ON SOME PARTIALLY DE RHAM GALOIS REPRESENTATIONS

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Abstract. In this note, we study some partially de Rham representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ for a finite Galois extension L of $\mathbb{Q}_p$. We study some related subspaces of Galois cohomology and cohomology of B-pairs. We prove partial non-criticalness implies partial de Rhamness for trianguline representations. As an application, we obtain a partial classicality result (in terms of Galois representations) for overconvergent Hilbert modular forms. We also associate a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(L)$ to a 2-dimensional (generic) trianguline $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$-representation, which generalizes some of Breuil's theory in crystalline case [8].

Contents

Introduction

Let L be a finite Galois extension of $\mathbb{Q}_p$ of degree d, E a finite extension of $\mathbb{Q}_p$ containing all the embeddings of L in $\overline{\mathbb{Q}_p}$, denote by Σ_L the set of embeddings of L in $\overline{\mathbb{Q}_p}$ (hence in E). Let V be a finite dimensional continuous representation of $G_L := \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ over E, by Fontaine's theory, one can associate to V an $L \otimes_{\mathbb{Q}_p} E$-module $D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_L}$. Using the isomorphism

$$L \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma \in \Sigma_L} E, \ a \otimes b \mapsto (\sigma(a)b)_{\sigma \in \Sigma_L},$$

one can decompose $D_{\mathrm{dR}}(V)$ as $D_{\mathrm{dR}}(V) \xrightarrow{\sim} \prod_{\sigma \in \Sigma_L} D_{\mathrm{dR}}(V)_\sigma$. For $\sigma \in \Sigma_L$, we say that V is *σ-de Rham* if $\dim_E D_{\mathrm{dR}}(V)_\sigma = \dim_E V$ (see also [31]). More generally, for $J \subseteq \Sigma_L$, V is called *J-de Rham* if V is σ-de Rham for all $\sigma \in J$.

Partially de Rham G_L-representations appear naturally in the study of p-adic families of G_L-representations (e.g. see [29] and [12]), and may play a role in the p-adic Langlands program (cf. [6]), initialized by Breuil. In fact, for a locally $\mathbb{Q}_p$-analytic representation V of $\mathrm{GL}_2(L)$ (or other p-adic L-analytic groups) over E and for any $J \subseteq \Sigma_L$, one can consider the *J-classical* vectors of V (cf. §4.2, for example, all the locally J-analytic vectors (cf. [28, §2]) are $\Sigma_L \setminus J$-classical, the locally algebraic vectors are Σ_L-classical). In the author's thesis ([12]), we studied eigenvarieties X constructed from (the locally $\mathbb{Q}_p$-analytic vectors of) the completed cohomology group $\widehat{H}^1$ (cf. [14]) of unitary Shimura curves. It turned out that for any $J \subseteq \Sigma_L$, $J \neq \emptyset$, there exists a closed subspace X_J of X associated to the J-classical vectors of $\widehat{H}^1$ (cf. [12, §6.2.1]). Moreover, we showed that the (2-dimensional) Galois representations associated to points in X_J (which we call *J-classical* points) are always J-de Rham (cf. [12, Prop. 6.2.40]). We conjecture that the inverse is also true, i.e. for a closed point z of X, if the associated Galois representation is J-de Rham and the weight of z is J-dominant, then z is J-classical (cf. [12, Conj.6.2.41]). For example, when $J = \Sigma_L$, this is implied by Fontaine-Mazur conjecture. The "relation" between J-classical vectors and J-de Rham Galois representations is rather a new phenomenon in the p-adic Langlands program for $L \neq \mathbb{Q}_p$. Besides, the J-de Rham Galois representations may also be useful in the study of p-adic Hilbert modular forms (e.g. see Thm.0.5).

In this note, using a Lubin-Tate version of p-adic Hodge theory developed in [17], we study some related subspaces of Galois cohomology and of cohomology of B-pairs (or equivalently cohomology of (φ, Γ)-modules by [3]), and thus some partially de Rham trianguline representations. We summarize some results in 2-dimensional case in the introduction.

Firstly consider extensions of Galois characters. Let χ_1, χ_2 be two de Rham characters of G_L over E, of respective Hodge-Tate weights $(-k_{\chi_1,\sigma})_{\sigma \in \Sigma_L}$, $(-k_{\chi_2,\sigma})_{\sigma \in \Sigma_L}$ (where we use the convention that the Hodge-Tate weight of the p-adic cyclotomic character χ_{cyc} is -1). Consider the (finite dimensional) E-vector space $\mathrm{Ext}^1(\chi_2, \chi_1)$ of G_L-extensions of χ_2 by χ_1. For $J \subseteq \Sigma_L$, we put

$$\mathrm{Ext}^1_{g,J}(\chi_2, \chi_1) := \big\{[V] \in \mathrm{Ext}^1(\chi_2, \chi_1) \mid V \text{ is } J\text{-de Rham}\big\} \tag{1}$$

which is an E-vector subspace of $\mathrm{Ext}^1(\chi_2, \chi_1)$. Suppose $D_{\mathrm{cris}}(\chi_2 \chi_1^{-1} \chi_{\mathrm{cyc}})^{\varphi=1} = 0$, then one has

Theorem 0.1 (cf. §1). $\dim_E \mathrm{Ext}^1_{g,J}(\chi_2, \chi_1) = \dim_E \mathrm{Ext}^1(\chi_2, \chi_1) - \big|\{\sigma \in J \mid k_{\chi_1,\sigma} - k_{\chi_2,\sigma} < 1\}\big|$.

