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On O'Grady's Generalized Franchetta Conjecture

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We study relative zero cycles on the universal polarized $K3$ surface $X \to \mathcal{F}_g$ of degree $2g-2$. It was asked by O'Grady if the restriction of any class in $\mathrm{CH}^2(X)$ to a closed fiber X_s is a multiple of the Beauville–Voisin canonical class $c_{X_s} \in \mathrm{CH}_0(X_s)$. Using Mukai models, we give an affirmative answer to this question for $g \leq 10$ and $g = 12, 13, 16, 18, 20$.

1 Introduction

Throughout, we work over the complex numbers. Let S be a projective $K3$ surface. In [2], Beauville and Voisin studied the Chow ring $\mathrm{CH}^*(S)$ of S. They showed that there is a canonical class $c_S \in \mathrm{CH}_0(S)$ represented by a point on a rational curve in S, which satisfies the following properties:

(i) The intersection of two divisor classes on S always lies in $\mathbb{Z}c_S \subset \mathrm{CH}_0(S)$.
(ii) The second Chern class $c_2(T_S)$ equals $24c_S \in \mathrm{CH}_0(S)$.

This result is rather surprising since the Chow group $\mathrm{CH}_0(S)$ is infinite-dimensional by Mumford's theorem [11].

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Let $\mathcal{F}_g$ denote the moduli space of (primitively) polarized $K3$ surfaces of degree $2g-2$. For $g \geq 3$, let $\mathcal{F}_g^0 \subset \mathcal{F}_g$ be the open dense subset parametrizing polarized $K3$ surfaces with trivial automorphism groups, which carries a universal family $X \to \mathcal{F}_g^0$. Motivated by Franchetta's conjecture on the moduli spaces of curves (see [1]), O'Grady asked the following question in [12], referred to as the generalized Franchetta conjecture.

Question 1.1 (Generalized Franchetta conjecture)**.** Given a class $\alpha \in \mathrm{CH}^2(X)$ and a closed point $s \in \mathcal{F}_g^0$, is it true that $\alpha|_{X_s} \in \mathbb{Z}c_{X_s}$? □

The goal of this article is to give an affirmative answer to Question 1.1 for a list of small values of g. By the work of Mukai [7–10], for these g a general polarized $K3$ surface can be realized in a variety with "small" Chow groups as a complete intersection with respect to a vector bundle.

Theorem 1.2. The generalized Franchetta conjecture holds for $g \leq 10$ and $g = 12, 13, 16, 18, 20$. □

This article is organized as follows. In Section 2 we review Mukai's constructions and make some comments about Question 1.1. In Section 3 we prove Theorem 1.2 for all cases except $g = 13, 16$. Two independent proofs are presented, one using Voisin's result [17], the other via a direct calculation. The cases $g = 13, 16$ have a different flavor and are treated in Section 4.

This work is inspired by a recent preprint of Pedrini [13]. However, contrary to what was claimed there, it does not suffice to show that $\mathrm{CH}^2(X)_{\mathbb{Q}}$ is finite-dimensional. Our proof relies deeply on the result of Beauville–Voisin [2].

2 Mukai Models and the Basic Setting

In this section we review Mukai's work [7–10] on the projective models of general polarized $K3$ surfaces of small degrees. Using Mukai's models, we set up the framework for the proof of Theorem 1.2.

The following table summarizes the ambient varieties $\mathbb{G}_g$ and vector bundles $\mathcal{U}_g$ involved in the constructions. It is also accompanied by a glossary.

