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# The Generic Nontriviality of the Faber–Pandharipande Cycle

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We present a simple and characteristic-free proof of a result of Green and Griffiths, which states that for the generic curve $C$ of genus $g \geq 4$, the Faber–Pandharipande cycle $K \times K - (2g-2)K_\Delta$ is nontorsion in $\mathrm{CH}^2(C \times C)$.

## 1 Introduction

Let $C$ be a smooth algebraic curve of genus $g$ over a field $k$. Denote by $K \in \mathrm{CH}^1(C)$ the class of a canonical divisor of $C$. Faber and Pandharipande introduced the 0-cycle

$$Z := K \times K - (2g-2)K_\Delta \in \mathrm{CH}^2(C \times C), \tag{1}$$

where $K_\Delta$ is the divisor $K$ on the diagonal $\Delta \subset C \times C$. This cycle is of degree 0 and lies in the kernel of the Albanese map.

It is easy to see that $Z = 0$ when $g = 0, 1, 2$. Faber and Pandharipande showed that it is also the case when $g = 3$, using the fact that genus 3 curves are either hyperelliptic or plane curves. They asked if $Z$ vanishes in general. Green and Griffiths [8] answered this in the negative (over $\mathbb{C}$).

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**Theorem 1.1** ([8, Theorem 2]). If $C$ is the generic curve of genus $g \geq 4$, then $Z \neq 0$ in $\mathrm{CH}^2(C \times C)_{\mathbb{Q}}$. □

Their proof involves lengthy calculations of a Hodge-theoretic infinitesimal invariant. In this note, we give a simple proof of this result, which also works in positive characteristic. It consists of two separate steps.

(i) *A problem on the Jacobian.* We observe that $Z$ is symmetric, so it naturally lives on the second symmetric power $C^{[2]}$ of $C$. The latter is closely related to the Jacobian $J$ of $C$ via the map $C^{[2]} \to J$ (with respect to a point $x_0 \in C(k)$). We show that $Z$ is the pull-back of an explicit codimension 2 cycle $W$ on $J$, which is tautological in the sense of Polishchuk [11]. In particular, the $\mathfrak{sl}_2$-action studied in [11] gives a new proof that $Z = 0$ for $g = 3$ (without classifying genus 3 curves).

(ii) *A degeneration argument.* We consider $W$ in the relative setting, where it lives on the universal Jacobian $\pi : \mathscr{J} \to \mathscr{M}_{g,1}$. Although Abel–Jacobi trivial fiberwise, the cycle $W$ gives a class $\mathrm{cl}(W)$ in $H^2(\mathscr{M}_{g,1}, R^2\pi_*\mathbb{Q})$ (over $\mathbb{C}$, or $H^2(\mathscr{M}_{g,1}, R^2\pi_*\mathbb{Q}_\ell(2))$ in general). If $W$ is trivial on the generic fiber, then there should exist an open subset $U \subset \mathscr{M}_{g,1}$ such that the restriction of $\mathrm{cl}(W)$ is zero in $H^2(U, R^2\pi_*\mathbb{Q})$. So it remains to show that such a $U$ does not exist. A key lemma by Fakhruddin (cf. [7, Lemma 4.1]) reduces this to an argument on the boundary of $\mathscr{M}_{g,1}$. There we construct explicit families of stable curves and study the cycle class of $W$. It turns out that even the simplest families of "test curves" will suffice for the proof.

### 1.1 Philosophical note

It is in general very difficult to detect nontrivial cycles in the kernel of the Abel–Jacobi map. Results in this direction are mostly variational, often obtained by calculating infinitesimal invariants on the generic fiber. The invariants are essentially Hodge-theoretic objects associated with certain Leray filtrations, and the calculation is usually difficult.

Now since we work with an abelian scheme $\mathscr{J}$, the classical Leray filtration is actually a multiplicative decomposition (cf. [13, Corollary 2.2]). Its compatibility with the Beauville decomposition in $\mathrm{CH}(\mathscr{J})_{\mathbb{Q}}$ tells us exactly in which cohomology group lies the cycle class. Finally, via the degeneration argument we take full advantage of the boundary of $\mathscr{M}_g$ (or $\mathscr{M}_{g,1}$), which is missing in the infinitesimal approach.

This method can also be used to detect nontrivial cycles that lie deeper in the conjectural Bloch–Beilinson filtration (cf. [14]).

### 1.2 Notation and conventions

We work over a field $k$ of arbitrary characteristic. Since the main result is a geometric statement (see note after Theorem 3.3), we assume $k$ to be algebraically closed. From now on, Chow groups are with $\mathbb{Q}$-coefficients. By a "cycle", we mean the rational equivalence class of a cycle. The word "generic" is taken in the schematic sense. Over $\mathbb{C}$ (or any uncountable field), the term "very general" is often used, which means outside a countable union of Zariski-closed proper subsets of the base scheme (see Corollary 3.5).

