Separable elements in Weyl groups

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Overview

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   - separable permutations
   - results by Fan Wei
   - root systems and Weyl groups

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3. Classification via pattern avoidance
   - Case 1: Simply-laced
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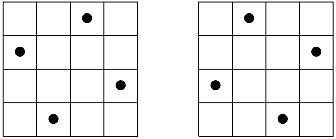

Figure: Permutations 3142 and 2413.

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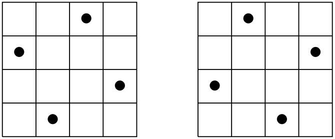

Figure: Permutations 3142 and 2413.

Lemma

If $w \in \mathfrak{S}_n$ is separable, then there exists $1 < m < n$ such that either

- *$w_1 \cdots w_m$ is a separable permutation on $\{1, \ldots, m\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{m+1, \ldots, n\}$;*
- *or $w_1 \cdots w_{n-m}$ is a separable permutation on $\{m+1, \ldots, n\}$ and $w_{n-m+1} \cdots w_n$ is a separable permutation on $\{1, \ldots, m\}$.*

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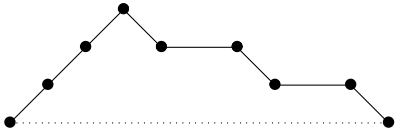

Figure: A Schröder path: lattice path from $(0,0)$ to $(2n,0)$ using steps $(1,1)$, $(1,-1)$, $(2,0)$ that never goes below the x-axis.

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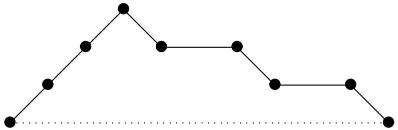

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If a collection of distinct real polynomials all have equal values at some number x, then the permutation that describes how the numerical ordering of the polynomials changes at x is separable, and every separable permutation can be realized in this way.

Notations on ranked posets

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For $x \in P$, let

- $V_x := \{y \in P : y \geq x\}$ be the principal upper order ideal at x,
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Let

$$F(P) = F(P, q) := \sum_{x \in P} q^{\mathrm{rk}(x)}$$

be the rank generating function of P.

Background on the weak (Bruhat) order

The right weak (Bruhat) order R_n is generated by

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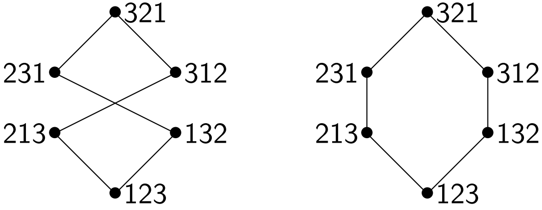

Figure: The left weak order and the right weak order on $\mathfrak{S}_3$.

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Her proof relies on the following lemma.

Lemma (Wei 2012)

Let $\pi = uv$ as words where u and v are separable. Then

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We will be generalizing these results to other types.

Root systems and Weyl groups

Root systems and Weyl groups

Definition (Root system)

Let $E = \mathbb{R}^n$. A root system $\Phi \subset E$ is a finite set of vectors, such that

- Φ spans E;
- for $\alpha \in \Phi$, $k\alpha \in \Phi$ iff $k \in \{\pm 1\}$;
- for $\alpha, \beta \in \Phi$, $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$;
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A choice of Φ^+ corresponds to a unique set of *simple roots* Δ such that

- $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a basis for E;
- every $\alpha \in \Phi^+$ is written as $\sum_{i=1}^n c_i \alpha_i$ where $c_i \in \mathbb{Z}_{\geq 0}\ \forall i$.

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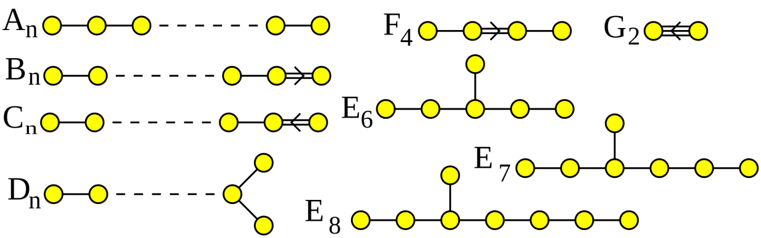

Figure: Irreducible root systems (Wikipedia)

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Definition (Root poset and support)

For $\alpha, \beta \in \Phi^+$, $\alpha \leq \beta$ if $\beta - \alpha$ is written as a nonnegative linear combination of simple roots. For $\alpha \in \Phi^+$, its support is defined as $\mathrm{Supp}(\alpha) := \{\alpha_i \in \Delta : \alpha_i \leq \alpha\}$.

