# The weak Bruhat order on the symmetric group is Sperner

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# Overview

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   - An $\mathfrak{sl}_2$-action on the weak Bruhat order of $S_n$
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   - Padded Schubert polynomials
   - Weighted enumeration of chains in the (strong) Bruhat order

# The Sperner property

Let $P$ be a ranked poset with rank decomposition $P_0 \sqcup P_1 \sqcup \cdots \sqcup P_r$.

**Definition**

$P$ is called *k-Sperner* if no union of its $k$ antichains is larger than the union of its largest $k$ ranks.
$P$ is called *Sperner* if it is 1-Sperner.
$P$ is called *strongly Sperner* if it is $k$-Sperner for any $k \in \mathbb{Z}_{\geq 1}$.

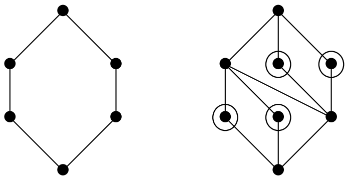


Figure: A Sperner poset (left) and a non-Sperner poset (right)

# The Sperner property

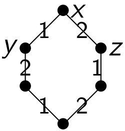
Further assume that $P = P_0 \sqcup \cdots \sqcup P_r$ is

- rank symmetric: $|P_i| = |P_{r-i}|$ for all $i$,
- rank unimodal: there exists $m$ such that $|P_0| \leq |P_1| \leq \cdots \leq |P_m| \geq \cdots \geq |P_{r-1}| \geq |P_r|$.

**Definition**

An *order lowering operator* is a linear map $D : \mathbb{C}P \to \mathbb{C}P$ such that

$$D \cdot x = \sum_{y \lessdot x} \mathrm{wt}(y, x) \cdot y, \quad x \in P_i.$$




$$Dx = y + 2z$$

Figure: An example of an order lowering operator.

# The Sperner property (via linear isomorphism)

Recall $P = P_0 \sqcup \cdots \sqcup P_r$ is rank symmetric and rank unimodal.

**Lemma (Stanley 1980)**

*If there exists an order lowering operator $D$ such that*

$$D^{r-2i} : \mathbb{C}P_{r-i} \to \mathbb{C}P_i$$

*is an isomorphism for any $0 \leq i \leq \lfloor r/2 \rfloor$, then $P$ is strongly Sperner.*

Together with the hard Lefschetz theorem in algebraic geometry, Stanley proved the following:

**Theorem (Stanley 1980)**

*Let $(W, S)$ be a Coxeter system for which $W$ is a Weyl group. Then the (strong) Bruhat order on $W$ or any parabolic quotient $W^J$ is rank symmetric, rank unimodal and strongly Sperner.*

# The Sperner property (via $\mathfrak{sl}_2$ representations)

## Definition

An $\mathfrak{sl}_2$ representation on $P$ consists of the following data:

- an **order** lowering operator $D : \mathbb{C}P_i \to \mathbb{C}P_{i-1}$, $\forall i$,
- a raising operator $U : \mathbb{C}P_i \to \mathbb{C}P_{i+1}$, $\forall i$, ($U$ doesn't need to respect the order)
- a modified rank function $H : \mathbb{C}P_i \to \mathbb{C}P_i$, $x \mapsto (2i - r)x$,

such that $UD - DU = H$.

In fact, $U, D, H$ make $\mathbb{C}P$ an $\mathfrak{sl}_2$ representation.

## Theorem (Proctor 1982)

*A ranked poset $P$ admits an $\mathfrak{sl}_2$ representation if and only if $P$ is rank symmetric, rank unimodal and strongly Sperner.*

# The weak and strong Bruhat orders (on $S_n$)

For $w \in S_n$, let $\ell(w)$ denote the usual Coxeter length.
The (right) weak (Bruhat) order $W_n$ is generated by

$$w \lessdot_W ws_i \quad \text{if } \ell(ws_i) = \ell(w) + 1, \text{ where } s_i = (i, i+1).$$

The (strong) Bruhat order $S_n$ is generated by

$$w \lessdot_S wt_{ij} \quad \text{if } \ell(wt_{ij}) = \ell(w) + 1, \text{ where } t_{ij} = (i, j).$$

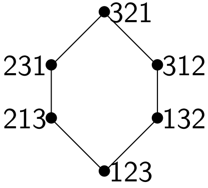



Figure: The weak and strong order on $S_3$.

# The weak and strong Bruhat orders (on $S_n$)

Stanley (1980) showed that the strong Bruhat order (on any Weyl group) is strongly Sperner, and has a symmetric chain decomposition for types $A_n, B_n, D_n$.

Björner (1984) conjectured that the weak Bruhat order is strongly Sperner.