## 2 Connections with the Jacobian

We briefly review Polishchuk's work [11] on the tautological ring of a Jacobian $J$. Let $C$ be a smooth curve of genus $g$ over $k$. Denote by $J$ the Jacobian of $C$, and by $\theta$ the class of a symmetric theta divisor on $J$. Recall the Beauville decomposition $\mathrm{CH}^i(J) = \oplus_{j=i-g}^{i} \mathrm{CH}^i_{(j)}(J)$, with

$$\mathrm{CH}^i_{(j)}(J) := \{\alpha \in \mathrm{CH}^i(J) : [n]^*(\alpha) = n^{2i-j}\alpha \text{ for all } n \in \mathbb{Z}\}.$$

We identify $J$ with its dual $J^t$ via the canonical principal polarization, and denote by $\mathscr{P}$ the Poincaré line bundle on $J \times J$. There is the Fourier transform $\mathscr{F} \colon \mathrm{CH}^i_{(j)}(J) \xrightarrow{\sim} \mathrm{CH}^{g-i+j}_{(j)}(J)$ given by $\alpha \mapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\alpha) \cdot \mathrm{ch}(\mathscr{P}))$, where $\mathrm{pr}_1, \mathrm{pr}_2 \colon J \times J \to J$ are the two projections. We refer the reader to Beauville's paper [2] for more details about the Beauville decomposition and the Fourier transform.

Choose a base point $x_0 \in C(k)$, and let $\iota \colon C \hookrightarrow J$ be the embedding given by $x \mapsto \mathscr{O}_C(x - x_0)$. Consider the 1-cycle $[C] := [\iota(C)]$ and its components $[C]_{(j)} \in \mathrm{CH}^{g-1}_{(j)}(J)$. Define

$$p_i := \mathscr{F}([C]_{(i-1)}) \in \mathrm{CH}^i_{(i-1)}(J) \quad \text{for } i \geq 1,$$

$$q_i := \mathscr{F}(\theta \cdot [C]_{(i)}) \in \mathrm{CH}^i_{(i)}(J) \quad \text{for } i \geq 0.$$

The $\mathbb{Q}$-subalgebra of $\mathrm{CH}(J)$ generated by $\{p_i\}_{i\geq 1}$ and $\{q_i\}_{i\geq 0}$ is called the tautological ring of $J$, denoted by $\mathscr{T}(J)$. Polishchuk proved that it is stable under $\mathscr{F}$ and the Pontryagin product "$*$".

An important tool is the $\mathfrak{sl}_2$-action on $\mathrm{CH}(J)$ and on $\mathscr{T}(J)$ (here $\mathfrak{sl}_2 = \mathbb{Q}\cdot e + \mathbb{Q}\cdot f + \mathbb{Q}\cdot h$), defined by

$$e(\alpha) := p_1 \cdot \alpha, \quad f(\alpha) := -[C]_{(0)} * \alpha,$$

$$h(\alpha) := (2i - j - g)\alpha \quad \text{for } \alpha \in \mathrm{CH}^i_{(j)}(J).$$

Polishchuk showed that $f$ acts on $\mathscr{T}(J)$ via the differential operator $\mathscr{D}$ given by

$$\mathscr{D} := -\frac{1}{2}\sum_{i,j\geq 1}\binom{i+j}{j} p_{i+j-1}\partial p_i \partial p_j - \sum_{i,j\geq 1}\binom{i+j-1}{j} q_{i+j-1}\partial q_i \partial p_j + \sum_{i\geq 1} q_{i-1}\partial p_i, \tag{2}$$

where $\partial p_i$ (respectively, $\partial q_i$) is the partial derivative with respect to $p_i$ (respectively, $q_i$) (cf. [11, Theorem 0.2]).

Now consider the map $\phi\colon C\times C \to J$ given by $(x, y) \mapsto \mathscr{O}_C(x + y - 2x_0)$. We would like to express the cycle $Z \in \mathrm{CH}^2(C\times C)$ in (1) as the pull-back of a certain cycle $W \in \mathscr{T}^2(J)$ under $\phi$. Since $Z$ is Abel–Jacobi trivial, we should look for $W$ in $\mathscr{T}^2_{(2)}(J)$, which is spanned by $q_1^2$ and $q_2$.

**Proposition 2.1.** Define $W := 2(q_1^2 - (2g-2)q_2) \in \mathscr{T}^2_{(2)}(J)$. We have $Z = \phi^*(W)$. □

**Proof.** This is done by an explicit calculation. The essential ingredients are the pull-back of $\theta$ via $\iota\colon C \to J$, and the pull-back of $c_1(\mathscr{P})$ via $(\iota, \phi)\colon C\times(C\times C) \to J\times J$. Write $\eta := \iota^*(\theta)$ and $\ell := (\iota,\phi)^*(c_1(\mathscr{P}))$. Then we have identities

$$\eta = \frac{1}{2}K + [x_0], \qquad ([11], (1.1))$$

$$\ell = [\Delta_1] + [\Delta_2] - 2\mathrm{pr}_1^*([x_0]) - \mathrm{pr}_2^*([x_0\times C] + [C\times x_0]), \qquad ([12], (2.1))$$

where $\Delta_1 = \{(x, x, y) : x, y \in C\}$ and $\Delta_2 = \{(x, y, x) : x, y \in C\}$, and $\mathrm{pr}_1\colon C\times(C\times C) \to C$, $\mathrm{pr}_2\colon C\times(C\times C) \to C\times C$ are the two projections.

