq 0. \end{aligned} \tag{14}$$

Since $\phi$ is quasi-Jacobi, the coefficient $\left[\phi\right]_{q^n y^k}$ only depends on $4n - k^2$, hence we may assume $k \in \{0, 1\}$. The result now follows from (14) by a direct check. □

## 2.2. Beauville–Bogomolov form

Let $X$ be a holomorphic symplectic variety of dimension $2n$. The Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$ induces an embedding

$$H^2(X,\mathbb{Z}) \hookrightarrow H_2(X,\mathbb{Z}), \quad \alpha \mapsto (\alpha, -),$$

which is an isomorphism after tensoring with $\mathbb{Q}$. Let

$$(-,-) : H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Q} \tag{15}$$

denote the unique $\mathbb{Q}$-valued extension of the Beauville–Bogomolov form.

If $X$ is of $K3^{[n]}$ type with $n \geq 2$, there is an isomorphism of abelian groups

$$r : H_2(X,\mathbb{Z})/H^2(X,\mathbb{Z}) \to \mathbb{Z}/(2n-2)\mathbb{Z}$$

such that $r(\alpha) = 1$ for some $\alpha \in H_2(X,\mathbb{Z})$ with $(\alpha,\alpha) = \frac{1}{2-2n}$. The morphism $r$ is unique up to multiplication by $\pm 1$.

## 2.3. *Curve classes*

Consider a pair $(X,\beta)$ where $X$ is a holomorphic symplectic variety of $K3^{[n]}$ type, and $\beta \in H_2(X,\mathbb{Z})$ is a primitive curve class. The curve class $\beta$ has the following invariants:

(i) the Beauville–Bogomolov norm $(\beta,\beta) \in \mathbb{Q}$, and
(ii) the residue $[\beta] \in H_2(X,\mathbb{Z})/H^2(X,\mathbb{Z})$.

The *residue set* of $\beta$ is the subset

$$\pm[\beta] = \{\pm r([\beta])\} \subset \mathbb{Z}/(2n-2)\mathbb{Z}.$$

It is independent of the choice of map $r$. If $n = 1$, we set $\pm[\beta] = 0$.

Given a (quasi-)Jacobi form $\phi$ of index $m = n - 1$, we define

$$\phi_\beta = \phi[(\beta,\beta), \pm[\beta]].$$

By Markman [18] (see also [21, Lemma 23]), two pairs $(X,\beta)$ and $(X',\beta')$ are deformation equivalent through a family of holomorphic symplectic manifolds which keeps the curve class of Hodge type if and only if the norms and the residue sets of $\beta$ and $\beta'$ agree. Hence, by identifying $H^*(X)$ with $H^*(X')$ via parallel transport and by property (c) of the virtual fundamental class, the Gromov–Witten invariants of the pairs $(X,\beta)$ and $(X',\beta')$ are equal.[9]

[9] The (reduced) virtual fundamental class can also be defined via symplectic geometry and the twistor space of $X$; see [6]. Hence, the Gromov–Witten invariants are invariant also under (nonnecessarily algebraic) symplectic deformations of $(X,\beta)$ which keep $\beta$ of Hodge type. The invariance under nonalgebraic deformations is not needed for our application to the Fano variety of lines in a cubic 4-fold.

**Table 1**
The first few multiplicities of uniruled divisors for $K3^{[2]}$.

| $(\beta,\beta)$ | $-\frac{5}{2}$ | $-2$ | $-\frac{1}{2}$ | $0$ | $\frac{3}{2}$ | $2$ | $\frac{7}{2}$ | $4$ | $\frac{11}{2}$ | $6$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_\beta$ | 0 | 1 | 4 | 30 | 120 | 504 | 1980 | 6160 | 23576 | 60720 |

### *2.4. Proof of Theorem 0.1*

Recall from (13) the quasi-Jacobi form $\phi$.

**Theorem 2.3** *([22]). Let $X$ be a holomorphic symplectic variety of $K3^{[n]}$ type, and let $\beta \in H_2(X,\mathbb{Z})$ be a primitive curve class. Then we have*

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} = \left(\frac{\phi^{n-1}}{\Delta}\right)_\beta h \in H^2(X,\mathbb{Q})$$

*where $h = (\beta, -) \in H^2(X,\mathbb{Q})$ is the dual of $\beta$ with respect to (15).*

For the readers' convenience, we provide a proof of Theorem 2.3 at the end of this section. Theorem 2.3 together with the positivity of the Fourier coefficients of $\phi$ implies Theorem 0.1.