g	$\mathbb{G}_g$	$\mathcal{U}_g$	g	$\mathbb{G}_g$	$\mathcal{U}_g$
2	$\mathbb{P}(1,1,1,2)$	$\mathcal{O}(6)$	9	$\mathbb{G}(3,6)$	$\mathcal{O}(1)^{\oplus 4} \oplus \wedge^2 \mathcal{Q}$
3	$\mathbb{P}^3$	$\mathcal{O}(4)$	10	$\mathbb{G}(2,7)$	$\mathcal{O}(1)^{\oplus 3} \oplus \wedge^4 \mathcal{Q}$
4	$\mathbb{P}^4$	$\mathcal{O}(2) \oplus \mathcal{O}(3)$	12	$\mathbb{G}(3,7)$	$\mathcal{O}(1) \oplus (\wedge^2 \mathcal{E}^\vee)^{\oplus 3}$
5	$\mathbb{P}^5$	$\mathcal{O}(2)^{\oplus 3}$	13	$\mathbb{G}(3,7)$	$(\wedge^2 \mathcal{E}^\vee)^{\oplus 2} \oplus \wedge^3 \mathcal{Q}$
6	$\mathbb{G}(2,5)$	$\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$	16	$\mathbb{G}(2,3,4)$	$\mathcal{V}_{16}^{\oplus 2} \oplus \widetilde{\mathcal{V}}_{16}^{\oplus 2}$
7	$\mathbb{OG}(5,10)$	$\mathcal{V}_7^{\oplus 8}$	18	$\mathbb{OG}(3,9)$	$\mathcal{V}_{18}^{\oplus 5}$
8	$\mathbb{G}(2,6)$	$\mathcal{O}(1)^{\oplus 6}$	20	$\mathbb{G}(4,9)$	$(\wedge^2 \mathcal{E}^\vee)^{\oplus 3}$

$\mathbb{P}(1,1,1,2)$	3-dimensional weighted projective space with weights $(1,1,1,2)$
$\mathbb{G}(r,n)$	Grassmannian parametrizing r-dimensional subspaces of a fixed n-dimensional vector space
$\mathcal{O}(i)$	Line bundle on $\mathbb{G}(r,n)$ with respect to the Plücker embedding
$\mathbb{OG}(r,n)$	Orthogonal Grassmannian parametrizing r-dimensional isotropic subspaces of a fixed n-dimensional vector space equipped with a nondegenerate symmetric 2-form
$\mathcal{V}_7$	Line bundle on $\mathbb{OG}(5,10)$ corresponding to a spin representation
$\mathcal{Q}$	Universal quotient bundle on $\mathbb{G}(r,n)$
$\mathcal{E}$	Universal subbundle on $\mathbb{G}(r,n)$
$\mathbb{G}(2,3,4)$	Ellingsrud–Piene–Strømme moduli space of twisted cubic curves, constructed as the GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by the action of $GL_2 \times GL_3$ (see [3])
$\mathcal{V}_{16}$	Rank 3 tautological vector bundle on $\mathbb{G}(2,3,4)$
$\widetilde{\mathcal{V}}_{16}$	Rank 2 tautological vector bundle on $\mathbb{G}(2,3,4)$
$\mathcal{V}_{18}$	Rank 2 vector bundle on $\mathbb{OG}(3,9)$ corresponding to a spin representation

For all g listed above, Mukai showed that a general $K3$ surface over $\mathcal{F}_g$ is given as the zero locus of a general global section of $\mathcal{U}_g$ (the cases $g \leq 5$ are classical).

Let

$$\mathbb{P}_g = \mathbb{P}H^0(\mathbb{G}_g, \mathcal{U}_g)$$

be the projectivization of the space of global sections of $\mathcal{U}_g$, and let

$$Y = \{(s, x) \in \mathbb{P}_g \times \mathbb{G}_g \mid s(x) = 0\}$$

be the incidence scheme. We have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & \mathbb{G}_g \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}_g & & \end{array}$$

where π, ι are the two projections.

The discussion above shows that a general fiber of $\pi : Y \to \mathbb{P}_g$ is a polarized $K3$ surface of degree $2g-2$ and that $\mathbb{P}_g$ rationally dominates the moduli space $\mathcal{F}_g$. Moreover, since $\mathcal{U}_g$ is globally generated, we know that $\iota : Y \to \mathbb{G}_g$ is a projective bundle. Its fiber over a point $x \in \mathbb{G}_g$ is given by

$$\mathbb{P}H^0(\mathbb{G}_g, \mathcal{U}_g \otimes \mathcal{I}_x),$$

where $\mathcal{I}_x$ is the ideal sheaf of x. We have the following lemma regarding the Chow group $\mathrm{CH}^2(Y)$ and its restriction to a general fiber of π.