Stanley (2017) suggested an order lowering operator

$$D \cdot w = \sum_{\ell(ws_i)=\ell(w)-1} i \cdot (ws_i).$$

## Conjecture (Stanley 2017)

*For $D$ defined as above, $D^{\binom{n}{2}-2i} : \mathbb{C}(W_n)_{\binom{n}{2}-i} \to \mathbb{C}(W_n)_i$ has nonzero determinant for $0 \leq i \leq \binom{n}{2}/2$. Thus, the weak Bruhat order $W_n$ is strongly Sperner.*

# An $\mathfrak{sl}_2$ action on the weak Bruhat order $W_n$

**Proposition (Gaetz and G. 2018)**

*The following data give an $\mathfrak{sl}_2$ action on $W_n$:*

- *the order lowering operator suggested by Stanley*

$$D \cdot w = \sum_{\ell(ws_i)=\ell(w)-1} i \cdot (ws_i),$$

- *a raising operator defined by*

$$U \cdot w = \sum_{w \lessdot_S u} ||\mathrm{code}(w) - \mathrm{code}(u)||_{L^1} \cdot u,$$

- $H \cdot w = \left(2\ell(w) - \binom{n}{2}\right) \cdot w.$

Recall $\mathrm{code}(w)_i = \{j > i : w(j) < w(i)\}$.

# An $\mathfrak{sl}_2$ action on the weak Bruhat order $W_n$

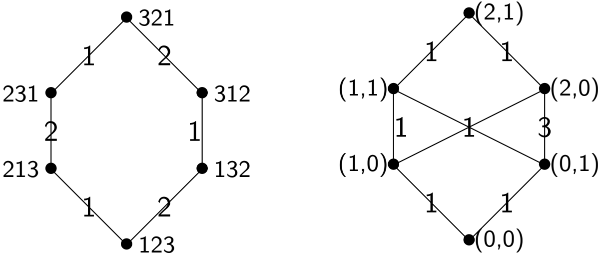



Figure: The order lowering operator $D$ and the raising operator $U$

The (unique) raising operator $U$ that corresponds to $D$ doesn't need to be supported on the strong order. It's just nice combinatorics.

**Corollary (Gaetz and G. 2018)**

*The weak order $W_n$ on the symmetric group is strongly Sperner.*

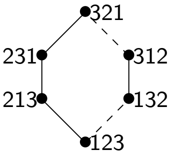
# Open Problems

**Conjecture**

*The weak Bruhat order is strongly Sperner for any Coxeter group.*

**Conjecture**

*The weak Bruhat order of type A has a symmetric chain decomposition.*




**Example (Leclerc 1994)**

The weak order of $H_3$ doesn't have a symmetric chain decomposition, but is strongly Sperner.

# Formulas by Hamaker, Pechenik, Speyer and Weigandt

Hamaker, Pechenik, Speyer and Weigandt resolved the full determinant conjecture by Stanley.

**Theorem (Hamaker et al. 2018, conjectured by Stanley 2017)**

$$\det D^{\binom{n}{2}-2k} = \left(\binom{n}{2} - k\right)!^{\#(W_n)_k} \prod_{i=0}^{k-1} \left( \frac{\binom{n}{2} - k - i}{k - i} \right)^{\#(W_n)_i}$$

# Formulas by Hamaker, Pechenik, Speyer and Weigandt

**Definition (Schubert Polynomials)**

The *Schubert Polynomials* $\mathfrak{S}_w$, for $w \in S_n$, can be defined as follows:

- $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$,
- $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$ if $\ell(w) = \ell(ws_i) - 1$,

where $\partial_i f = (f - s_i f)/(x_i - x_{i+1})$ is the $i$th divided difference operator.

**Proposition (Hamaker et al. 2018)**

*Let $\nabla = \sum_i \partial/\partial x_i$. Then*

$$\nabla \mathfrak{S}_{w^{-1}} = \sum_{i:\ \ell(w)=\ell(ws_i)+1} i \cdot \mathfrak{S}_{s_i w^{-1}}.$$

**Corollary (Macdonald's Identity)**

$$\sum_{\text{reduced } s_{a_1}\cdots s_{a_N} = w_0} a_1 \cdots a_N = \binom{n}{2}!.$$

# Padded Schubert Polynomials

Recall that $\{\mathfrak{S}_w\}_{w \in S_n}$ form a basis of $\mathrm{span}_{\mathbb{C}}\{x^\alpha : \alpha \le \rho\}$ where $\rho = (n-1, \ldots, 1)$ is the staircase partition.

**Definition (Gaetz and G. 2018)**

The *padded Schubert polynomial* $\widetilde{\mathfrak{S}}_w$ is the image of $\mathfrak{S}_w$ under

$$x^\alpha \mapsto x^\alpha y^{\rho - \alpha}.$$

Define the following linear operators

$$\nabla = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} y_i, \qquad \Delta = \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i} x_i.$$

**Proposition (Hamaker et al. 2018; Gaetz and G. 2018)**

1. $\nabla \widetilde{\mathfrak{S}}_{w^{-1}} = \sum_{i:\ \ell(w) = \ell(ws_i)+1} i \cdot \widetilde{\mathfrak{S}}_{s_i w^{-1}}$.
2. $\Delta \widetilde{\mathfrak{S}}_{w^{-1}} = \sum_{u:\ u \ge_S w} ||\mathrm{code}(u) - \mathrm{code}(w)||_{L^1} \cdot \widetilde{\mathfrak{S}}_{u^{-1}}$.