Example 0.2. *Let $\chi : G_L \to L^\times$ be a Lubin-Tate character, $\sigma \in \Sigma_L$, we put $\chi_\sigma := \sigma \circ \chi : G_L \to E^\times$. By this theorem, $\mathrm{Ext}^1_{g,\sigma}(E, \chi_\sigma) = \mathrm{Ext}^1(E, \chi_\sigma)$, thus any extension of the trivial character of G_L by χ_σ is σ-de Rham, which generalizes the well-known result: extensions of the trivial character by cyclotomic character are always de Rham.*

One can also define *J-de Rham* (φ, Γ)-modules over the Robba ring $\mathcal{R}_E := B^\dagger_{\mathrm{rig},L} \otimes_{\mathbb{Q}_p} E$ (or equivalently *J-de Rham E-B-pairs*, cf. Def. 3.10). For a continuous character χ of $L^\times$ over E, we denote by $(k_{\chi,\sigma})_{\sigma \in \Sigma_L} \in E^d$ the weights of χ (cf. §3.3) and $\mathcal{R}_E(\chi)$ the associated rank 1 (φ, Γ)-module over $\mathcal{R}_E$ (cf. [24, §1.4], $k_{\chi,\sigma}$ is in fact the inverse of the generalized Hodge-Tate weights of $\mathcal{R}_E(\chi)$). Let χ_1, χ_2 be two continuous characters of $L^\times$ over E, denote by $\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\chi_2), \mathcal{R}_E(\chi_1)\big)$ the (finite dimensional) E-vector space of extensions (of (φ, Γ)-modules over $\mathcal{R}_E$) of $\mathcal{R}_E(\chi_2)$ by $\mathcal{R}_E(\chi_1)$.

Suppose $k_{\chi_i,\sigma} \in \mathbb{Z}$ for all $\sigma \in \Sigma_L$, $i = 1, 2$, and for $J \subseteq \Sigma_L$, denote by $\mathrm{Ext}^1_{(\varphi,\Gamma),g,J}\big(\mathcal{R}_E(\chi_2), \mathcal{R}_E(\chi_1)\big)$ the E-vector subspace of $\mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\chi_2), \mathcal{R}_E(\chi_1)\big)$ generated by *J-de Rham* extensions of $\mathcal{R}_E(\chi_2)$ by $\mathcal{R}_E(\chi_1)$. Suppose moreover $\chi_2\chi_1^{-1} \neq \mathrm{unr}(q)\prod_{\sigma\in\Sigma_L}\sigma^{k_\sigma}$ for any $(k_\sigma)_{\sigma\in\Sigma_L} \in \mathbb{Z}^d$ (where $\mathrm{unr}(z)$ denotes the unramified character of $L^\times$ sending uniformizers to z).

Theorem 0.3 (cf. Cor.3.20)**.** *Keep the above notation, one has*

$$\dim_E \mathrm{Ext}^1_{(\varphi,\Gamma),g,J}\big(\mathcal{R}_E(\chi_2), \mathcal{R}_E(\chi_1)\big) = \dim_E \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\chi_2), \mathcal{R}_E(\chi_1)\big) - \big|\{\sigma \in J \mid k_{\chi_1,\sigma} - k_{\chi_2,\sigma} < 1\}\big|.$$

Let ρ be a 2-dimensional trianguline G_L-representation (cf. [11]) over E such that $[D_{\mathrm{rig}}(\rho)] \in \mathrm{Ext}^1_{(\varphi,\Gamma)}\big(\mathcal{R}_E(\chi_2), \mathcal{R}_E(\chi_1)\big)$ (where $D_{\mathrm{rig}}(\rho)$ denotes the (φ,Γ)-module associated to ρ, and the characters χ_1, χ_2 are as above), for $\sigma \in \Sigma_L$, we call ρ *non-σ-critical* if $k_{\chi_1,\sigma} - k_{\chi_2,\sigma} \in \mathbb{Z}_{\leq 0}$ (cf. Def.3.23). We can thus deduce from Thm.0.3:

Corollary 0.4 (cf. Prop.3.24)**.** *Let $\sigma \in \Sigma_L$, if ρ is non-σ-critical, then ρ is σ-de Rham.*

This corollary generalizes the "non-critical trianguline representations are de Rham" result, which might be seen as a Galois version of Coleman's classicality result: "small slope modular forms are classical". Indeed, this corollary allows to get a (Galois version of) partial classicality result of Hilbert modular forms: "partially small slope Hilbert modular forms are partially classical", giving evidence to Breuil's conjectures in [7]:

Let F be a totally real number field with p inert (for simplicity), $w \in \mathbb{Z}$ and $k_\sigma \in \mathbb{Z}_{\geq 2}$, $k_\sigma \equiv w \pmod 2$, for all $\sigma \in \Sigma_{F_p}$, h an overconvergent Hilbert eigenform of weights $(\underline{k}_{\Sigma_{F_p}}; w)$ (where we adopt Carayol's convention of weights as in [10]) of tame level N ($N \geq 4$, $p \nmid N$) with the U_p-eigenvalue $a_p \in E^\times$. Denote by ρ_h the associated Galois representation of $\mathrm{Gal}(\overline{F}/F)$, and $\upsilon_p(\cdot)$ the additive valuation on $\overline{\mathbb{Q}_p}$ normalized by $\upsilon_p(p) = 1$.

Theorem 0.5 (cf. §4.1)**.** *Keep the above notation, and let $\emptyset \neq S \subseteq \Sigma_{F_p}$.*

(1) If $\upsilon_p(a_p) < \inf_{\sigma\in S}\{k_\sigma - 1\} + \sum_{\sigma\in\Sigma_\wp}\frac{w-k_\sigma+2}{2}$, then $\rho_{h,p} := \rho_h|_{G_{F_p}}$ is S-de Rham.

(2) If $\upsilon_p(a_p) < \sum_{\sigma\in S}(k_\sigma - 1) + \sum_{\sigma\in\Sigma_\wp}\frac{w-k_\sigma+2}{2}$, then there exists $\sigma \in S$ such that $\rho_{h,p}$ is σ-de Rham.

Note that the statement in (1) in the case $S = \Sigma_{F_p}$ (and F_p unramified) is implied by the classicality results of [30, Thm.1] (note that the convention of weights in *loc. cit.* is slightly different from ours).

The notion "partially de Rham" allows to get a more precise classification of trianguline representations. As a result, in §4.2, we associate a semi-simple locally $\mathbb{Q}_p$-analytic representation $\Pi(\rho)$ of $\mathrm{GL}_2(L)$ to a 2-dimensional trianguline representation ρ of $\mathrm{Gal}(\overline{L}/L)$, generalizing some of Breuil's theory in crystalline case [8]. We expect $\Pi(\rho)$ to be the socle of the "right" representation associated to ρ in the p-adic Langlands program (cf. Conj.4.9). The corollary 0.4 plays an important role in the construction of $\Pi(\rho)$ (see Rem.4.6 (2)). A main philosophy in our construction is that if ρ is σ-de Rham, then $\Pi(\rho)$ should have σ-classical vectors, which more or less motivates our study of σ-de Rham Galois representations.

The results in this paper also find applications in establishing Colmez-Greenberg-Stevens formulas (on $\mathcal{L}$-invariants) in critical case [13], thus might find applications in p-adic L-functions in $L \neq \mathbb{Q}_p$ case. Besides, these results would also be useful for investigation of local behavior of the eigenvarieties constructed in [12] (or of certain deformation spaces), which we leave for future work.

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1. Notations and some p-adic Hodge theory

Recall some results in p-adic Hodge theory and introduce some notations, our main reference is [17]. Let L be a finite Galois extension of $\mathbb{Q}_p$ of degree d with $\mathcal{O}_L$ the ring of integers and ϖ_L a uniformizer, L_0 the maximal unramified sub-extension of $\mathbb{Q}_p$ in L of degree d_0 with $\mathcal{O}_{L_0}$ the ring of integers, $q := |\mathcal{O}_L/\varpi_L| = p^{d_0}$, Σ_L the set of $\mathbb{Q}_p$-embeddings of L inside $\overline{\mathbb{Q}_p}$, $\upsilon : \mathbb{C}_p \to \mathbb{Z} \cup \{+\infty\}$ the p-adic additive valuation on $\mathbb{C}_p$ normalized by sending ϖ_L to 1.

Recall Fontaine's ring

$$R := \big\{(x_n)_{n\in\mathbb{Z}_{\geq 0}} \mid x_{n+1}^p = x_n,\ x_n \in \mathcal{O}_{\mathbb{C}_p}/p,\ \forall n \in \mathbb{Z}_{\geq 0}\big\},$$

which is of characteristic p, equipped with a natural action of $G_{\mathbb{Q}_p}$: $g((x_n)_{n\in\mathbb{Z}_{\geq 0}}) = (g(x_n)_{n\in\mathbb{Z}_{\geq 0}})$. Let $x = (x_n)_{n\in\mathbb{Z}_{\geq 0}} \in R$, let $\widetilde{x_n} \in \mathcal{O}_{\mathbb{C}_p}$ be an arbitrary lift of x_n for all n, one can prove the sequence $\{\widetilde{x_n}^{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$ converges in $\mathcal{O}_{\mathbb{C}_p}$, and put $x^{(0)} := \lim_{n\to+\infty} \widetilde{x_n}^{p^n}$. The map $R \to \mathcal{O}_{\mathbb{C}_p}$, $x \mapsto x^{(0)}$ is multiplicative (and $G_{\mathbb{Q}_p}$-invariant), and we define a valuation $\upsilon_R : R \to \mathbb{Q}_{\geq 0} \cup \{+\infty\}$ with $\upsilon_R(x) := \upsilon(x^{(0)})$. Let $\varepsilon := (\zeta_{p^n})_{n\in\mathbb{Z}_{\geq 0}} \in R$, where ζ_{p^n} is a primitive p^n-th root of unity.