Lemma 2.1. Given a closed point $s \in \mathbb{P}_g$ with $K3$ fiber Y_s, let $\phi_s : Y_s \hookrightarrow Y$ and $\iota_s : Y_s \hookrightarrow \mathbb{G}_g$ be the natural embeddings. Then we have

$$\mathrm{Im}(\phi_s^* : \mathrm{CH}^2(Y)_{\mathbb{Q}} \to \mathrm{CH}_0(Y_s)_{\mathbb{Q}}) = \mathrm{Im}(\iota_s^* : \mathrm{CH}^2(\mathbb{G}_g)_{\mathbb{Q}} \to \mathrm{CH}_0(Y_s)_{\mathbb{Q}}). \qquad \square$$

Proof. Let $\xi \in \mathrm{CH}^1(Y)$ be the relative hyperplane class of $\iota : Y \to \mathbb{G}_g$. By the projective bundle formula, we have for $k \geq 0$,

$$\mathrm{CH}^k(Y) = \xi^k \cdot \iota^*\mathrm{CH}^0(\mathbb{G}_g) \oplus \xi^{k-1} \cdot \iota^*\mathrm{CH}^1(\mathbb{G}_g) \oplus \cdots \oplus \iota^*\mathrm{CH}^k(\mathbb{G}_g). \tag{2.1}$$

Let $h \in \mathrm{CH}^1(\mathbb{P}_g)$ be the hyperplane class. Then we have

$$\pi^* h = a \cdot \xi + \iota^* \beta$$

for some $a \in \mathbb{Z}$ and $\beta \in \mathrm{CH}^1(\mathbb{G}_g)$. We claim that $a \neq 0$, otherwise

$$\pi^*(h^{\dim \mathbb{P}_g}) = \iota^*(\beta^{\dim \mathbb{P}_g}).$$

Since $\dim \mathbb{P}_g > \dim \mathbb{G}_g$, the right-hand side vanishes, but the left-hand side is the pullback of a point class and is nonzero. Contradiction. Hence

$$\xi = \frac{1}{a}(\pi^* h - \iota^* \beta) \in \mathrm{CH}^1(Y)_{\mathbb{Q}}.$$

The lemma then follows from (2.1) for $k = 2$ and the fact that $\phi_s^* \pi^* h = 0$. ■

We end this section by a few remarks on the generalized Franchetta conjecture.

(i) By a standard "spreading out" argument (see [16, Chapter 1]), it is equivalent to answer Question 1.1 for general (in fact, very general) fibers X_s over $\mathcal{F}_g^0$. Moreover, classes in $\mathrm{CH}^2(X)$ supported over a proper closed subset of $\mathcal{F}_g^0$ vanish when restricted to a fiber X_s.
Hence one may work with a family $Y \to B$ such that a general fiber Y_s is a polarized $K3$ surface of degree $2g - 2$ and that B rationally dominates $\mathcal{F}_g$ via the natural rational map $B \dashrightarrow \mathcal{F}_g$. It then suffices to answer (the analog of) Question 1.1 for classes in $\mathrm{CH}^2(Y)$ and $K3$ fibers Y_s. See Section 4 for an even more precise statement.
One may also formulate Question 1.1 in terms of the Chow group $\mathrm{CH}_0(X_\eta)$ of the generic fiber X_η, but we omit this point of view.

(ii) By Roĭtman's theorem [14], the Chow group $\mathrm{CH}_0(S)$ of a complex $K3$ surface S is torsion-free. Hence in Question 1.1 it is equivalent to work with $\mathbb{Q}$-coefficients. This also means that under Lemma 2.1, we have

$$\mathrm{Im}(\phi_s^* : \mathrm{CH}^2(Y) \to \mathrm{CH}_0(Y_s)) \subset \mathbb{Z} c_{Y_s}$$

if and only if

$$\mathrm{Im}(\iota_s^* : \mathrm{CH}^2(\mathbb{G}_g) \to \mathrm{CH}_0(Y_s)) \subset \mathbb{Z} c_{Y_s}.$$

(iii) Instead of restricting to $\mathcal{F}_g^0$, one may work with the moduli stack and the universal family over it. Question 1.1 can then be formulated using the Chow groups of smooth Deligne–Mumford stacks with $\mathbb{Q}$-coefficients. This notably covers the case $g = 2$, where a general $K3$ surface over $\mathcal{F}_2$ carries an involution. Our proof in Section 3 works in this case without change.