By chasing through the following cartesian squares (here $\mathrm{pr}_1, \mathrm{pr}_2$ stand for the two projections in all cases)

$$\begin{array}{ccccc}
C\times(C\times C) & \xrightarrow{\iota\times\mathrm{id}} & J\times(C\times C) & \xrightarrow{\mathrm{pr}_2} & C\times C \\
\downarrow{\scriptstyle \mathrm{id}\times\phi} & & \downarrow{\scriptstyle \mathrm{id}\times\phi} & & \downarrow{\scriptstyle \phi} \\
C\times J & \xrightarrow[\iota\times\mathrm{id}]{} & J\times J & \xrightarrow[\mathrm{pr}_2]{} & J \\
\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\
C & \xrightarrow[\iota]{} & J & &
\end{array}$$

we find

$$\begin{aligned}\phi^*(\mathscr{F}(\theta\cdot[C])) &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\eta)\cdot\exp(\ell))\\ &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot\exp([\Delta_1]+[\Delta_2]-2\,\mathrm{pr}_1^*([x_0])))\cdot\exp(-[x_0\times C]-[C\times x_0])\\ &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*((\tfrac{1}{2}K+[x_0])\cdot\exp(-2[x_0]))\cdot\exp([\Delta_1]+[\Delta_2]))\cdot\exp(-[x_0\times C]-[C\times x_0])\\ &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot\exp([\Delta_1]+[\Delta_2]))\cdot\exp(-[x_0\times C]-[C\times x_0]).\end{aligned}$$

Then by expanding the exponentials while keeping track of the codimension, we obtain

$$\begin{aligned}\phi^*(q_1) &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot([\Delta_1]+[\Delta_2]))-\mathrm{pr}_{2,*}\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot([x_0\times C]+[C\times x_0])\\ &= \tfrac{1}{2}(K\times[C]+[C]\times K)-(g-1)([x_0\times C]+[C\times x_0]),\\ \phi^*(q_2) &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot\tfrac{1}{2}([\Delta_1]+[\Delta_2])^2)\\ &\quad-\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot([\Delta_1]+[\Delta_2]))\cdot([x_0\times C]+[C\times x_0])\\ &\quad+\mathrm{pr}_{2,*}\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot\tfrac{1}{2}([x_0\times C]+[C\times x_0])^2\\ &= \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\tfrac{1}{2}K+[x_0])\cdot([\Delta_1]\cdot[\Delta_2]))-\tfrac{1}{2}(K\times[x_0]+[x_0]\times K)+(g-2)[x_0\times x_0]\\ &= \tfrac{1}{2}K_\Delta-\tfrac{1}{2}(K\times[x_0]+[x_0]\times K)+(g-1)[x_0\times x_0].\end{aligned}$$

Hence

$$\phi^*(q_1^2)=\frac{1}{2}K\times K-(g-1)(K\times[x_0]+[x_0]\times K)+2(g-1)^2[x_0\times x_0],$$

and we obtain $\phi^*(2(q_1^2-(2g-2)q_2))=K\times K-(2g-2)K_\Delta$. ■

**Corollary 2.2.**

(i) We have $Z=0$ if and only if $W=0$. In particular, whether $W$ vanishes or not is independent of $x_0$.

(ii) If $g=3$, then $Z=0$ in $\mathrm{CH}^2(C\times C)$. □

**Proof.** For (i), we calculate the push-forward $\phi_*(Z)$. First, we have

$$
\begin{aligned}
Z &= K \times K - (2g-2)K_\Delta \\
&= 4(\tfrac{1}{2}K + [x_0]) \times (\tfrac{1}{2}K + [x_0]) - 4(\tfrac{1}{2}K + [x_0]) \times [x_0] - 4[x_0] \times (\tfrac{1}{2}K + [x_0]) \\
&\quad - (4g-4)(\tfrac{1}{2}K + [x_0])_\Delta + 4g[x_0 \times x_0].
\end{aligned}
$$

By the projection formula, we find

$$
\begin{aligned}
\phi_*(Z) &= 4(\theta \cdot [C]) * (\theta \cdot [C]) - 8\theta \cdot [C] - (4g-4)[2]_*(\theta \cdot [C]) + 4g[0] \\
&= 0(\theta \cdot [C]_{(0)}) + 0(\theta \cdot [C]_{(1)}) + 4(\theta \cdot [C]_{(1)}) * (\theta \cdot [C]_{(1)}) - (8g-8)(\theta \cdot [C]_{(2)}) \\
&\quad + (\text{terms in } \oplus_{j\geq 3} \mathscr{T}^g_{(j)}(J)) \\
&= 4((\theta \cdot [C]_{(1)}) * (\theta \cdot [C]_{(1)}) - (2g-2)(\theta \cdot [C]_{(2)})) + (\text{terms in } \oplus_{j\geq 3} \mathscr{T}^g_{(j)}(J)).
\end{aligned}
$$

Observe that the Fourier dual of $(\theta \cdot [C]_{(1)}) * (\theta \cdot [C]_{(1)}) - (2g-2)(\theta \cdot [C]_{(2)})$ is exactly $W$. If $Z = 0$, then every Beauville component of $\phi_*(Z)$ is zero, and hence $W = 0$.

For (ii), consider the cycle $p_2^2 \in \mathscr{T}^4_{(2)}(J)$. When $g = 3$, we have $p_2^2 = 0$ for dimension reasons. By applying the differential operator $\mathscr{D}$ in (2) twice, we obtain

$$\mathscr{D}^2(p_2^2) = \mathscr{D}(-6p_3 + 2q_1 p_2) = 2(q_1^2 - 4q_2) = 0.$$

So $W = 0$, and thus $Z = 0$. ■

**Remark 2.3.**

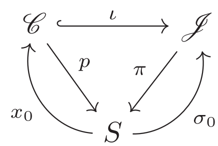
(i) By a classical theorem of Rojtman (cf. [10, Theorem 0.1]), the vanishing of $Z$ with $\mathbb{Q}$-coefficients implies its vanishing with $\mathbb{Z}$-coefficients.