# Padded Schubert polynomials

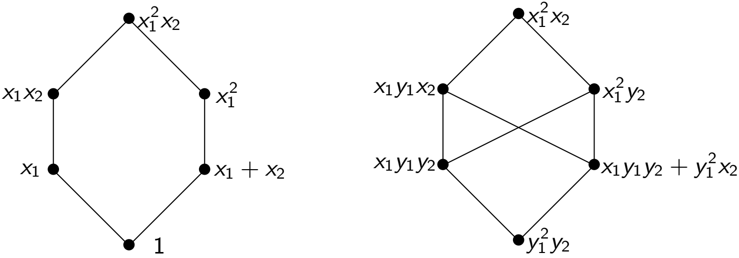



Figure: Schubert polynomials and padded Schubert polynomials on $S_3$

We see that

$$\left(\sum \frac{\partial}{\partial y_i} x_i\right)(x_1y_1y_2+y_1^2x_2) = 3x_1y_1x_2 + x_1^2y_2.$$

# Weights on the strong Bruhat order

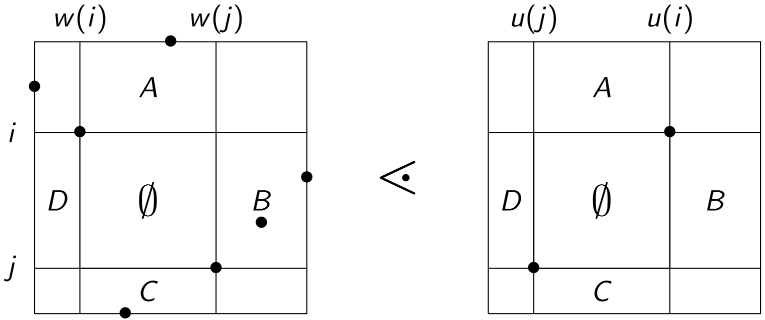



Figure: Weights on the strong Bruhat order

Let $a_{w \lessdot u} = \{k < i : w(i) < w(k) < w(j)\}$ and similarly define $b_{w \lessdot u}$, $c_{w \lessdot u}$ and $d_{w \lessdot u}$.
For example, when $w = 4127653$, $u = 4157623$,

$$a_{w \lessdot u} = 1,\ b_{w \lessdot u} = 2,\ c_{w \lessdot u} = 1,\ d_{w \lessdot u} = 0.$$

# Weighted enumeration of maximal chains

If $\mathrm{wt} : E \to R$ is a weight function on covering relations, where $R$ is a commutative ring, we can define, for $x \le y$,

$$m_{\mathrm{wt}}(x, y) = \sum_{\substack{C: x \to y \\ \text{maximal chain}}} \prod_{e \in C} \mathrm{wt}(e).$$

**Theorem (Gaetz and G. 2019)**

*Let $z_A, z_B, z_C, z_D$ be indeterminates and define a weight function on the covering relations on the strong Bruhat order of $S_n$ as follows:*

$$\mathrm{wt}(w \lessdot u) = 1 + z_A a_{w \lessdot u} + z_B b_{w \lessdot u} + z_C c_{w \lessdot u} + z_D d_{w \lessdot u}.$$

*Then if $\{z_A, z_B, z_C, z_D\} = \{0, 0, z, 2 - z\}$ as multisets,*

$$m_{\mathrm{wt}}(\mathrm{id}, w_0) = \binom{n}{2}!.$$

# Weighted enumeration of maximal chains

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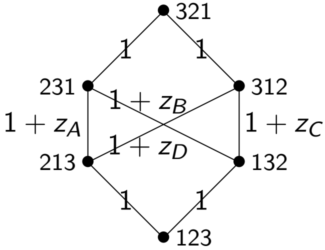


Figure: Weights on covering relations of $S_3$

Then $m_{\mathrm{wt}}(123, 321) = 4 + z_A + z_B + z_C + z_D$, which is $6 = 3!$ if $\{z_A, z_B, z_C, z_D\} = \{0, 0, z, 2 - z\}$.

# Weighted enumeration of maximal chains

**Theorem (Gaetz and G. 2019)**

*Let $z_A, z_B, z_C, z_D$ be indeterminates and define a weight function on the covering relations on the strong Bruhat order of $S_n$ as follows:*

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*Then if $\{z_A, z_B, z_C, z_D\} = \{0, 0, z, 2-z\}$ as multisets,*

$$m_{\mathrm{wt}}(\mathrm{id}, w_0) = \binom{n}{2}!.$$

Special cases:

1. $(z_A, z_B, z_C, z_D) = (0, 1, 0, 1)$, $\mathrm{wt}(w \lessdot wt_{ij}) = j - i$,
2. $(z_A, z_B, z_C, z_D) = (0, 0, 2, 0)$, $\mathrm{wt}(w \lessdot u) = ||\mathrm{code}(w) - \mathrm{code}(u)||_{L^1}$.

The "$j-i$" weight is commonly known as the Chevalley weight, which is investigated by Stembridge (2002) and further by Postnikov and Stanley (2009). It is still open to find a combinatorial proof.

# References


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# Thanks

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Thank you for listening!