3 Polarized $K3$ Surfaces as Unique Complete Intersections

In this section we deal with the cases $g \leq 10$ and $g = 12, 18, 20$. For these g, the Mukai model embeds a general polarized $K3$ surface of degree $2g-2$ in $\mathbb{G}_g$ as a complete intersection with respect to $\mathcal{U}_g$, and the embedding is unique up to automorphisms of $\mathbb{G}_g$ and $\mathcal{U}_g$. Moreover, the variety $\mathbb{G}_g$ is a Grassmannian or an orthogonal Grassmannian.

Since $\mathbb{P}_g$ rationally dominates the moduli space $\mathcal{F}_g$, to prove Theorem 1.2 it suffices to show that the restriction of any class in $\mathrm{CH}^2(Y)$ to a $K3$ fiber Y_s lies in $\mathbb{Z}c_{Y_s}$. By Lemma 2.1, it is equivalent to show that

$$\mathrm{Im}(\iota_s^* : \mathrm{CH}^2(\mathbb{G}_g) \to \mathrm{CH}^2(Y_s)) \subset \mathbb{Z}c_{Y_s}.$$

This allows us to work with a single $K3$ surface S with an embedding

$$i : S \hookrightarrow \mathbb{G}_g.$$

If $g \leq 5$, the variety $\mathbb{G}_g$ is a projective space and its Chow ring is generated by the hyperplane class. Thus Theorem 1.2 follows from property (i) of c_S in Section 1.

Now assume that $\mathbb{G}_g$ is not a projective space. It is well known that the Chow group $\mathrm{CH}^2(\mathbb{G}(r,n))$ of the Grassmannian is generated by the Chern classes $c_1(\mathcal{Q})^2$ and $c_2(\mathcal{Q})$, where $\mathcal{Q}$ is the universal quotient bundle. For the orthogonal Grassmannians, we have instead

$$\mathrm{CH}^2(\mathbb{OG}(5,10)) = \mathbb{Z}\Big(\frac{1}{2}c_2(\mathcal{Q})\Big) \oplus \mathbb{Z}\Big(\frac{1}{4}c_1(\mathcal{Q})^2\Big)$$

and

$$\mathrm{CH}^2(\mathbb{OG}(3,9)) = \mathbb{Z}\Big(\frac{1}{2}c_2(\mathcal{Q})\Big) \oplus \mathbb{Z}c_1(\mathcal{Q})^2,$$

where $\mathcal{Q}$ is the corresponding universal quotient bundle (see [15]). Hence in all cases, a class $\alpha \in \mathrm{CH}^2(\mathbb{G}_g)$ can be uniquely expressed as

$$\alpha = a \cdot c_2(\mathcal{Q}) + b \cdot c_1(\mathcal{Q})^2,$$

with $a \in \mathbb{Z}$ if $\mathbb{G}_g$ is a Grassmannian, or $a \in \frac{1}{2}\mathbb{Z}$ if $\mathbb{G}_g$ is an orthogonal Grassmannian. For convenience we define the index $I(\alpha)$ of $\alpha \in \mathrm{CH}^2(\mathbb{G}_g)$ to be the coefficient a.

By property (i) of c_S in Section 1, we have $i^*(c_1(\mathcal{Q})^2) \in \mathbb{Z}c_S$. Hence the following proposition implies Theorem 1.2 for $g = 6, 7, 8, 9, 10, 12, 18, 20$.

Proposition 3.1. With the notation as above, we have $i^*c_2(\mathcal{Q}) \in \mathbb{Z}c_S$. □

We give two independent proofs of the proposition.

First proof. Mukai showed in [7, 8] that the restriction of either $\mathcal{Q}$ or $\mathcal{E}^\vee$ to a general S is simple and rigid, where $\mathcal{E}$ is the universal subbundle. In fact, the rigidity ensures that the embedding of S in $\mathbb{G}_g$ is unique. The proposition follows from a strong result of Voisin [17, Corollary 1.10] that the second Chern class of any simple rigid vector bundle on a $K3$ surface S lies in $\mathbb{Z}c_S$, which was conjectured by Huybrechts earlier in [5]. ■

Since part of the original motivation of the generalized Franchetta conjecture was to make Huybrechts' conjecture as its consequence (see [12, Section 5]), we give a direct proof of Proposition 3.1 without using Voisin's result.