**Proof of Theorem 0.1.** By Lemma 2.2 the criterion in Theorem 0.1 holds if and only if

$$\left(\frac{\phi^{n-1}}{\Delta}\right)_\beta > 0,$$

hence by Theorem 2.3 if and only if the pushforward $\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}$ is nontrivial. Since the pushforward is a class in $H^2(X,\mathbb{Q})$ supported on a uniruled subvariety, the first claim follows. The second claim follows from Proposition 1.1 and property (b) of the virtual fundamental class. □

In the $K3^{[2]}$ case, we define

$$f = \frac{\phi}{\Delta} = \left(-\wp + \frac{1}{12}E_2\right)\frac{\Theta^2}{\Delta}.$$

The first few values of $f_\beta$ are listed in Table 1.[10]

### *2.5. Gromov–Witten correspondence*

In this section, we specialize to the $K3^{[2]}$ case. Recall the Gromov–Witten correspondence $\mathsf{GW}_\beta$ in (8). We also define

[10] When $n = 2$, the value $(\beta,\beta) \in \mathbb{Q}$ uniquely determines $\pm[\beta] \subset \mathbb{Z}/2\mathbb{Z}$.

**Table 2**
The first eigenvalues of $\mathsf{GW}_\beta$ for $K3^{[2]}$.

| $(\beta,\beta)$ | $-\frac{5}{2}$ | $-2$ | $-\frac{1}{2}$ | 0 | $\frac{3}{2}$ | 2 | $\frac{7}{2}$ | 4 | $\frac{11}{2}$ | 6 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | 0 | $-2$ | $-2$ | 0 | 180 | 1008 | 6930 | 24640 | 129668 | 364320 |
| $\lambda_2$ | 3 | 0 | 0 | 0 | 945 | 3840 | 53760 | 138240 | 1237005 | 2661120 |

$$g = \left(-\frac{12}{5}\wp - E_2\right)\frac{\Theta^2}{\Delta}.$$

**Theorem 2.4** *([22]). Let $X$ be a holomorphic symplectic 4-fold of $K3^{[2]}$ type, and let $\beta \in H_2(X,\mathbb{Z})$ be a primitive curve class. If $(\beta,\beta) \neq 0$, then $\mathsf{GW}_\beta$ is diagonalizable with eigenvalues*

$$\lambda_0 = 0, \quad \lambda_1 = (\beta,\beta) f_\beta, \quad \lambda_2 = (\beta,\beta) g_\beta,$$

*and eigenspaces*

$$V_{\lambda_1} = \mathbb{Q}\langle h, h^3, (he_i)_{i=1,\ldots,22}\rangle, \quad V_{\lambda_2} = \mathbb{Q}v.$$

*Here $h = (\beta, -) \in H^2(X,\mathbb{Q})$ is the dual of $\beta$ with respect to (15), $\{e_i\}_{i=1,\ldots,22}$ is a basis of the orthogonal of $h$ in $H^2(X,\mathbb{Q})$, and*

$$v = 5h^2 - \frac{1}{6}(\beta,\beta)c_2(X) \in H^4(X,\mathbb{Q}).$$

One can show that the eigenvalues $\lambda_1, \lambda_2$ are integral, and if $(\beta,\beta) > 0$ then $\lambda_2 > \lambda_1 > 0$. The first few eigenvalues are listed in Table 2.

## *2.6. Proof of Theorem 2.3*

A very general pair $(X,\beta)$ has Picard rank 1.[11] Hence there exists $N_\beta \in \mathbb{Q}$ such that

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}} = N_\beta h \in H^2(X,\mathbb{Q}).$$

By specialization, this also holds for any pair $(X,\beta)$ as in Theorem 2.3.

We will evaluate $N_\beta$ on the Hilbert scheme of $n$ points on an elliptic $K3$ surface $\mathsf{S}$ with a section. By Section 2.3, we may assume

$$\beta = \mathsf{B} + (d+1)\mathsf{F} + r\mathsf{A} \in H_2(\mathrm{Hilb}^n(\mathsf{S}),\mathbb{Z}), \quad d \geq -1, \; r \in \mathbb{Z}, \tag{16}$$

where $\mathsf{B}, \mathsf{F} \in H_2(\mathsf{S},\mathbb{Z})$ are the classes of the section and fiber of the elliptic fibration, and $\mathsf{A} \in H_2(\mathrm{Hilb}^n(\mathsf{S}),\mathbb{Z})$ is the class of an exceptional curve (for $n \geq 2$). Here we apply the natural identification

[11] In this statement, we allow $X$ to be a holomorphic symplectic manifold.

$$H_2(\mathrm{Hilb}^n(\mathsf{S}), \mathbb{Z}) \simeq H_2(\mathsf{S}, \mathbb{Z}) \oplus \mathbb{Z}\mathsf{A}.$$

Let $\mathsf{F}_0 \subset \mathsf{S}$ be a nonsingular fiber, and let $x_1, \ldots, x_{n-1} \in \mathsf{S} \setminus \mathsf{F}_0$ be distinct points. Consider the curve

$$\mathsf{C} = \{x_1 + \cdots + x_{n-1} + x' : x' \in \mathsf{F}_0\} \subset \mathrm{Hilb}^n(\mathsf{S}).$$

Then $\int_{[\mathsf{C}]} h = 1$ and hence by the first equation in [22, Theorem 2], we find

$$N_\beta = \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*[\mathsf{C}] = \left[\frac{\phi^{n-1}}{\Delta}\right]_{q^d y^r} = \left(\frac{\phi^{n-1}}{\Delta}\right)_\beta . \quad \square$$

