Fall 2020: Topics in Number Theory: Fermat's Last Theorem

Instructor: Liang Xiao (肖梁)
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Meeting time: Monday 1–2, and Wednesday 7–8(even week)
Lecture room: Lecture Building #3, Room 206 (三教 206)
Office hours: Friday 10–11am, at 78101–1
Webpage: `http://bicmr.pku.edu.cn/~lxiao/20fall/index.htm`

Goal of this course: The focus will be on understanding the Langlands correspondence and the Taylor-Wiles method. The course consists of five parts:

(1) Review basic facts on structures of Galois group of local and global number fields, and discuss Galois cohomology and duality theory.
(2) A light introduction to p-adic Hodge theory.
(3) Discuss deformation of Galois representations, local and global.
(4) Discuss modular forms and associated Galois representations.
(5) Prove a modularity lifting theorem, through which we deduce Fermat's Last Theorem.

Our main reference includes various lecture notes by Richard Taylor, and a lecture note by Sug Woo Shin at Berkeley. For modularity lifting theorems, we will partially follow Toby Gee's notes at Arizona Winter School.

Prerequisite:

- Class field theory, especially descent understanding of Galois cohomology (but will recall some in first several lectures).
- Homology theory (in topology).
- Algebraic geometry (Hartshorne Chap 2, can concurrently learning Chap 3)
- Modular forms.
- Very basic knowledge of elliptic curves.

Grade Distribution:

Homeworks: 40%, every two-weeks starting week 4, in total 7 times, with lowest grade dropped
Midterm exam: 25%, tentatively on November 16
Take-home final exam: 35%, to be announced

Homework: *Homework problems are posted on the course webpage, and are usually due on the Wednesday of odd weeks.* You are welcome and encouraged to work with other students on the problems, but you should write up your homework independently.

Syllabus (Tentative)

Lecture	Dates	Content
1	9/21	Introduction and Background in Number Theory I: ℓ-adic representations, structure of local Galois group.
2	9/28	Background in Number Theory II: Grothendieck's ℓ-adic monodromy theorem, Weil–Deligne representations, higher ramification groups, local class field theory in terms of Artin maps.
3	9/30	Background in Number Theory III: Global class field theory, Galois cohomology.
Happy National's Day and Mid-Autumn Festival!		
4	10/12	Background in Number Theory IV: Local duality theorems for Galois cohomology.
5	10/14	Background in Number Theory V: Global duality theorems for Galois cohomology, and Selmer complexes.
6	10/19	Overflow material from duality theorems
7	10/26	Galois deformation I: Framed deformation, computation of tangent space.
8	10/28	Galois deformation II: Quotient by free actions, existence of deformation ring.
9	11/2	Galois deformation III: Relations of deformation rings, and some examples.
10	11/9	Galois deformation IV: Relative deformation problem, Galois deformation with local conditions, more examples.
11	11/11	Galois deformation IV: Taylor-Wiles primes and Galois patching.
12	11/16	Galois deformation V: Conditioned local Galois deformation.
13	11/23	Galois deformation VI: Conditioned local Galois deformation at $\ell \neq p$
14	11/25	p-adic Hodge theory I: A quick overview.
15	11/30	p-adic Hodge theory II: Fontaine–Laffaille deformation.

16	12/7	Galois representations appearing in geometry.
17	12/9	Modular forms I: Jacobian of curves.
18	12/14	Modular forms II: Galois representations attached to modular forms.
19	12/21	Modular forms III: Geometric modular forms.
20	12/23	Modular forms IV: Eichler–Shimura relations.
21	12/28	Modularity lifting theorem I: preparation.
22	1/4	Modularity lifting theorem II: patching argument.
23	1/6	Modularity lifting theorem III: Wiles' 3-5 trick and finish of the proof.
	TBA	**Final Exam**