Second proof. We first consider the cases where $\mathbb{G}_g$ is a Grassmannian. The standard exact sequence of normal bundles

$$0 \to T_S \to i^*T_{\mathbb{G}_g} \to i^*\mathcal{U}_g \to 0$$

yields the following relation in $\mathrm{CH}_0(S)$:

$$i^*c_2(T_{\mathbb{G}_g}) = c_2(T_S) + i^*c_2(\mathcal{U}_g). \tag{3.1}$$

Here $T_{\mathbb{G}_g}$ and T_S are the corresponding tangent bundles. Using the index of classes in $\mathrm{CH}^2(\mathbb{G}_g)$, the relation (3.1) can be written as

$$\Big(I(c_2(T_{\mathbb{G}_g})) - I(c_2(\mathcal{U}_g))\Big) \cdot i^*c_2(\mathcal{Q}) = c_2(T_S) + \gamma, \tag{3.2}$$

where γ can be expressed in terms of divisor classes on S. By properties (i) and (ii) of c_S in Section 1, both $c_2(T_S)$ and γ lie in $\mathbb{Z}c_S$. Hence it suffices to verify that

$$I(c_2(T_{\mathbb{G}_g})) - I(c_2(\mathcal{U}_g)) \neq 0. \tag{3.3}$$

The tangent bundle $T_{\mathbb{G}(r,n)}$ of the Grassmannian is $\mathcal{H}om(\mathcal{E}, \mathcal{Q})$, where $\mathcal{E}$ is the universal subbundle. By computing the Chern character

$$\mathrm{ch}(\mathcal{H}om(\mathcal{E}, \mathcal{Q})) = \mathrm{ch}(\mathcal{E}^\vee \otimes \mathcal{Q}) = \mathrm{ch}(\mathcal{E}^\vee) \cdot \mathrm{ch}(\mathcal{Q})$$

and the standard relation $c(\mathcal{E}) \cdot c(\mathcal{Q}) = 1$ between the total Chern classes, we have

$$I(c_2(T_{\mathbb{G}(r,n)})) = 2r - n.$$

The following is a case-by-case study:

$g = 6, 8$ Here $\mathcal{U}_g$ is a direct sum of line bundles. Hence $I(c_2(\mathcal{U}_g)) = 0$ and

$$I(c_2(T_{\mathbb{G}_g})) - I(c_2(\mathcal{U}_g)) = 2r - n \neq 0.$$

$g = 9$ We have $I(c_2(T_{\mathbb{G}_9})) = 0$ and $I(c_2(\mathcal{U}_9)) = I(c_2(\wedge^2 \mathcal{Q})) = 1$. Hence

$$I(c_2(T_{\mathbb{G}_9})) - I(c_2(\mathcal{U}_9)) = -1 \neq 0.$$

$g = 10$ We have $I(c_2(T_{\mathbb{G}_{10}})) = -3$ and $I(c_2(\mathcal{U}_{10})) = I(c_2(\wedge^4 \mathcal{Q})) = 1$. Hence

$$I(c_2(T_{\mathbb{G}_{10}})) - I(c_2(\mathcal{U}_{10})) = -4 \neq 0.$$

$g = 12, 20$ We have $I(c_2(T_{\mathbb{G}_g})) = -1$ and $I(c_2(\mathcal{U}_g)) = 3I(c_2(\wedge^2 \mathcal{E}^\vee))$. Hence

$$I(c_2(T_{\mathbb{G}_g})) - I(c_2(\mathcal{U}_g)) = -1 - 3I(c_2(\wedge^2 \mathcal{E}^\vee)) \neq 0.$$

The orthogonal Grassmannian cases ($g = 7, 18$) are similar. The relation (3.2) still holds, and it suffices to show (3.3). Here the left-hand side of (3.3) may be a half integer.

The natural embedding $j : \mathbb{OG}(r, n) \hookrightarrow \mathbb{G}(r, n)$ can be realized as the zero locus of a smooth section of the vector bundle $\mathcal{W}$, which is given by the cohomology group $H^0(\mathbb{P}^{r-1}, \mathcal{O}(2))$ over every closed point

$$[\mathbb{P}^{r-1} \subset \mathbb{P}^{n-1}] \in \mathbb{G}(r, n).$$

Hence we have

$$I(c_2(T_{\mathbb{OG}(r,n)})) = I(j^* c_2(T_{\mathbb{G}(r,n)})) - I(j^* c_2(\mathcal{W})).$$

The term $I(j^* c_2(T_{\mathbb{G}(r,n)}))$ was already calculated, and the term $I(j^* c_2(\mathcal{W}))$ can be determined by the following Grothendieck–Riemann–Roch calculation.