*2.7. Proof of Theorem 2.4*

Consider the 2-pointed class

$$\mathsf{Z}_\beta = \mathrm{ev}_{12*}[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}} \in H^8(X \times X, \mathbb{Q}).$$

By the divisor equation [10] and Theorem 2.3, we have

$$\int_{\mathsf{Z}_\beta} \gamma \otimes \delta = \left( \int_\beta \delta \int_\gamma h \right) f_\beta$$

for all $\delta \in H^2(X, \mathbb{Q})$ and $\gamma \in H^6(X, \mathbb{Q})$.[12] Hence

$$\mathsf{GW}_\beta(\delta) = \left( \int_\beta \delta \right) f_\beta h \in H^2(X, \mathbb{Q}),$$

$$\mathsf{GW}_\beta(\gamma) = \left( \int_\gamma h \right) f_\beta \beta \in H^6(X, \mathbb{Q}).$$

Now consider the $(4,4)$-Künneth factor of $\mathsf{Z}_\beta$,

$$\mathsf{Z}_\beta^{4,4} \in H^4(X) \otimes H^4(X).$$

By monodromy invariance under the group $\mathrm{SO}(H^2(X, \mathbb{C}), h)$, we have

12 We have suppressed an application of Poincaré duality here.

$$\begin{aligned}\mathsf{Z}_\beta^{4,4} =& a_\beta h^2 \otimes h^2 + b_\beta(h^2 \otimes c_2(X) + c_2(X) \otimes h^2) + c_\beta c_2(X) \otimes c_2(X) \\ &+ d_\beta (h \otimes h) c_{BB} + e_\beta [\Delta_X]^{4,4}\end{aligned}$$

for some $a_\beta, b_\beta, c_\beta, d_\beta, e_\beta \in \mathbb{Q}$; see [13, Section 4]. Here

$$c_{BB} \in \mathrm{Sym}^2(H^2(X,\mathbb{Q})) \subset H^2(X,\mathbb{Q}) \otimes H^2(X,\mathbb{Q})$$

is the inverse of the Beauville–Bogomolov class.

Since $\int_{\mathsf{Z}_\beta} \sigma^2 \otimes \bar{\sigma}^2 = 0$, we have $e_\beta = 0$. Also, since the Gromov–Witten correspondence is equivariant with respect to multiplication by $\sigma$, we find

$$\mathsf{GW}_\beta(h\sigma) = \mathsf{GW}_\beta(h)\sigma = (\beta,\beta) f_\beta h\sigma.$$

Hence $d_\beta = f_\beta$. Together with Proposition 1.2 and $\int_X v^2 = 48(\beta,\beta)^2 \neq 0$, this implies

$$\mathsf{Z}_\beta^{4,4} = \psi_\beta \frac{v \otimes v}{48(\beta,\beta)^2} + f_\beta (h \otimes h)\left( c_{BB} - \frac{h \otimes h}{(\beta,\beta)} \right) \tag{17}$$

for some $\psi_\beta \in \mathbb{Q}$. It remains to determine $\psi_\beta$.

As in the proof of Theorem 2.3, let $\mathsf{S}$ be an elliptic $K3$ surface with a section, and let $\beta$ be as in (16). Consider the fiber class of the Lagrangian fibration $\mathrm{Hilb}^2(\mathsf{S}) \to \mathbb{P}^2$ induced by the elliptic fibration $\mathsf{S} \to \mathbb{P}^1$,

$$\mathsf{L} \in H^4(\mathrm{Hilb}^2(\mathsf{S}), \mathbb{Q}).$$

We have

$$\int_{\mathrm{Hilb}^2(\mathsf{S})} h^2 \mathsf{L} = 2, \qquad \int_{\mathrm{Hilb}^2(\mathsf{S})} v\mathsf{L} = 10, \qquad \int_{\mathrm{Hilb}^2(\mathsf{S}) \times \mathrm{Hilb}^2(\mathsf{S})} (h\mathsf{L} \otimes h\mathsf{L}) c_{BB} = 0.$$

Then [22, Theorem 1] and (17) imply the relation

$$\left( \frac{\Theta^2}{\Delta} \right)_\beta = \int_{\mathsf{Z}_\beta} \mathsf{L} \otimes \mathsf{L} = \frac{10^2}{48(\beta,\beta)^2} \psi_\beta - \frac{2^2}{(\beta,\beta)} f_\beta.$$

Hence

$$\psi_\beta = \frac{12(\beta,\beta)}{25} \left( 4f + \mathcal{H}_1 \left( \frac{\Theta^2}{\Delta} \right) \right)_\beta$$

where

$$\mathcal{H}_m = 2q \frac{d}{dq} - \frac{1}{2m} \left( y \frac{d}{dy} \right)^2, \quad m \geq 1$$

is the *heat operator*. Explicit formulas for the derivatives of Jacobi forms can be found in [22, Appendix B], and this yields $\psi_\beta = (\beta, \beta) g_\beta$ as desired. □

## 3. Rational curves in the Fano varieties of lines

We give the proof of Theorem 0.2. From now on, let $F$ be the Fano variety of lines in a very general cubic 4-fold $Y$, and let $\beta \in H_2(F, \mathbb{Z})$ be the primitive curve class.