(ii) It would be interesting to study the vanishing locus of $Z$ when $g \geq 4$. Also conjecturally (by Bloch and Beilinson) $Z$ vanishes if the curve $C$ is defined over $\bar{\mathbb{Q}}$. □

## 3 Fakhruddin's Degeneration Argument

### 3.1 The relative setting

We include stable curves of compact type in the relative setting. The base scheme $S$ is a smooth connected variety of dimension $d$ over $k$. Let $p\colon \mathscr{C} \to S$ be a relative curve with a section (marked point) $x_0\colon S\to\mathscr{C}$, such that the fibers are stable 1-pointed curves of compact type in the sense of Deligne and Mumford. Write $g$ for the genus of the fibers. Denote by $\pi\colon\mathscr{J}\to S$ the relative Jacobian of $\mathscr{C}$, which is a principally polarized abelian scheme of relative dimension $g$. As in the absolute case, the section $x_0$ induces an embedding $\iota\colon\mathscr{C}\hookrightarrow\mathscr{J}$ that gives the following diagram (here $\sigma_0$ is the zero section of $\pi$):




We refer the reader to [9, Section 6] for the precise definition of $\iota$ in the compact type case. Roughly speaking, the Jacobian of a curve of compact type is a product of Jacobians, and $\iota$ is obtained by taking the product of the embeddings in the factors.

On $\mathrm{CH}(\mathscr{J})$ we again have a decomposition $\mathrm{CH}^i(\mathscr{J})=\oplus\mathrm{CH}^i_{(j)}(\mathscr{J})$, such that $[n]^*$ is the multiplication by $n^{2i-j}$ on $\mathrm{CH}^i_{(j)}(\mathscr{J})$ (cf. [6, Theorem 2.19]). At the cohomology level, there is a canonical decomposition

$$R\pi_*\mathbb{Q}_\ell(r)\simeq\sum_i R^i\pi_*\mathbb{Q}_\ell(r)[-i],$$

with $[n]^*$ acting on $R^i\pi_*\mathbb{Q}_\ell(r)$ by the multiplication by $n^i$ (cf. [5, 2.19]). This decomposition is multiplicative, that is, compatible with the multiplicative structure

$$R\pi_*\mathbb{Q}_\ell(r)\otimes R\pi_*\mathbb{Q}_\ell(r')\to R\pi_*\mathbb{Q}_\ell(r+r')$$

given by the cup product (cf. [13, Corollary 2.2]). It follows that we have a multiplicative decomposition

$$H^m(\mathscr{J},\mathbb{Q}_\ell(r))\simeq\bigoplus_{i+j=m}H^j(S,R^i\pi_*\mathbb{Q}_\ell(r)).$$

Comparing the action of $[n]$ on Chow groups and on cohomology, we know that the cycle class map $\mathrm{cl}\colon\mathrm{CH}^i(\mathscr{J})\to H^{2i}(\mathscr{J},\mathbb{Q}_\ell(i))$ decomposes as a sum of maps

$$\mathrm{cl}\colon\mathrm{CH}^i_{(j)}(\mathscr{J})\to H^j(S,R^{2i-j}\pi_*\mathbb{Q}_\ell(i)),\tag{3}$$

which respect the multiplicative structures on both sides. Note that if the base field $k=\mathbb{C}$, one may work with singular cohomology with coefficients in $\mathbb{Q}$.

### 3.2 The degeneration

Let $\alpha \in \mathrm{CH}^i_{(j)}(\mathscr{J})$. Denote by $J_\eta$ the generic fiber of $\mathscr{J} \to S$, and by $\alpha_\eta \in \mathrm{CH}^i_{(j)}(J_\eta)$ the restriction of $\alpha$ to $J_\eta$. Now suppose $\alpha_\eta = 0$. By the standard "spreading-out" procedure, there exists a nonempty open subset $U \subset S$ such that $\alpha_U = 0 \in \mathrm{CH}^i_{(j)}(\mathscr{J}_U)$, where $\mathscr{J}_U := \mathscr{J} \times_S U$ and $\alpha_U := \alpha|_{\mathscr{J}_U}$. Combining with the cycle class map (3), we have the following implication.

**Proposition 3.1.** If $\alpha_\eta = 0$, then there exists a nonempty open subset $U \subset S$ such that

$$\mathrm{cl}(\alpha_U) = 0 \in H^j(U, R^{2i-j}\pi_*\mathbb{Q}_\ell(i)). \qquad \square$$

We consider the cycles $q_1^2, q_2$, and $W$ in the relative setting (3). More precisely, denote by $\theta \in \mathrm{CH}^1_{(0)}(\mathscr{J})$ the divisor class corresponding to the canonical principal polarization $\lambda\colon \mathscr{J} \xrightarrow{\sim} \mathscr{J}^t$ (so $2\theta$ is the pull-back of the first Chern class of the Poincaré bundle $\mathscr{P}$ under the map $\mathrm{id} \times \lambda\colon \mathscr{J} \to \mathscr{J} \times \mathscr{J}^t$, and fiberwise $\theta$ is the class of a symmetric theta divisor). Again we identify $\mathscr{J}$ with $\mathscr{J}^t$ and regard the Fourier transform $\mathscr{F}$ as an endomorphism of $\mathrm{CH}(\mathscr{J})$. Generalizing the definitions in Section 2, we write $[\mathscr{C}] := [\iota(\mathscr{C})]$ and let

$$q_i := \mathscr{F}(\theta \cdot [\mathscr{C}]_{(i)}) \in \mathrm{CH}^i_{(i)}(\mathscr{J}) \quad \text{for } i \geq 0.$$

As before, we define $W := 2(q_1^2 - (2g-2)q_2) \in \mathrm{CH}^2_{(2)}(\mathscr{J})$.