Let $W(R)$ be the Witt ring of R, and $W_{\mathcal{O}_L}(R) := W(R)\otimes_{\mathcal{O}_{L_0}} \mathcal{O}_L$. One can also construct $W_{\mathcal{O}_L}(R)$ more intrinsically by the theory of Witt $\mathcal{O}_L$-vectors (cf. [17, §5.1]). Any element $x \in W_{\mathcal{O}_L}(R)$ can be uniquely written as

$$x = \sum_{n\in\mathbb{Z}_{\geq 0}} [x_n]\varpi_L^n$$

where $[\cdot] : R \to W_{\mathcal{O}_L}(R)$ denotes the Teichimüller lifting. The Witt $\mathcal{O}_L$-ring $W_{\mathcal{O}_L}(R)$ is naturally equipped with an action of G_L and φ_L (which commutes):

$$\begin{aligned} g\Big(\sum_{n\in\mathbb{Z}_{\geq 0}} [x_n]\varpi_L^n\Big) &:= \sum_{n\in\mathbb{Z}_{\geq 0}} [g(x_n)]\varpi_L^n,\ \forall g \in G_L, \\ \varphi_L\Big(\sum_{n\in\mathbb{Z}_{\geq 0}} [x_n]\varpi_L^n\Big) &:= \sum_{n\in\mathbb{Z}_{\geq 0}} [x_n^q]\varpi_L^n. \end{aligned}$$

Let $B_L^{+,b} := W_{\mathcal{O}_L}(R)\big[\frac{1}{p}\big]$, thus an element $x \in B_L^{+,b}$ can be uniquely written as $x = \sum_{n\gg-\infty}[x_n]\varpi_L^n$. The action of G_L and φ_L on $W_{\mathcal{O}_L}(R)$ extends naturally to $B_L^{+,b}$. For any $r \in \mathbb{R}_{\geq 0}$, one has a valuation υ_r on $B_L^{+,b}$:

$$\upsilon_r : B_L^{+,b} \to \mathbb{R} \cup \{+\infty\},\ \upsilon_r\Big(\sum_{n\gg-\infty} [x_n]\varpi_L^n\Big) := \inf_{n\in\mathbb{Z}}\{\upsilon_R(x_n) + rn\}.$$

Denote by $B_{r,L}^+$ the completion of $B_L^{+,b}$ via the valuation υ_r, which turns out to be a Banach space over L, moreover, for $r' \geq r$, one has a natural inclusion $B_{r',L}^+ \subseteq B_{r,L}^+$ (cf. [17, §5.2.3]). Put $B_L^+ := \cap_{r>0} B_{r,L}^+$. All these L-algebras are equipped with a natural action of G_L and φ_L $\big($which extends that on $B_L^{+,b}\big)$, moreover $\varphi_L(B_{r,L}^+) = B_{qr,L}^+$ (cf. [17, §5.2.3]), thus φ_L is bijective on B_L^+. One has $B_L^+ \cong B_{\mathbb{Q}_p}^+ \otimes_{L_0} L$, and $\varphi_L = \varphi_{\mathbb{Q}_p}^{d_0} \otimes \mathrm{id}$ respectively (cf. [17, §5.2.4]).

Put $P_{L,\varpi_L} := \bigoplus_{n\geq 0}(B_L^+)^{\varphi_L=\varpi_L^n}$ (cf. [17, Def.9.1]), which is a graded L-algebra, and let $X_L := \mathrm{Proj}(P_{L,\varpi_L})$ (cf. [17, Def.10.1], which turns out to be independent of the choice of ϖ_L, see the discussion after [17, Def.10.1]). By [17, Thm.10.2 (1)], X_L is a completed curve defined over L, and $X_L \cong X_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$. One has a one-to-one correspondence between L-lines in $(B_L^+)^{\varphi_L=\varpi_L}$ and closed points of X_L (cf. [17, Thm.10.2 (4)]).

Consider the $\mathbb{Q}_p$-case, let $t = \log([\varepsilon]) = \sum_{n\geq 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \in (B_{\mathbb{Q}_p}^+)^{\varphi_{\mathbb{Q}_p}=p}$, which is usual cyclotomic period element $\big(g(t) = \chi_{\mathrm{cyc}}(g)t$ for $g \in G_{\mathbb{Q}_p}\big)$ and corresponds to a closed point ∞ of $X_{\mathbb{Q}_p}$. The completion of the curve $X_{\mathbb{Q}_p}$ at the point ∞ is Fontaine's ring B_{dR}^+. Let $B_e := \big(B_{\mathbb{Q}_p}^+\big[\frac{1}{t}\big]\big)^{\varphi_{\mathbb{Q}_p}=\mathrm{id}} = B_{\mathrm{cris}}^{\varphi=\mathrm{id}}$, by [17, Thm.10.2 (6)], one has $D^+(t) := X_{\mathbb{Q}_p} \setminus \{\infty\} = \mathrm{Spec}(B_e)$, B_e is a PID. There is an exact sequence ($G_{\mathbb{Q}_p}$-invariant)

$$0 \to \mathbb{Q}_p \to B_e \oplus B_{\mathrm{dR}}^+ \xrightarrow{(x,y)\mapsto x-y} B_{\mathrm{dR}} \to 0$$

which is the so-called *p-adic fundamental exact sequence*.