We consider $p : \mathbb{P}(\mathcal{E}) \to \mathbb{G}(r, n)$ the projective bundle on $\mathbb{G}(r, n)$ associated to the universal subbundle $\mathcal{E}$. Let $L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and let ξ be the relative hyperplane class $c_1(L)$.

We have $R^k p_* L = 0$ for $k > 0$. Hence by the Grothendieck–Riemann–Roch theorem, we have

$$\mathrm{ch}(\mathcal{W}) = \mathrm{ch}(Rp_* L^{\otimes 2}) = p_*(\exp(2\xi) \cdot \mathrm{td}(T_p)).$$

Together with the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to p^*\mathcal{E} \otimes L \to T_p \to 0,$$

we obtain for $r = 3, 5$,

$$I(c_2(\mathcal{W})) = -(r+2).$$

We finish the proof of Proposition 3.1:

$g = 7$ Here $\mathcal{U}_7$ is a direct sum of line bundles. Hence $I(c_2(\mathcal{U}_7)) = 0$ and

$$I(c_2(T_{\mathbb{G}_7})) - I(c_2(\mathcal{U}_7)) = 0 - (-7) = 7 \neq 0.$$

$g = 18$ We have $I(c_2(T_{\mathbb{G}_{18}})) = -3 - (-5)$ and $I(c_2(\mathcal{U}_{18})) = 5I(c_2(\mathcal{V}_{18}))$. Hence

$$I(c_2(T_{\mathbb{G}_{18}})) - I(c_2(\mathcal{U}_{18})) = 2 - 5I(c_2(\mathcal{V}_{18})) \neq 0.$$ ■

4 Polarized $K3$ Surfaces as Nonunique Complete Intersections

In this section we treat the remaining cases $g = 13, 16$. In both cases, the embedding of a polarized $K3$ surface S of degree $2g-2$ in $\mathbb{G}_g$ is not unique and the restriction of the tautological bundles to S may not be rigid. Hence the methods in Section 3 break down.

We keep the notation of Section 2 and write $\Phi : \mathbb{P}_g \dashrightarrow \mathcal{F}_g$ for the dominant rational map. Let $t \in \mathcal{F}_g^0$ be a closed point outside the indeterminacy locus of Φ in $\mathcal{F}_g$. Given two closed points $s_1, s_2 \in \mathbb{P}_g$ with $\Phi(s_1) = \Phi(s_2) = t$, there are canonical isomorphisms

$$Y_{s_1} \cong Y_{s_2} \cong X_t.$$

We identify $\mathrm{CH}_0(Y_{s_1}), \mathrm{CH}_0(Y_{s_2})$ with $\mathrm{CH}_0(X_t)$, and define

$$\mathrm{CH}^2(Y)_{\mathrm{inv}} = \{\alpha \in \mathrm{CH}^2(Y) \mid \phi_{s_1}^*\alpha = \phi_{s_2}^*\alpha \text{ for all } s_1, s_2 \in \mathbb{P}_g \text{ above}\}.$$

Recall that $\phi_s : Y_s \hookrightarrow Y$ is the natural embedding for $s \in \mathbb{P}_g$.

Again by the "spreading out" argument and the fact that classes supported over a proper closed subset of $\mathcal{F}_g^0$ do not contribute, to prove Theorem 1.2 it suffices show that

$$\mathrm{Im}(\phi_s^* : \mathrm{CH}^2(Y)_{\mathrm{inv}} \to \mathrm{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}$$

for all (or general, or very general) $K3$ fibers Y_s.