Our main focus is the case $S = \mathscr{M}^{\mathrm{ct}}_{g,1}$, that is, the moduli stack of stable 1-pointed curves of genus $g$ and of compact type. The fact that $\mathscr{M}^{\mathrm{ct}}_{g,1}$ is a stack plays no role in the discussion. In fact, since the Chow groups are with $\mathbb{Q}$-coefficients, for our purpose (Theorem 3.3) it is equivalent to work over a finite cover of the moduli stack that is an honest variety (cf. [1, Theorem 7.6.4] for the existence of such a cover).

The goal is to prove that for $g \geq 4$, we have $W \neq 0$ generically over $\mathscr{M}^{\mathrm{ct}}_{g,1}$. In view of Proposition 3.1, we would like to show that for all nonempty open subsets $U \subset \mathscr{M}^{\mathrm{ct}}_{g,1}$, we have

$$\mathrm{cl}(W_U) \neq 0 \in H^2(U, R^2\pi_*\mathbb{Q}_\ell(2)).$$

Using the following lemma by Fakhruddin, we can reduce the proof to a calculation on the boundary of $\mathscr{M}^{\mathrm{ct}}_{g,1}$.

**Lemma 3.2** ([7, Lemma 4.1])**.** Let $X, S$ be smooth connected varieties over $k$ and $\pi\colon X \to S$ be a smooth proper map. Consider a class $h \in H^m(X, \mathbb{Q}_\ell(r))$. Suppose there exists a nonempty subvariety $T \subset S$ such that for all nonempty open subsets $V \subset T$, we have $h_V \neq 0$, where $h_V := h|_{X_V}$. Then for all nonempty open subsets $U \subset S$, we have $h_U \neq 0$. □

### 3.3 **Proof of Theorem 1.1**

Now to prove Theorem 1.1, it suffices to construct a family of "test curves" over a variety $T$ on the boundary of $\mathscr{M}_{g,1}^{\mathrm{ct}}$, and to show that the class of $W$ does not vanish over any nonempty open subset of $T$. In fact, we can prove a slightly stronger result.

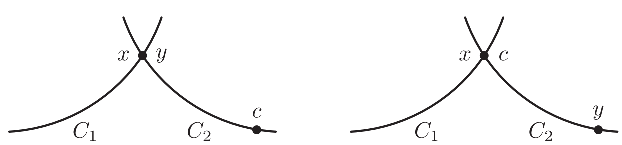
**Theorem 3.3.** When $g \geq 4$, the cycles $q_1^2$ and $q_2$ are linearly independent on the generic Jacobian (over $\mathscr{M}_{g,1}^{\mathrm{ct}}$). In particular, we have $W \neq 0$ on the generic Jacobian. □

Note that Theorem 3.3 is of geometric nature: if the statement is true over the base field $k$, then it is automatically true over any base field $k' \subset k$. This means the theorem still holds over an arbitrary field (not necessarily algebraically closed). Together with Corollary 2.2(i), it implies Theorem 1.1.

The rest of this paper is devoted to the construction of the "test curves" and the proof of Theorem 3.3. We shall construct two families of curves over the same base scheme $T$. We show that for any nontrivial linear combination of $q_1^2$ and $q_2$, at least one of the two families will give a cohomology class that does not vanish when restricted to nonempty open subsets of $T$. For simplicity, we begin with the case $g=4$, while the proof for the general case is almost identical (see end of proof).

#### 3.3.1 *Case* $g=4$

Take two smooth curves $C_1$ and $C_2$ of genus 2 over $k$, with Jacobians $(J_1, \theta_1)$ and $(J_2, \theta_2)$. Let $x$ (respectively, $y$) be a varying point on $C_1$ (respectively, $C_2$), and $c$ be a fixed point on $C_2$. We construct the first family of stable curves by joining $x$ and $y$ and using $c$ as the marked point, and then the second family by joining $x$ and $c$ and using $y$ as the marked point, as is shown in the picture below.




With $x$ and $y$ varying, both families have the same base scheme $T := C_1 \times (C_2 \backslash \{c\})$. We denote them by $\mathscr{C} \to T$ and $\mathscr{C}' \to T$, respectively. Observe that $\mathscr{C}$ and $\mathscr{C}'$ have also the same relative Jacobian $\mathscr{J} := J_1 \times J_2 \times T$, a constant abelian scheme over $T$ via the last projection.