### *3.1. Degeneracy locus*

The variety $F$ is naturally embedded in the Grassmannian $\mathrm{Gr}(2, 6)$. Let $\mathcal{U}$ and $\mathcal{Q}$ be the tautological bundles of ranks 2 and 4 with the short exact sequence

$$0 \to \mathcal{U} \to \mathbb{C}^6 \otimes \mathcal{O}_{\mathrm{Gr}(2,6)} \to \mathcal{Q} \to 0.$$

We use $\mathcal{U}_F, \mathcal{Q}_F$ to denote the restriction of $\mathcal{U}, \mathcal{Q}$ on $F$. Let $H = c_1(\mathcal{U}_F^*)$ be the hyperplane class on $F$ with respect to the Plücker embedding. By [4], the primitive curve class $\beta \in H_2(F, \mathbb{Z})$ is characterized by $\int_\beta H = 3$.

The indeterminacy locus $S$ of the rational map (2) consists of lines $l \subset Y$ with normal bundle

$$\mathcal{N}_{l/Y} = \mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)^{\oplus 2}.$$

For every line $l \subset Y$ corresponding to $s \in S$, there is a pencil of planes tangent to $Y$ along $l$. The residual lines of this pencil form the rational curve $\phi(p^{-1}(s)) \subset F$. By [1, Proposition 6], we have

$$\int_{[\phi(p^{-1}(s))]} H = 3.$$

Hence the curve $\phi(p^{-1}(s))$ lies in the primitive curve class $\beta$. Moreover, by the calculations in [1, Theorem 8], we find

$$\phi_*[D] = 60H \in H^2(F, \mathbb{Q}). \tag{18}$$

In [1], the surface $S$ is shown to be nonsingular, and is expressed as the degeneracy locus of the (sheafified) Gauss map

$$g : \mathrm{Sym}^2(\mathcal{U}_F) \to \mathcal{Q}_F^*$$

associated to the cubic $Y$. Let $\pi : \mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F) \to F$ be the $\mathbb{P}^2$-bundle and let $h$ be the relative hyperplane class. Then $S$ is isomorphic to the zero locus $S'$ of a section of the rank 4 vector bundle $\pi^*\mathcal{Q}_F^* \otimes \mathcal{O}(h)$ on $\mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F)$. Let $H_{S'}, h_{S'}$ be the restrictions of the divisor classes $\pi^*H, h$ on $S'$. There is the following calculation of intersection numbers.

**Lemma 3.1.** *We have*

$$\int_{S'} H_{S'}^2 = \int_{S'} H_{S'} h_{S'} = \int_{S'} h_{S'}^2 = 315.$$

**Proof.** Let $c = c_2(\mathcal{U}_F^*) \in H^4(F, \mathbb{Q})$. Since $S' \subset \mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F)$ is the zero locus of a section of the vector bundle $\pi^* \mathcal{Q}_F^* \otimes \mathcal{O}(h)$, a direct calculation yields

$$\begin{aligned}[S'] &= c_4(\mathcal{Q}_F^* \otimes \mathcal{O}(h)) = 5(\pi^* H^2 - \pi^* c)h^2 - \frac{35}{6}\pi^* H^3 \cdot h + \frac{10}{3}\pi^* H^4 \\ &\in H^8(\mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F), \mathbb{Q}).\end{aligned}$$

The lemma follows from the projection formula, the intersection numbers calculated in [1, Lemma 4], and the projective bundle formula associated to $\pi : \mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F) \to F$,

$$h^3 = 3\pi^* H \cdot h^2 - (2\pi^* H^2 + 4\pi^* c)h + \frac{5}{3}\pi^* H^3 \in H^6(\mathbb{P}\mathrm{Sym}^2(\mathcal{U}_F), \mathbb{Q}). \quad \square$$

*3.2. Connectedness*

Now we prove that $S$ is connected and calculate its first Chern class.

Let $\mathbb{G}$ be the total space of the projective bundle $\mathbb{P}\mathrm{Sym}^2(\mathcal{U})$ over the Grassmannian $\mathrm{Gr}(2, 6)$, and let

$$\tilde{\pi} : \mathbb{G} \to \mathrm{Gr}(2, 6)$$

be the projection. For convenience, we also write $H$ for the hyperplane class on $\mathrm{Gr}(2, 6)$, and $h$ for the relative hyperplane class of $\tilde{\pi}$. We define

$$\mathcal{V} = \tilde{\pi}^* \mathrm{Sym}^3(\mathcal{U}^*) \oplus \tilde{\pi}^* \mathcal{Q}^* \otimes \mathcal{O}(h)$$

to be the rank 8 tautological vector bundle on $\mathbb{G}$. Then $S$ is isomorphic to the zero locus of a section of $\mathcal{V}$. We consider the universal zero locus of all sections of $\mathcal{V}$,

$$W = \{(s, x) : s(x) = 0\} \subset \mathbb{P}H^0(\mathbb{G}, \mathcal{V}) \times \mathbb{G}$$

together with the two projections

$$\begin{array}{ccc} W & \xrightarrow{\quad \iota \quad} & \mathbb{G} \\ \big\downarrow{\scriptstyle q} & & \\ \mathbb{P}H^0(\mathbb{G}, \mathcal{V}). & & \end{array}$$