First we consider the case $g = 13$. As described by the Mukai model, let

$$i : S \hookrightarrow \mathbb{G}(3,7)$$

be the embedding of a $K3$ surface S in $\mathbb{G}(3,7)$. The restriction of $\mathcal{E}^\vee$ (dual of the universal subbundle) to S is semi-rigid, which carries a 2-dimensional deformation. Let M_S be the moduli space of stable vector bundles on S with Mukai vector $(3, H, 4)$, where H is the polarization class. A general point of M_S is represented by $i^*\mathcal{E}^\vee$ for some i; see [9] for details. Note that M_S is also a polarized $K3$ surface with $g = 13$.

Let $s \in \mathbb{P}_{13}$ be a closed point with $K3$ fiber Y_s, and let $\iota_s : Y_s \hookrightarrow \mathbb{G}(3,7)$ be as in Section 2. By Lemma 2.1, the restriction $\phi_s^*\alpha$ of a class $\alpha \in \mathrm{CH}^2(Y)_{\mathrm{inv}}$ can be expressed as

$$\phi_s^*\alpha = a \cdot \iota_s^* c_2(\mathcal{Q}) + b \cdot \iota_s^*(c_1(\mathcal{Q})^2), \tag{4.1}$$

where $\mathcal{Q}$ is the universal quotient bundle and $a, b \in \mathbb{Q}$ are constants independent of $s \in \mathbb{P}_{13}$. By property (i) of c_{Y_s} in Section 1, we have $\iota_s^*(c_1(\mathcal{Q})^2) \in \mathbb{Z}c_{Y_s}$.

Theorem 1.2 for $g = 13$ is a direct consequence of the following lemma.

Lemma 4.1. In the expression (4.1), the coefficient a is zero. □

Proof. We choose closed points $s_1, s_2 \in \mathbb{P}_{13}$ with $\Phi(s_1) = \Phi(s_2) = t \in \mathcal{F}_{13}^0$, such that the vector bundles $\iota_{s_1}^*\mathcal{E}^\vee, \iota_{s_2}^*\mathcal{E}^\vee$ represent different point classes in $\mathrm{CH}_0(M_{X_t})$. This is possible by [9, Theorem 2], which shows that $\mathbb{P}_{13}$ rationally dominates the moduli space of triples (S, H, E), where S is a $K3$ surface, H is a polarization with $H^2 = 24$, and E is a stable vector bundle with Mukai vector $(3, H, 4)$.

Since $\alpha \in \mathrm{CH}^2(Y)_{\mathrm{inv}}$, we have by definition $\phi_{s_1}^*\alpha = \phi_{s_2}^*\alpha$ and hence

$$a \cdot \iota_{s_1}^* c_2(\mathcal{Q}) = a \cdot \iota_{s_2}^* c_2(\mathcal{Q}), \tag{4.2}$$

viewed as an equality in $\mathrm{CH}_0(X_t)_{\mathbb{Q}}$.

On the other hand, let $\mathbb{F}$ be a universal sheaf over $M_{X_t} \times X_t$ (which exists by the numerics of the Mukai vector; see [6, Corollary 4.6.7]). The correspondence

$$\mathrm{ch}(\mathbb{F}) \cdot \sqrt{\mathrm{td}(T_{M_{X_t} \times X_t})} \in \mathrm{CH}^*(M_{X_t} \times X_t)_{\mathbb{Q}}$$

induces an isomorphism of (ungraded) Chow groups

$$\theta : \mathrm{CH}^*(M_{X_t}) \xrightarrow{\simeq} \mathrm{CH}^*(X_t).$$

Here for $[E] \in M_{X_t}$, we have

$$\theta([E]) = 3[X_t] + H + 15c_{X_t} - c_2(E) \in \mathrm{CH}^*(X_t).$$

According to our choice of $s_1, s_2 \in \mathbb{P}_{13}$, the vector bundles $\iota_{s_1}^*\mathcal{E}^\vee, \iota_{s_2}^*\mathcal{E}^\vee$ represent different classes in $\mathrm{CH}_0(M_{X_t})$. By applying θ, we find

$$\iota_{s_1}^* c_2(\mathcal{E}^\vee) \neq \iota_{s_2}^* c_2(\mathcal{E}^\vee)$$

in $\mathrm{CH}_0(X_t)$, and together with (4.2) we obtain $a = 0$. ■

Finally we consider the case $g = 16$. The variety $\mathbb{G}_{16} = \mathbb{G}(2, 3, 4)$ is realized as a GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by the obvious action of $\mathrm{GL}_2 \times \mathrm{GL}_3$ on the first two factors. As described in [3] (see also [10]), there are two tautological vector bundles $\mathcal{V}_{16}$ and $\widetilde{\mathcal{V}}_{16}$ of ranks 3 and 2, respectively, as well as a morphism

$$\mathcal{V}_{16} \otimes (\mathbb{C}^4)^\vee \to \widetilde{\mathcal{V}}_{16}.$$