Consider the embeddings $\mathscr{C} \hookrightarrow \mathscr{J}$ with respect to $c$, and $\mathscr{C}' \hookrightarrow \mathscr{J}$ with respect to $y$. An important fact is that both embeddings naturally extend over $C_1 \times C_2 \supset T$. More precisely, we have

$$\psi_1 \colon C_1 \times C_1 \times C_2 \hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } \quad (z, x, y) \mapsto (\mathscr{O}_{C_1}(z - x), \mathscr{O}_{C_2}(y - c), x, y),$$

$$\psi_2 \colon C_2 \times C_1 \times C_2 \hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } \quad (w, x, y) \mapsto (0, \mathscr{O}_{C_2}(w - c), x, y),$$

$$\psi_1' \colon C_1 \times C_1 \times C_2 \hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } \quad (z, x, y) \mapsto (\mathscr{O}_{C_1}(z - x), \mathscr{O}_{C_2}(c - y), x, y),$$

$$\psi_2' \colon C_2 \times C_1 \times C_2 \hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } \quad (w, x, y) \mapsto (0, \mathscr{O}_{C_2}(w - y), x, y).$$

We take $\bar{T} := C_1 \times C_2$ as the base scheme and view the other schemes as $\bar{T}$-schemes through the projections onto the last two factors. We also write $\bar{\mathscr{J}} := J_1 \times J_2 \times \bar{T}$. The divisor $\theta$ corresponding to the polarization of $\bar{\mathscr{J}} \to \bar{T}$ is $\theta := \theta_1 \times [J_2] \times [\bar{T}] + [J_1] \times \theta_2 \times [\bar{T}]$.

Let $\bar{\mathscr{C}} \subset \bar{\mathscr{J}}$ be the union of the images of $\psi_1$ and $\psi_2$; similarly, let $\bar{\mathscr{C}}' \subset \bar{\mathscr{J}}$ be the union of the images of $\psi_1'$ and $\psi_2'$. We see that the restriction of $\bar{\mathscr{C}}$ (respectively, $\bar{\mathscr{C}}'$) to $T$ is exactly $\mathscr{C}$ (respectively, $\mathscr{C}'$). Define

$$\bar{q}_i := \mathscr{F}(\theta \cdot [\bar{\mathscr{C}}]_{(i)}) \in \mathrm{CH}^i_{(i)}(\bar{\mathscr{J}}), \quad \bar{q}_i' := \mathscr{F}(\theta \cdot [\bar{\mathscr{C}}']_{(i)}) \in \mathrm{CH}^i_{(i)}(\bar{\mathscr{J}}).$$

Again, the restriction of $\bar{q}_i$ (respectively, $\bar{q}_i'$) to $T$ is exactly the $q_i$ of $\mathscr{C}$ (respectively, $\mathscr{C}'$).

As $\bar{\mathscr{J}}$ is a constant abelian scheme over $\bar{T}$, we have a Künneth decomposition

$$H^m(\bar{\mathscr{J}}) = \bigoplus_{a_1+b_1+a_2+b_2=m} H^{a_1}(J_1) \otimes H^{b_1}(C_1) \otimes H^{a_2}(J_2) \otimes H^{b_2}(C_2).$$

Here, and in what follows, we omit the coefficients of the cohomology groups. Also on the right-hand side we have sorted the factors in the order $J_1$-$C_1$-$J_2$-$C_2$, as this turns out to be convenient in our calculations. Given a class $h \in H^m(\bar{\mathscr{J}})$, we denote by $h^{[a_1,b_1,a_2,b_2]}$ its Künneth component in the indicated degrees.

In this case, the cycle class map (3) takes the form

$$\mathrm{cl} \colon \mathrm{CH}^i_{(j)}(\bar{\mathscr{J}}) \to \bigoplus_{\substack{a_1+a_2=2i-j \\ b_1+b_2=j}} H^{a_1}(J_1) \otimes H^{b_1}(C_1) \otimes H^{a_2}(J_2) \otimes H^{b_2}(C_2). \tag{4}$$

Then for $\alpha = \bar{q}_1^2, \bar{q}_2, \bar{q}_1'^2, \bar{q}_2' \in \mathrm{CH}^2_{(2)}(\bar{\mathscr{J}})$, we can only have $\mathrm{cl}(\alpha)^{[a_1,b_1,a_2,b_2]} \neq 0$ if $a_1 + a_2 = 2$ and $b_1 + b_2 = 2$. Moreover, remark that $H^2(C_1)$ (respectively, $H^2(C_2)$) is supported on a point of $C_1$ (respectively, $C_2$). As we should like to have the cycle classes after restriction to open subsets $V \subset T \subset \bar{T}$, the only interesting components are $\mathrm{cl}(\alpha)^{[a_1,1,a_2,1]}$ with $a_1 + a_2 = 2$ (in fact, we will see in the proof of Proposition 3.4 that for $\mathrm{cl}(\alpha)^{[a_1,1,a_2,1]}$ to be nonzero, we also have $a_1 = a_2 = 1$).

The following elementary calculation is the key point in the proof of Theorem 3.3.