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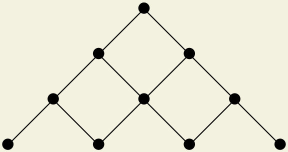

Figure: Root system of type A_4

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Example: restriction map in type A

Let $w = 6347215 \in W(A_6)$.

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Consider $E' = \mathrm{span}(e_2 - e_4, e_4 - e_5)$. Then Φ' is of type A_2. And the set of simple roots for Φ' is $\Delta' = \{e_2 - e_4, e_4 - e_5\}$.

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Compare the following equivalent definition of separable permutations.

Definition

Let $w \in \mathfrak{S}_n$. Then w is separable if one of the following holds:

- $n \leq 2$;
- there exists $1 < m < n$ such that either
  - $w_1 \cdots w_m$ is a separable permutation on $\{1, \ldots, m\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{m+1, \ldots, n\}$;
  - or $w_1 \cdots w_m$ is a separable permutation on $\{n-m+1, \ldots, n\}$ and $w_{m+1} \cdots w_n$ is a separable permutation on $\{1, \ldots, n-m\}$.

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Example (separable elements in $W(B_2)$)

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. $\Delta = \{\alpha_1, \alpha_2\}$. Dynkin diagram  .

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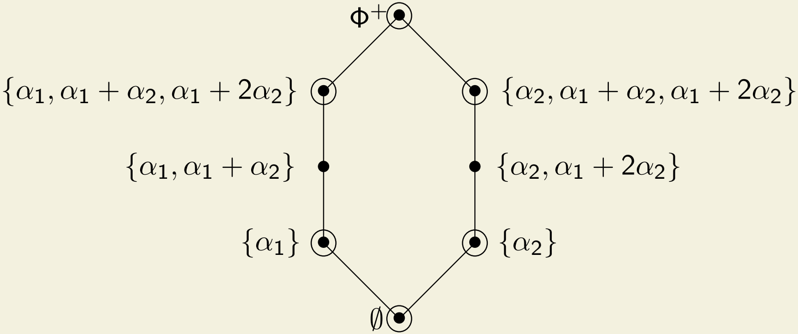

Figure: Weak order of type B_2 labeled by inversion sets, where separable elements are circled.

Properties of separable elements

Theorem (Gaetz and G. 2019)

Let $w \in W(\Phi)$ be separable. Then the upper order ideal V_w and the lower order ideal Λ_w in the (left) weak order are both rank symmetric and rank unimodal. Moreover, $F(V_w)F(\Lambda_w) = F(W(\Phi))$.

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The longest element $w_0^J \in W^J$ is separable, for which the above theorem is known, because of the rank-preserving decomposition $W = W^J \cdot W_J$.

Classification via pattern avoidance

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An element $w \in W(\Phi)$ is separable if (and only if) it avoids:

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Open question

Is there a nice proof?

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The following lemma is heavily used in all steps of the proof.

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Remark

$|W(E_8)| = 696,729,600$ and $|\Phi^+_{E_8}| = 120$.

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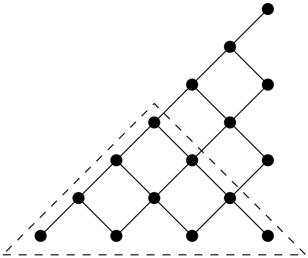

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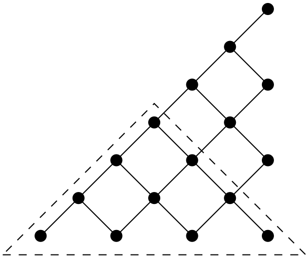

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Step 2: show that whether $\alpha \in I_\Phi(w)$ depends only on its support, using induction on the height and bad patterns in B_2.

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Faces of the graph associahedron of (the graph of) the Dynkin diagram of Φ can be labeled by exactly half (or the other half) of the separable elements in $W(\Phi)$.

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Open question

Can we label the faces of any graph associahedron analogously?

Thanks

Thanks: Alex Postnikov, Victor Reiner, and Anders Björner.

Thank you for listening!