Since the morphism $q$ has a fiber isomorphic to the surface $S$, a general fiber $W_s \to s \in \mathbb{P}H^0(\mathbb{G}, \mathcal{V})$ is also of dimension 2 by upper semi-continuity.

**Proposition 3.2.** *For $s \in \mathbb{P}H^0(\mathbb{G}, \mathcal{V})$ very general, the surface $W_s$ is nonsingular of Picard rank* 1.

**Proof.** Over a point $x \in \mathbb{G}$, the fiber of $\iota$ is the projective space

$$\mathbb{P}H^0(\mathbb{G}, \mathcal{V} \otimes \mathcal{I}_x)$$

where $\mathcal{I}_x$ is the ideal sheaf of $x$. By the projection formula, we have

$$H^0(\mathbb{G}, \mathcal{V} \otimes \mathcal{I}_x) = H^0(\mathrm{Gr}(2,6), \mathrm{Sym}^3(\mathcal{U}^*) \otimes \tilde{\pi}_*\mathcal{I}_x \oplus \mathcal{Q}^* \otimes \tilde{\pi}_*\mathcal{I}_x(h)).$$

In particular, the dimension of $H^0(\mathbb{G}, \mathcal{V} \otimes \mathcal{I}_x)$ only depends on the projection $\tilde{\pi}(x) \in \mathrm{Gr}(2,6)$. The homogeneity of $\mathrm{Gr}(2,6)$ implies that $\iota : W \to \mathbb{G}$ is a projective bundle.

Since $W$ is nonsingular, a general fiber $W_s$ is also nonsingular. For $W_s$ very general, an identical argument as in [24, Lemma 2.1] yields

$$\mathrm{Pic}(W_s)_{\mathbb{Q}} = \mathrm{Im}\left(\iota^* : \mathrm{Pic}(\mathbb{G})_{\mathbb{Q}} \to \mathrm{Pic}(W_s)_{\mathbb{Q}}\right).$$

Hence the Picard group $\mathrm{Pic}(W_s)_{\mathbb{Q}}$ is spanned by $\tilde{\pi}^*H$ and $h$. The calculation in Lemma 3.1 and the Hodge index theorem imply that

$$(\tilde{\pi}^*H - h)|_{W_s} = 0 \in H^2(W_s, \mathbb{Q}).$$

Hence the classes $\tilde{\pi}^*H$ and $h$ coincide in the Néron–Severi group of $W_s$. □

**Corollary 3.3.** *The surface $S$ in* (3) *is connected. If $H_S$ is the restriction of $H$ to $S$, then we have*

$$c_1(S) = -3H_S \in H^2(S, \mathbb{Q}).$$

**Proof.** The surface $S$ is isomorphic to the zero locus $S'$ of a section of $\mathcal{V}$ via the natural projection $\pi|_{S'} : S' \xrightarrow{\sim} S$. This isomorphism identifies the divisor classes $H_{S'}$ and $H_S$.

By Proposition 3.2, a very general $W_s$ is connected, which then implies that $S$ is connected. Moreover, Proposition 3.2 shows that $c_1(S)$ is proportional to $H_S$ in $H^2(S, \mathbb{Q})$. The coefficient is determined by a calculation of intersection numbers; see [1, Remark in Section 2]. □

*3.3. Divisorial contribution*

By Proposition 1.1, the moduli space of stable maps $\overline{M}_{0,1}(F, \beta)$ is pure of dimension 3. Recall the decomposition (1),

$$\overline{M}_{0,1}(F, \beta) = M^0 \cup M^1,$$

such that a general fiber of $\mathrm{ev} : M^i \to \mathrm{ev}(M^i) \subset F$ is of dimension $i$. We first analyze the component $M^0$.

By construction, the family of maps $p : D \to S$ in (3) has a factorization

$$\phi : D \to M^0 \xrightarrow{\mathrm{ev}} F.$$

We have seen in (18) that

$$\phi_*[D] = 60H \in H^2(F, \mathbb{Q}).$$

On the other hand, by Theorem 2.3[13] together with property (b) of the virtual fundamental class, we find

$$\mathrm{ev}_*[M^0] = \mathrm{ev}_*[\overline{M}_{0,1}(F,\beta)] = \mathrm{ev}_*[\overline{M}_{0,1}(F,\beta)]^{\mathrm{vir}} = 60H \in H^2(F, \mathbb{Q}).$$

To conclude $M^0 = D$, it suffices to prove the following proposition.