**Proposition 3.4.** There exist nonzero classes

$$h_1 \in H^1(J_1) \otimes H^1(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

$$h_2, h_4 \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

$$h_3 \in H^1(J_1) \otimes H^1(C_1) \otimes H^0(J_2) \otimes H^0(C_2),$$

such that

$$\mathrm{cl}(\bar{q}_2)^{[1,1,1,1]} = h_1, \quad \mathrm{cl}(\bar{q}_1^2)^{[1,1,1,1]} = 2h_2 \smile h_3,$$

$$\mathrm{cl}(\bar{q}_2')^{[1,1,1,1]} = -h_1, \quad \mathrm{cl}(\bar{q}_1'^2)^{[1,1,1,1]} = -2h_2 \smile h_3 + 2h_3 \smile h_4.$$

Moreover, the classes $h_2 \smile h_3$ and $h_3 \smile h_4$ are also nonzero. □

**Proof.** The proof is just a careful analysis of the embeddings $\psi_1, \psi_2, \psi_1', \psi_2'$. We first calculate the relevant Künneth components of $\mathrm{cl}([\bar{\mathscr{C}}]_{(i)})$ and $\mathrm{cl}([\bar{\mathscr{C}}']_{(i)})$. Then by intersecting with $\mathrm{cl}(\theta)$ and applying $\mathscr{F}$ in cohomology, we obtain the relevant components of $\mathrm{cl}(\bar{q}_i)$ and $\mathrm{cl}(\bar{q}_i')$.

We start with the cycle classes of $[\bar{\mathscr{C}}]_{(1)}$ and $[\bar{\mathscr{C}}]_{(2)}$. Observe that the image of $\psi_2$ only gives a class in $H^4(J_1) \otimes H^0(C_1) \otimes H^2(J_2) \otimes H^0(C_2)$, which by (4), does not contribute to either $[\bar{\mathscr{C}}]_{(1)}$ or $[\bar{\mathscr{C}}]_{(2)}$. Regarding $\psi_1$, we may view it as the product of

$$\psi_3\colon C_1 \times C_1 \hookrightarrow J_1 \times C_1, \quad \psi_4\colon C_2 \hookrightarrow J_2 \times C_2,$$

$$(z, x) \mapsto (\mathscr{O}_{C_1}(z - x), x), \quad y \mapsto (\mathscr{O}_{C_2}(y - c), y).$$

The class of $\mathrm{Im}(\psi_3)$ has components in $H^2(J_1) \otimes H^0(C_1)$, $H^1(J_1) \otimes H^1(C_1)$, and $H^0(J_1) \otimes H^2(C_1)$. The third component is irrelevant due to the appearance of $H^2(C_1)$. We claim that the other two components are both nonzero. For the first, we regard $J_1 \times C_1$

as a constant family over $C_1$. Then $C_1 \times C_1$ is fiberwise an ample divisor, which gives a nonzero class in $H^2(J_1) \otimes H^0(C_1)$. For the component in $H^1(J_1) \otimes H^1(C_1)$, we consider

$$\begin{array}{ccccccc} C_1 \times C_1 & \xrightarrow{\mathrm{id}\times\Delta} & C_1 \times C_1 \times C_1 & \xrightarrow{\sigma\times\mathrm{id}} & C_1^{[2]} \times C_1 & \xrightarrow{\varphi\times\mathrm{id}} & J_1 \times C_1 \\ (z, x) & \mapsto & (z, x, x) & \mapsto & ((z, x), x) & \mapsto & (\mathscr{O}_{C_1}(z + x - 2x), x). \end{array}$$

The class of the diagonal in $C_1 \times C_1$ has a component in $H^1(C_1) \otimes H^1(C_1)$ which, viewed as a correspondence, gives the identity $H^1(C_1) \xrightarrow{\sim} H^1(C_1)$. It follows that the class of $\mathrm{Im}(\mathrm{id} \times \Delta)$ has a nonzero component in $H^0(C_1) \otimes H^1(C_1) \otimes H^1(C_1)$. Moreover, we have isomorphisms

$$\sigma_*\colon H^0(C_1) \otimes H^1(C_1) \xrightarrow{\sim} H^1(C_1^{[2]}), \quad \varphi_*\colon H^1(C_1^{[2]}) \xrightarrow{\sim} H^1(J_1),$$

the latter due to the fact that $C_1^{[2]}$ is obtained by blowing up a point in $J_1$. So $\mathrm{Im}(\psi_3)$ as a correspondence gives an isomorphism $H^1(J_1) \xrightarrow{\sim} H^1(C_1)$, which implies a nonzero component in $H^1(J_1) \otimes H^1(C_1)$.

Similarly, the class of $\mathrm{Im}(\psi_4)$ has nonzero components in $H^4(J_2) \otimes H^0(C_2)$ and $H^3(J_2) \otimes H^1(C_2)$. Now we collect all nonzero contributions to the classes of $[\bar{\mathscr{C}}]_{(1)}$ and $[\bar{\mathscr{C}}]_{(2)}$ that do not involve either $H^2(C_1)$ or $H^2(C_2)$. For $[\bar{\mathscr{C}}]_{(2)}$, there is only one nonzero class

$$h_1^0 \in H^1(J_1) \otimes H^1(C_1) \otimes H^3(J_2) \otimes H^1(C_2).$$

By intersecting with $\mathrm{cl}(\theta)$ and applying $\mathscr{F}$, we obtain a nonzero class

$$h_1 := \mathscr{F}(\mathrm{cl}(\theta) \smile h_1^0) \in H^1(J_1) \otimes H^1(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

For $[\bar{\mathscr{C}}]_{(1)}$, there are two nonzero classes

$$h_2^0 \in H^2(J_1) \otimes H^0(C_1) \otimes H^3(J_2) \otimes H^1(C_2), \quad h_3^0 \in H^1(J_1) \otimes H^1(C_1) \otimes H^4(J_2) \otimes H^0(C_2).$$

Again by intersecting with $\mathrm{cl}(\theta)$ and applying $\mathscr{F}$, we obtain nonzero classes

$$h_2 := \mathscr{F}(\mathrm{cl}(\theta) \smile h_2^0) \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

$$h_3 := \mathscr{F}(\mathrm{cl}(\theta) \smile h_3^0) \in H^1(J_1) \otimes H^1(C_1) \otimes H^0(J_2) \otimes H^0(C_2).$$

By the definition of $\bar{q}_i$, we have $\mathrm{cl}(\bar{q}_2)^{[1,1,1,1]} = h_1$ and $\mathrm{cl}(\bar{q}_1^2)^{[1,1,1,1]} = h_2 \smile h_3 + h_3 \smile h_2 = 2h_2 \smile h_3$.