Consider the natural covering $\pi : X_L \to X_{\mathbb{Q}_p}$, one sees $\pi^{-1}(\infty)$ is of cardinality d. We fix an embedding $\iota : L \hookrightarrow B_{\mathrm{dR}}^+$ in this note (thus an embedding $\iota : L \hookrightarrow \mathbb{C}_p$), in other words, we fix a closed point $\infty_\iota \in \pi^{-1}(\infty)$, which induces an embedding $\iota : B_L^+ \hookrightarrow B_{\mathrm{dR}}^+$. Let $t_L \in (B_L^+)^{\varphi_L=\varpi}$ be the element corresponding to ∞_ι (which is unique up to scalars in $L^\times$), thus $t_L \in tB_{\mathrm{dR}}^+ \setminus t^2B_{\mathrm{dR}}^+$ (cf. [17, Thm.10.2 (5)]). Let $B_{e,L} := \big(B_L^+\big[\frac{1}{t_L}\big]\big)^{\varphi_L=\mathrm{id}}$, by [17], $D^+(\infty_\iota) = \mathrm{Spec}\, B_{e,L}$, and $B_{e,L}$ is a PID. The L-line $L\cdot t_L$ is stable by G_L. One gets in fact a Lubin-Tate character $\chi_{\mathcal{LT}} : G_L \to L^\times$, $\chi_{\mathcal{LT}}(g) = g(t_L)/t_L$. One has a G_L-invariant exact sequence

$$0 \to L \to B_{e,L} \oplus B_{\mathrm{dR}}^+ \xrightarrow{(x,y)\mapsto x-y} B_{\mathrm{dR}} \to 0, \tag{2}$$

called *the p-adic fundamental exact sequence of* $B_{e,L}$. Indeed, it's clear that $L \subseteq B_{e,L} \cap B_{\mathrm{dR}}^+$; let $x \in B_{e,L} \cap B_{\mathrm{dR}}^+$ with $z := xt_L^n \in B_L^+$, thus $z \in (B_L^+)^{\varphi_L=\varpi_L^n} \cap t_L^nB_{\mathrm{dR}}^+$, which is zero if $n \geq 1$ by [17, Thm.9.10], one can thus assume $n = 0$, so $x \in (B_L^+)^{\varphi_L=\mathrm{id}} = L$ (cf. [17, Prop.7.1]). To see $B_{\mathrm{dR}} = B_{e,L} + B_{\mathrm{dR}}^+$, one can use induction on $t_L^{-n}B_{\mathrm{dR}}^+$ for $n \in \mathbb{Z}_{\geq 0}$: suppose $t_L^{-n}B_{\mathrm{dR}}^+ \subseteq B_{e,L} + B_{\mathrm{dR}}^+$, let $x \in t_L^{-(n+1)}B_{\mathrm{dR}}^+ \setminus t_L^{-n}B_{\mathrm{dR}}^+$, thus $x = t_L^{-(n+1)}y$ with $y \in B_{\mathrm{dR}}^+ \setminus t_LB_{\mathrm{dR}}^+$. By [17, Thm.9.10], there exists $z \in (B_L^+)^{\varphi_L=\varpi_L^{n+1}} \subset t_L^{n+1}B_{e,L}$ and $\lambda \in t_LB_{\mathrm{dR}}^+$ such that $y = z + \lambda$, thus $x = t_L^{-(n+1)}(z+\lambda) \in B_{e,L} + t_L^{-n}B_{\mathrm{dR}}^+ = B_{e,L} + B_{\mathrm{dR}}^+$ (by assumption).