**Proposition 3.4.** *For a very general $F$, each $s \in S$ yields a distinct rational curve $\phi(p^{-1}(s)) \subset F$.*

**Proof.** Let $s_1, s_2 \in S$ be two distinct points and suppose

$$\phi(p^{-1}(s_1)) = \phi(p^{-1}(s_2)) \subset F.$$

For $i = 1, 2$, let $l_i \subset Y$ be the line corresponding to $s_i$, and let $P_i \subset \mathbb{P}^5$ be the 3-dimensional linear subspace spanned by the tangent planes along $l_i$. Then necessarily $P_1 = P_2$. Otherwise, the intersection $P_1 \cap P_2$ is a plane that contains all lines in $Y$ corresponding to the points on $\phi(p^{-1}(s_i))$. The fact that $Y$ contains a plane violates the very general assumption. We also know $l_1 \cap l_2 = \emptyset$. Otherwise, the plane spanned by $l_1$ and $l_2$ is tangent to $Y$ along both $l_1$ and $l_2$, which is impossible.

Consider the Gauss map[14] associated to the cubic $Y$,

$$\mathcal{D} : \mathbb{P}^5 \to \mathbb{P}^{5*}.$$

By definition, the image $\mathcal{D}(l_i) \subset \mathbb{P}^{5*}$ is a line which is dual to $P_i \subset \mathbb{P}^5$. Following the argument of Clemens and Griffiths [7, Section 6], we may assume that $l_1, l_2$ are given by the equations

$$X_2 = X_3 = X_4 = X_5 = 0,$$
$$X_0 = X_1 = X_4 = X_5 = 0.$$

[13] By [4], we have $(\beta, \beta) = \frac{3}{2}$ and $(\beta, -) = \frac{1}{2}H \in H^2(F, \mathbb{Q})$.

[14] It is called the *dual mapping* in [7].

Then the condition $P_1 = P_2$ forces $\mathcal{D}(l_1) = \mathcal{D}(l_2)$ to be given by the equations

$$X_0^* = X_1^* = X_2^* = X_3^* = 0.$$

As a result, the cubic polynomial of $Y$ takes the form

$$\begin{aligned} &X_4 Q_4^1(X_0, X_1) + X_5 Q_5^1(X_0, X_1) \\ &\quad + X_4 Q_4^2(X_2, X_3) + X_5 Q_5^2(X_2, X_3) + R_1 + R_2. \end{aligned} \tag{19}$$

Here the $Q_i^j$ are quadratic polynomials, $R_1$ consists of terms of degree at least 2 in $\{X_4, X_5\}$, and $R_2$ consists of terms of degree 1 in each of $\{X_0, X_1\}, \{X_2, X_3\}, \{X_4, X_5\}$. The total number of possibly nonzero coefficients in (19) is

$$4 \cdot 3 + (4 \cdot 3 + 4) + 2 \cdot 2 \cdot 2 = 36.$$

On the other hand, the subgroup of $\mathrm{GL}(\mathbb{C}^6)$ fixing two disjoint lines in $\mathbb{P}^5$ is of dimension

$$4 + 4 + 3 \cdot 4 = 20,$$

resulting in a locus of dimension $36 - 20 = 16$ in the moduli space of cubic 4-folds. This again contradicts the very general assumption of $Y$. $\square$

### *3.4. Non-contribution*

We use the Gromov–Witten correspondence introduced in (8) to eliminate the component $M^1$. Recall that by property (b) of the virtual fundamental class, the class $[\overline{M}_{0,2}(F, \beta)]^{\mathrm{vir}}$ in (8) equals the ordinary fundamental class.

We begin by calculating the contribution of $M^0 = D$ to the Gromov–Witten correspondence

$$\mathsf{GW}_\beta : H^4(F, \mathbb{Q}) \to H^4(F, \mathbb{Q}). \tag{20}$$

Recall the diagram (3) and consider morphisms

$$\begin{aligned} \Phi_1^D &= p_* \phi^* : H^4(F, \mathbb{Q}) \to H^2(S, \mathbb{Q}), \\ \Phi_2^D &= \phi_* p^* : H^2(S, \mathbb{Q}) \to H^4(F, \mathbb{Q}). \end{aligned}$$

Comparing with (9) and (10), we see that $\Phi_2^D \circ \Phi_1^D = \phi_* p^* p_* \phi^*$ gives the contribution of $D$ to the Gromov–Witten correspondence (20).

Let $c = c_2(\mathcal{U}_F^*) \in H^4(F, \mathbb{Q})$. Using the short exact sequence

$$0 \to T_F \to T_{\mathrm{Gr}(2,6)}|F \to \mathrm{Sym}^3(\mathcal{U}_F^*) \to 0,$$

we find

$$8c = 5H^2 - c_2(F) = v_F \in H^4(F, \mathbb{Q})$$

where $v_F$ is the class defined in (11).[15] There is the following explicit calculation.