Further, it was shown in [3, Proposition 2] that the Chow ring $\mathrm{CH}^*(\mathbb{G}(2, 3, 4))$ is generated by the Chern classes of $\mathcal{V}_{16}, \widetilde{\mathcal{V}}_{16}$. To prove Theorem 1.2 we have to take care of the second Chern classes of both tautological bundles.

Let $i : S \hookrightarrow \mathbb{G}(2, 3, 4)$ be the embedding of a $K3$ surface S in $\mathbb{G}(2, 3, 4)$ as in the Mukai model. By the same reasoning as in Section 3, we have the following relation in $\mathrm{CH}_0(S)$:

$$i^* c_2(T_{\mathbb{G}(2,3,4)}) = c_2(T_S) + i^* c_2(\mathcal{U}_{16}). \tag{4.3}$$

Here $\mathcal{U}_{16} = \mathcal{V}_{16}^{\oplus 2} \oplus \widetilde{\mathcal{V}}_{16}^{\oplus 2}$. Using the exact sequence (see [4, (4-4)])

$$0 \to \mathcal{O}_{\mathbb{G}(2,3,4)} \to \mathcal{E}nd(\mathcal{V}_{16}) \oplus \mathcal{E}nd(\widetilde{\mathcal{V}}_{16}) \to \mathcal{H}om(\mathcal{V}_{16} \otimes (\mathbb{C}^4)^\vee, \widetilde{\mathcal{V}}_{16}) \to T_{\mathbb{G}(2,3,4)} \to 0,$$

the relation (4.3) can be written as

$$6c_2(i^*\widetilde{\mathcal{V}}_{16}) = c_2(T_S) + \gamma,$$

where γ can be expressed in terms of divisor classes on S. By properties (i) and (ii) of c_S in Section 1, this verifies that $i^*c_2(\widetilde{\mathcal{V}}_{16}) \in \mathbb{Z}c_S$.

Alternatively, by Mukai's results [10, Propositions 1.3 and 2.2], for a general S the vector bundle $i^*\widetilde{\mathcal{V}}_{16}$ is simple and rigid. The statement $i^*c_2(\widetilde{\mathcal{V}}_{16}) \in \mathbb{Z}c_S$ also follows from Voisin's result [17, Corollary 1.10].

Let $s \in \mathbb{P}_{16}$ be a closed point with $K3$ fiber Y_s, and let $\iota_s : Y_s \hookrightarrow \mathbb{G}(2,3,4)$ be as before. Again by Lemma 2.1 and property (i) of c_{Y_s} in Section 1, the restriction $\phi_s^*\alpha$ of a class $\alpha \in \mathrm{CH}^2(Y)_{\mathrm{inv}}$ can be expressed as

$$\phi_s^*\alpha = a \cdot \iota_s^*c_2(\mathcal{V}_{16}) + \tilde{a} \cdot \iota_s^*c_2(\widetilde{\mathcal{V}}_{16}) + b \cdot c_{Y_s},$$

where $a, \tilde{a}, b \in \mathbb{Q}$ are constants independent of $s \in \mathbb{P}_{16}$. Moreover, the fact that $\iota_s^*c_2(\widetilde{\mathcal{V}}_{16}) \in \mathbb{Z}c_{Y_s}$ implies

$$\phi_s^*\alpha = a \cdot \iota_s^*c_2(\mathcal{V}_{16}) + b' \cdot c_{Y_s} \tag{4.4}$$

for some $a, b' \in \mathbb{Q}$ independent of $s \in \mathbb{P}_{16}$. Since $\iota_s^*\mathcal{V}_{16}$ is semi-rigid with Mukai vector $(3, H, 5)$ by [10, Proposition 2.2], an identical argument as in the proof of Lemma 4.1 yields $a = 0$ in the expression (4.4).

This finishes the proof of Theorem 1.2 for $g = 16$.

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