Since $X_L \cong X_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$, for any affine subset U of $X_{\mathbb{Q}_p}$, one can equip $\mathcal{O}(\pi^{-1}(U)) \cong \mathcal{O}(U) \otimes_{\mathbb{Q}_p} L$ with an action of $\mathrm{Gal}(L/\mathbb{Q}_p)$ given by $\tau(u \otimes \lambda) := u \otimes \tau(\lambda)$ for $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, $u \in \mathcal{O}(U)$ and $\lambda \in L$. This action induces a regular $\mathrm{Gal}(L/\mathbb{Q}_p)$-action on $\pi^{-1}(z)$ for each closed point z of $X_{\mathbb{Q}_p}$ (cf. [17, Thm.10.2 (7)]). For $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, let $t_L^\tau \in (B_L^+)^{\varphi_L=\varpi}$ be the element (up to scalars in $L^\times$) corresponding to the closed point $\tau(\infty_\iota)$ (by [17, Thm.10.2 (4)]). If $\tau \neq 1$, then $\tau(\infty_\iota) \neq \infty_\iota$, thus $t_L^\tau \in (B_{\mathrm{dR}}^+)^\times$ $\big($since $\infty_\tau \notin V^+(t_L^\tau)$, recall we have fixed the embedding $\iota : B_L^+ \hookrightarrow B_{\mathrm{dR}}^+\big)$. Let $B'_{e,L} := B_e \otimes_{\mathbb{Q}_p} L$, so $X_L \setminus \pi^{-1}(\infty) = D^+(\pi^{-1}(\infty)) \cong \mathrm{Spec}\, B'_{e,L}$. The embedding ι induces also an embedding $\iota : B'_{e,L} \hookrightarrow B_{\mathrm{dR}}$. In the following, we view $B_{e,L'}$, $B_{e,L}$ etc. as L-subalgebras of B_{dR} via ι with no further mention.

Let $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau \neq 1$, the closed point $\tau(\infty_\iota)$ of $D^+(t_L)$ is defined by the maximal ideal $\mathfrak{m}_\tau := \big(\frac{t_L^\tau}{t_L}\big)$ of $B_{e,L}$. Since $D^+(\pi^{-1}(\infty)) = D^+(\infty_\iota) \setminus \{\tau(\infty_\iota)\}_{1\neq\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)}$, $B'_{e,L}$ is the localization of $B_{e,L}$ by inverting the elements $\{t_L^\tau/t_L\}_{1\neq\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)}$. For $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau \neq 1$, denote by $\mathrm{ord}_{\tau(\infty_\iota)} : B_{e,L} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ the valuation defined by $\frac{t_L^\tau}{t_L}$, which extends naturally to a valuation

$\mathrm{ord}_{\tau(\infty_\iota)}: B'_{e,L} \to \mathbb{Z} \cup \{+\infty\}$. Note that for $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau \neq 1$

$$\mathrm{ord}_{\tau(\infty_\iota)}(t_L^{\tau'}/t_L) = \begin{cases} 0 & \tau' \neq \tau \\ 1 & \tau' = \tau \end{cases}. \tag{3}$$

One has $B_{e,L} = \{x \in B'_{e,L} \mid \mathrm{ord}_{\tau(\infty_\iota)}(x) \geq 0, \ \forall \tau \in \mathrm{Gal}(L/\mathbb{Q}_p), \ \tau \neq 1\}$. In other words, one has

Lemma 1.1. $B_{e,L} = \{x \in B'_{e,L} \mid \tau(x) \in B_{\mathrm{dR}}^+, \ \forall \tau \in \mathrm{Gal}(L/\mathbb{Q}_p), \ \tau \neq 1\}$.

Let E be a finite extension of $\mathbb{Q}_p$ which contains all the embeddings of L in $\overline{\mathbb{Q}_p}$. For an L-algebra B (e.g. $B = B_{e,L}, B'_{e,L}$ etc.) and $\sigma \in \Sigma_L$, set $B_\sigma := B \otimes_{L,\sigma} E$. So we have

$$B \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} B \otimes_L (L \otimes_{\mathbb{Q}_p} E) \xrightarrow{\sim} B \otimes_L (\prod_{\sigma \in \Sigma_L} E) \xrightarrow{\sim} \prod_{\sigma \in \Sigma_L} B_\sigma. \tag{4}$$

For a $B \otimes_{\mathbb{Q}_p} E$-module M, using the above isomorphism, we can decompose M as

$$M \xrightarrow{\sim} \prod_{\sigma \in L} M_\sigma \tag{5}$$

where $M_\sigma \cong M \otimes_{B \otimes_{\mathbb{Q}_p} E} B_\sigma$ for $\sigma \in \Sigma_L$.

The $\mathrm{Gal}(L/\mathbb{Q}_p)$-action on $B'_{e,L}$ induces a $\mathrm{Gal}(L/\mathbb{Q}_p)$-action on $B'_{e,L} \otimes_{\mathbb{Q}_p} E$. Moreover, for any $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$, τ induces an isomorphism $\tau : B'_{e,L,\sigma} \xrightarrow{\sim} B'_{e,L,\sigma\circ\tau^{-1}}$. One can deduce easily from Lem.1.1:

Lemma 1.2. *Let* $\sigma \in \Sigma_L$, *then* $B_{e,L,\sigma} = \{x \in B'_{e,L,\sigma} \mid \tau(x) \in B^+_{\mathrm{dR},\sigma\circ\tau^{-1}}, \ \forall \tau \in \mathrm{Gal}(L/\mathbb{Q}_p), \ \tau \neq 1\}$.