**Proposition 3.5.** *We have*

$$\phi_* p^* p_* \phi^* c = 945c \in H^4(F, \mathbb{Q}).$$

**Proof.** The argument in Proposition 1.2 shows that $c$ is an eigenvector of $\phi_* p^* p_* \phi^*$. To determine the eigenvalue, it suffices to compute the intersection number

$$\int_F \phi_* p^* p_* \phi^* c \cdot H^2. \tag{21}$$

By the projection formula, we have

$$\begin{aligned}\int_F \phi_* p^* p_* \phi^* c \cdot H^2 &= \int_D p^* p_* \phi^* c \cdot \phi^* H^2 \\ &= \int_S p_* \phi^* c \cdot p_* \phi^* H^2 = \int_F \phi_* p^* p_* \phi^* H^2 \cdot c.\end{aligned}$$

Again by the argument in Proposition 1.2, we know that $\phi_* p^* p_* \phi^* H^2$ is proportional to $c$. Hence we can deduce the intersection number (21) by calculating instead

$$\int_F \phi_* p^* p_* \phi^* H^2 \cdot H^2 = \int_S (p_* \phi^* H^2)^2.$$

Let $\xi$ be the relative hyperplane class of the projective bundle

$$p : D = \mathbb{P}(\mathcal{N}_{S/F}) \to S.$$

By [1, Proposition 6] and the projective bundle formula, we find

$$p_* \phi^* H^2 = p_*(7p^* H_S + 3\xi)^2 = 42 H_S - 9 c_1(\mathcal{N}_{S/F}) \in H^2(S, \mathbb{Q}),$$

where $H_S$ is the restriction of $H$ to $S$. Moreover, Corollary 3.3 yields

$$c_1(\mathcal{N}_{S/F}) = -c_1(S) = 3H_S \in H^2(S, \mathbb{Q}).$$

[15] The proportionality of $c$ and $v_F$ also follows from the fact that $c$ is represented by a rational (hence Lagrangian) surface.

Hence we obtain

$$p_*\phi^* H^2 = 15 H_S \in H^2(S, \mathbb{Q}).$$

Applying Lemma 3.1, we find the intersection number

$$\int_F \phi_* p^* p_* \phi^* H^2 \cdot H^2 = \int_S (p_* \phi^* H^2)^2 = 15^2 \cdot 315 = 70875.$$

Finally, by the intersection numbers calculated in [1, Lemma 4], we have

$$\int_F \phi_* p^* p_* \phi^* c \cdot H^2 = \int_F \phi_* p^* p_* \phi^* H^2 \cdot c = 70875 \cdot \frac{27}{45} = 42525$$

and hence

$$\phi_* p^* p_* \phi^* c = \frac{42525}{45} c = 945 c \in H^4(F, \mathbb{Q}). \quad \square$$

The eigenvalue in Proposition 3.5 coincides with the one in Theorem 2.4,

$$\mathsf{GW}_\beta(c) = 945 c \in H^4(F, \mathbb{Q}).$$

Hence the final step is to show that if the component $M^1$ is nonempty, then it has to contribute nontrivially to the Gromov–Witten correspondence (20).

If $M' \subset M^1$ is a nonempty irreducible component, consider the restriction of (9)

$$\begin{array}{ccc} M' & \xrightarrow{\text{ev}} & F \\ \downarrow{\scriptstyle p} & & \\ T' & & \end{array}$$

where $T' \subset p(M^1) \subset \overline{M}_{0,0}(F, \beta)$ is the base of $M'$. We define morphisms

$$\Phi_1^{M'} : H^4(F, \mathbb{Q}) \to H_2(T', \mathbb{Q}), \quad \gamma \mapsto p_*(\text{ev}^* \gamma \cap [M']),$$
$$\Phi_2^{M'} = \text{ev}_* p^* : H_2(T', \mathbb{Q}) \to H^4(F, \mathbb{Q}).$$

By definition, the composition $\Phi_2^{M'} \circ \Phi_1^{M'}$ gives the contribution of $M'$ to the Gromov–Witten correspondence (20).

**Proposition 3.6.** *If $M' \subset M^1$ is a nonempty irreducible component, then we have*

$$\Phi_2^{M'} \circ \Phi_1^{M'}(c) = Nc \in H^4(F, \mathbb{Q})$$

*for some $N > 0$.*

**Proof.** Let $Z' = \mathrm{ev}(M')$ with $\iota : Z' \hookrightarrow F$ the embedding. Consider the following diagram

$$\begin{array}{ccccc} \widetilde{M'} & \xrightarrow{\widetilde{\mathrm{ev}}} & \widetilde{Z}' & & \\ \downarrow{\scriptstyle \tau} & & \downarrow & \searrow^{\tilde{\iota}} & \\ M' & \xrightarrow{\mathrm{ev}} & Z' & \xrightarrow{\iota} & F, \\ \downarrow{\scriptstyle p} & & & & \\ T' & & & & \end{array}$$

where $\widetilde{M'}$ and $\widetilde{Z}'$ are simultaneous resolutions of $M'$ and $Z'$.