For the cohomology classes of $\bar{q}_2'$ and $\bar{q}_1'^2$, we remark that the embedding $\psi_1'$ differs from $\psi_1$ only by an action of $[-1]$ on the $J_2$ factor. As a consequence, by repeating the same procedure we obtain classes $h'_1 = -h_1$, $h'_2 = -h_2$ and $h'_3 = h_3$, so that $2h'_2 \smile h'_3 = -2h_2 \smile h_3$. However, this time the embedding $\psi'_2$ makes an additional contribution. The class of $\mathrm{Im}(\psi'_2)$ has a nonzero component

$$h_4^0 \in H^4(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

which belongs to the class of $[\bar{\mathscr{C}}']_{(1)}$. By intersecting with $\mathrm{cl}(\theta)$ and applying $\mathscr{F}$, we get a nonzero class

$$h_4 := \mathscr{F}(\mathrm{cl}(\theta) \smile h_4^0) \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2).$$

This time we have $\mathrm{cl}(\bar{q}'_2)^{[1,1,1,1]} = -h_1$ and $\mathrm{cl}(\bar{q}'^2_1)^{[1,1,1,1]} = -2h_2 \smile h_3 + 2h_3 \smile h_4$.

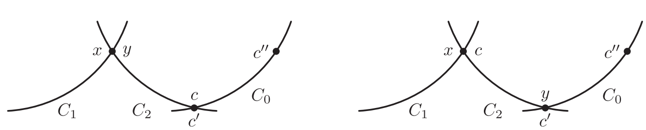
Finally, since the 0th cohomology groups $H^0(C_i)$ and $H^0(J_i)$ are generated by the unit of the ring structures, we see that both $h_2 \smile h_3$ and $h_3 \smile h_4$ are nonzero. ■

As $h_1 \neq 0$ and $h_3 \smile h_4 \neq 0$, it follows from Proposition 3.4 that for any $(r, s) \neq (0, 0) \in \mathbb{Q}^2$, at least one of $\mathrm{cl}(r\bar{q}_1^2 + s\bar{q}_2)^{[1,1,1,1]}$ and $\mathrm{cl}(r\bar{q}'^2_1 + s\bar{q}'_2)^{[1,1,1,1]}$ is nonzero in $H^1(C_1) \otimes H^1(J_1) \otimes H^1(C_2) \otimes H^1(J_2)$.

It remains to ensure that this nonzero cohomology class does not vanish when restricted to nonempty open subsets of $\bar{T} = C_1 \times C_2$, that is, not supported on a divisor of $C_1 \times C_2$. We can achieve this by imposing additional assumptions on $C_1$ and $C_2$. In positive characteristic, we choose $C_1$ to be ordinary and $C_2$ supersingular. Over $\bar{\mathbb{Q}}$, and hence for any $k = \bar{k}$ of characteristic 0, we take $C_1$ and $C_2$ such that $J_1$ and $J_2$ are both simple, and such that $\mathrm{End}(J_1) = \mathbb{Z}$ and $J_2$ is of CM type (cf. [3], Chapters 14 and 15 for explicit examples). In both situations, we have $\mathrm{Hom}(J_1, J_2) = 0$, which implies that there is no nonzero divisor class in $H^1(C_1) \otimes H^1(C_2)$. This completes the proof for $g = 4$.

### 3.3.2 *General case: end of proof*

When $g > 4$, we may attach to both families a constant curve $C_0$ of genus $g - 4$ via a fixed point $c' \in C_0$, and use another fixed point $c'' \in C_0$ as the marked point. The proof is exactly the same. Alternatively, we may apply Ceresa's degeneration argument (cf. [4, (3.10)]) and reduce to the case $g = 4$.



**Corollary 3.5.** When the base field $k$ is uncountable (e.g., $k=\mathbb{C}$) and when $g\geq 4$, the same statement as in Theorem 3.3 holds for the Jacobian of a very general curve (over $\mathscr{M}_{g,1}$). □

**Proof.** In fact, the data $\mathscr{M}_{g,1}$, $\mathscr{J}$, $q_1^2$, and $q_2$ are all defined over the prime field $k_0=\mathbb{Q}$ or $\mathbb{F}_p$. Then for any class $\alpha\in\mathbb{Q}\cdot q_1^2+\mathbb{Q}\cdot q_2$, if $\alpha$ is nonzero over the generic point $\eta\in\mathscr{M}_{g,1}/k_0$, by base change, it is also nonzero over any point $s\in\mathscr{M}_{g,1}(k)$ that maps to $\eta$. In other words, we have $\alpha\neq 0$ over any point $s\in\mathscr{M}_{g,1}(k)$ that does not lie in a subvariety of $\mathscr{M}_{g,1}/k_0$. Since $k_0$ is countable, there are only countably many such varieties. ■

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