Let V be a finite dimensional continuous representation of G_L over E, set $D_*(V) := (B_* \otimes_{\mathbb{Q}_p} V)^{G_L}$ where $* \in \{\mathrm{dR}, e, \{e, L\}\}$. When B_* is moreover an L-algebra, $D_*(V)$ is an $L \otimes_{\mathbb{Q}_p} E$-module, and we have $D_*(V)_\sigma \xrightarrow{\sim} (B_{*,\sigma} \otimes_E V)^{G_L}$ for any $\sigma \in \Sigma_L$. Put $D'_{e,L}(V) := (B'_{e,L} \otimes_{\mathbb{Q}_p} V)^{G_L}$, by the isomorphism $B'_{e,L} \cong B_e \otimes_{\mathbb{Q}_p} L$, one sees $D'_{e,L}(V)_\sigma \cong D_e(V)$ as E-vector spaces for all $\sigma \in \Sigma_L$. There also exists a $\mathrm{Gal}(L/\mathbb{Q}_p)$-action on $D'_{e,L}(V)$ (induced by that on $B_{e,L'}$), and one can easily deduce from Lem.1.1 the

Lemma 1.3. $D_{e,L}(V) = \{v \in D'_{e,L}(V) \mid \tau(v) \in D^+_{\mathrm{dR}}(V)$ *for all* $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p), \ \tau \neq 1\}$.

2. Galois cohomology

2.1. **Bloch-Kato Selmer groups.** Let V be a finite dimensional continuous representation of G_L over E, put (cf. [5, §3]):

$$\begin{aligned}
H^1_e(G_L, V) &:= \mathrm{Ker}\big[H^1(G_L, V) \to H^1(G_L, V \otimes_{\mathbb{Q}_p} B_e)\big],\\
H^1_g(G_L, V) &:= \mathrm{Ker}\big[H^1(G_L, V) \to H^1(G_L, V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})\big],\\
H^1_{e,\sigma}(G_L, V) &:= \mathrm{Ker}\big[H^1(G_L, V) \to H^1(G_L, V \otimes_E B_{e,L,\sigma})\big],\\
H^1_{g,\sigma}(G_L, V) &:= \mathrm{Ker}\big[H^1(G_L, V) \to H^1(G_L, V \otimes_E B_{\mathrm{dR},\sigma})\big],\\
H^1_{e',\sigma}(G_L, V) &:= \mathrm{Ker}\big[H^1(G_L, V) \to H^1(G_L, V \otimes_E B'_{e,L,\sigma})\big].
\end{aligned}$$

By definition, one has natural injections $H^1_e(G_L, V) \hookrightarrow H^1_g(G_L, V)$, $H^1_{e,\sigma}(G_L, V) \hookrightarrow H^1_{g,\sigma}(G_L, V)$, $H^1_{e,\sigma}(G_L, V) \hookrightarrow H^1_{e',\sigma}(G_L, V)$. Since $B'_{e,L} \cong B_e \otimes_{\mathbb{Q}_p} L$, one easily deduces $H^1_e(G_L, V) \xrightarrow{\sim} H^1_{e',\sigma}(G_L, V)$ for any $\sigma \in \Sigma_L$. Thus for $\sigma \in \Sigma_L$, one gets a natural injection

$$j_\sigma : H^1_{e,\sigma}(G_L, V) \longrightarrow H^1_e(G_L, V). \tag{6}$$

By the isomorphism $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \cong \prod_{\sigma\in\Sigma_L} B_{\mathrm{dR},\sigma} \otimes_E V$, one sees

$$\cap_{\sigma\in\Sigma_L} H^1_{g,\sigma}(G_L, V) = H^1_g(G_L, V).$$

Put $H^1_{g,J}(G_L, V) := \cap_{\sigma\in J} H^1_{g,\sigma}(G_L, V)$ for $J \subseteq \Sigma_L$, $J \neq \emptyset$. Assume that V is J-de Rham (i.e. $\dim_E D_{\mathrm{dR}}(V)_\sigma = \dim_E V$ for all $\sigma \in J$, see the introduction), and let $[W] \in H^1(G_L, V)$, where W is an extension of E (with the trivial action of G_L) by V:

$$0 \to V \to W \to E \to 0,$$

thus $[W] \in H^1_{g,J}(G_L, V)$ if and only if W is J-de Rham. Put $H^1_{e,J}(G_L, V) := \mathrm{Im}(\sum_{\sigma\in J} j_\sigma) \subseteq H^1_e(G_L, V) \subseteq H^1(G_L, V)$ (cf. (6)). For $\emptyset \neq J \subseteq J'$, one has thus

$$0 \subseteq H^1_{e,J}(G_L, V) \subseteq H^1_{e,J'}(G_L, V) \subseteq H^1_e(G_L, V) \\ \subseteq H^1_g(G_L, V) \subseteq H^1_{g,J'}(G_L, V) \subseteq H^1_{g,J}(G_L, V) \subseteq H^1(G_L, V).$$

Let $\sigma \in \Sigma_L$, by taking tensor products $- \otimes_{L,\sigma} E$, one deduces from (2) an exact sequence

$$0 \to E \to B_{e,L,\sigma} \oplus B^+_{\mathrm{dR},\sigma} \to B_{dR,\sigma} \to 0, \tag{7}$$

tensoring with V, one gets

$$0 \to V \to B_{e,L,\sigma} \otimes_E V \oplus B^+_{\mathrm{dR},\sigma} \otimes_E V \to B_{\mathrm{dR},\sigma} \otimes_E V \to 0. \tag{8}$$