We calculate $\Phi_1^{M'}(c) \in H_2(T', \mathbb{Q})$. By the projection formula, we have[16]

$$\begin{aligned} \Phi_1^{M'}(c) &= p_*(\mathrm{ev}^* \iota^* c \cap [M']) \\ &= p_* \tau_* \tau^* \mathrm{ev}^* \iota^* c \\ &= p_* \tau_* \widetilde{\mathrm{ev}}^* \tilde{\iota}^* c \in H_2(T', \mathbb{Q}). \end{aligned}$$

Since $Z'$ is Lagrangian, we find

$$[Z'] = \tilde{\iota}_* [\widetilde{Z}'] = N'c \in H^4(F, \mathbb{Q})$$

for some $N' > 0$. The intersection number $\int_F c^2 = 27$ calculated in [1, Lemma 4] then implies

$$\tilde{\iota}^* c = 27N'[\tilde{x}] \in H^4(\widetilde{Z}', \mathbb{Q})$$

for any point $\tilde{x} \in \widetilde{Z}'$. This yields

$$\Phi_1^{M'}(c) = 27N' p_* \tau_* \widetilde{\mathrm{ev}}^* [\tilde{x}] = 27N'[V_x] \in H_2(T', \mathbb{Q}),$$

where $V_x \subset T'$ parametrizes rational curves through a general point $x \in Z'$. In particular, we see that $\Phi_1^{M'}(c) \in H_2(T', \mathbb{Q})$ is an effective curve class.

As a result, the class

$$\Phi_2^{M'} \circ \Phi_1^{M'}(c) = \mathrm{ev}_* p^* \Phi_1^{M'}(c) \in H^4(F, \mathbb{Q})$$

is an effective sum of classes of Lagrangian surfaces, and hence a positive multiple of $c$. $\square$

We conclude $M^1 = \emptyset$, and the proof of Theorem 0.2 is complete.

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[16] Since $\widetilde{M'}$ is nonsingular, we have suppressed an application of Poincaré duality here.

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## Appendix A. Sketch of a classical proof of Theorem 0.2

We sketch a proof of Theorem 0.2 via the classical geometry of cubic hypersurfaces. Let $Y \subset \mathbb{P}^5$ be a very general cubic 4-fold, and let $F$ be the Fano variety of lines in $Y$.

Consider the correspondence given by the universal family

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{q_Y} & Y \\ \downarrow{\scriptstyle q_F} & & \\ F. & & \end{array}$$

A rational curve $R \subset F$ corresponds to a surface $Z = q_Y(q_F^{-1}(R)) \subset Y$. If $R$ lies in the primitive curve class of $F$, then we have

$$[Z] = H_Y^2 \in H^4(Y, \mathbb{Z})$$

with $H_Y$ the hyperplane class on $Y$.

*Step 1.* Let $j : Y \hookrightarrow \mathbb{P}^5$ be the embedding. Since the surface $j(Z) \subset \mathbb{P}^5$ is of degree 3, we know from [12, Page 173] that $j(Z)$ lies in a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. Hence $Z$ is contained in the hyperplane section

$$Y' = Y \cap \mathbb{P}^4 \subset \mathbb{P}^4.$$

*Step 2.* By [12, Page 525, Proposition], the surface $Z \subset Y'$ belongs to one of the following classes:

(i) a cubic rational normal scroll;
(ii) a cone over a twisted cubic curve;
(iii) a cubic surface given by a hyperplane section of $Y' \subset \mathbb{P}^4$.

Since (i) and (ii) cannot hold for a very general[17] cubic 4-fold, we find that $Z$ is a cubic surface of the form

$$Z = Y \cap \mathbb{P}^3.$$

*Step 3.* The singularities of cubic surfaces were classified long ago; see [8, Chapter 9] and [16, Section 2]. Since $Z$ is integral, it satisfies one of the following conditions:

(i) $Z$ has rational double point singularities;
(ii) $Z$ has a simple elliptic singularity;
(iii) $Z$ is integral but not normal.

By definition, the surface $Z$ is swept out by a 1-dimensional family of lines parameterized by a rational curve. Hence we may narrow down to case (iii).

*Step 4.* By further classification results (see [16, Section 2.3]), the surface $Z$ is projectively equivalent to one of the four surfaces with explicit equations:

$$X_0^2 X_1 + X_2^2 X_3 = 0,$$
$$X_0 X_1 X_2 + X_0^2 X_3 + X_1^3 = 0,$$
$$X_1^3 + X_2^3 + X_1 X_2 X_3 = 0,$$
$$X_1^3 + X_2^2 X_3 = 0.$$

In each of the four cases, the singular locus of $Z$ is a line $l \subset Z$, and the 1-dimensional family of lines covering $Z$ is given by the residual lines of the planes containing $l$. Hence we conclude that all rational curves in the primitive curve class of $F$ are given by the uniruled divisor (3). The uniqueness part of Theorem 0.2 follows from Proposition 3.4.

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[17] Case (i) corresponds to the divisor $\mathcal{C}_{12}$ in the moduli space of cubic 4-folds; see [14]. Case (ii) is a degeneration of (i), and can be argued by a similar dimension count.

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