By taking Galois cohomology, one gets

$$0 \to V^{G_L} \to D_{e,L}(V)_\sigma \oplus D^+_{\mathrm{dR}}(V)_\sigma \to D_{\mathrm{dR}}(V)_\sigma \xrightarrow{\delta_\sigma} H^1(G_L, V) \\ \to H^1(G_L, B_{e,L,\sigma} \otimes_E V) \oplus H^1(G_L, B^+_{\mathrm{dR},\sigma} \otimes_E V) \to H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E V), \tag{9}$$

By the same argument as in [5, Lem.3.8.1], one has

Lemma 2.1. *Suppose V is σ-de Rham, then $H^1(G_L, B^+_{\mathrm{dR},\sigma} \otimes_E V) \to H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E V)$ is injective.*

Consequently, in this case, one deduces from (9) an exact sequence

$$0 \to V^{G_L} \to D_{e,d}(V)_\sigma \oplus D^+_{\mathrm{dR}}(V)_\sigma \to D_{\mathrm{dR}}(V)_\sigma \xrightarrow{\delta_\sigma} H^1_{e,\sigma}(G_L, V) \to 0. \tag{10}$$

2.2. **Tate duality.** We use "$\cup$" to denote the cup-products (in Galois cohomology). For a finite dimensional continuous G_L-representation V over E, we denote by $V^\vee$ the dual representation, and by $V(1)$ the twist of V by the p-adic cyclotomic character. Let $\sigma \in \Sigma_L$, we denote by $\cup_\sigma$ the composition

$$\cup_\sigma : H^1(G_L, V) \times D_{\mathrm{dR}}(V^\vee(1))_\sigma \longrightarrow H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E V) \times D_{\mathrm{dR}}(V^\vee(1))_\sigma \\ \xrightarrow{\cup} H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E E(1)).$$

We have the following commutative diagram (see also [5, (3.8.6)])

$$\begin{array}{ccc} H^1(G_L, V) \times D_{\mathrm{dR}}(V^\vee(1))_\sigma & \xrightarrow{(\mathrm{id},\delta_\sigma)} & H^1(G_L, V) \times H^1(G_L, V^\vee(1)) \\ \downarrow{\scriptstyle \cup_\sigma} & & \downarrow{\scriptstyle \cup} \\ H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E E(1)) & \xrightarrow{\delta_\sigma} & H^2(G_L, E(1)) \cong E \end{array}, \tag{11}$$

where the δ_σ's are the connecting maps obtained by taking Galois cohomology of (8) (with V replaced by $V^\vee(1)$ or $E(1)$). This diagram, together with (10) (with V replaced by $V(1)$), shows that if V is σ-de Rham then $H^1_{g,\sigma}(G_L, V) \subseteq H^1_{e,\sigma}(G_L, , V^\vee(1))^\perp$ via the perfect pairing:

$$\cup : H^1(G_L, V) \times H^1(G_L, V^\vee(1)) \longrightarrow H^2(G_L, E(1)) \cong E. \tag{12}$$

Indeed, let $V' := \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$, one can equip V' with a natural E-action and G_L-action (these two actions commute). One has isomorphisms $V^\vee \cong \mathrm{Hom}_E(V, E) \xrightarrow[\sim]{\mathrm{tr}_{E/\mathbb{Q}_p}} \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p) \cong V'$. Thus the Tate pairing $H^1(G_L, V) \times H^1(G_L, V'(1)) \to H^2(G_L, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ equals to the composition of the pairing (12) with $\mathrm{tr}_{E/\mathbb{Q}_p}$, from which one deduces (12) is perfect.

Proposition 2.2. *If V is σ-de Rham, then we have*

$$H^1_{g,\sigma}(G_L, V) \xrightarrow{\sim} H^1_{e,\sigma}(G_L, V^\vee(1))^\perp.$$

Proof. This proposition follows by the same argument as in [5, Prop.3.8]. We give the proof for the convenience of the reader. It's sufficient to prove

$$\delta_\sigma \circ \cup : H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E V) \times D_{\mathrm{dR}}(V^\vee(1))_\sigma \longrightarrow H^2(G_L, E(1)) \tag{13}$$

is a perfect pairing. Consider the following commutative diagram (deduced from (2), where $B^1_{\mathrm{dR}} := tB^+_{\mathrm{dR}}$, θ denotes the morphism $B^+_{\mathrm{dR}} \to \mathbb{C}_p$)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L(1) & \longrightarrow & (tB_{e,L}) \oplus B^1_{\mathrm{dR}} & \longrightarrow & B_{\mathrm{dR}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L(1) & \longrightarrow & (tB_{e,L} \cap B^+_{\mathrm{dR}}) \oplus B^1_{\mathrm{dR}} & \longrightarrow & B^+_{\mathrm{dR}} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(1) & \longrightarrow & (tB_{e,L}) \cap B^+_{\mathrm{dR}} & \xrightarrow{\theta} & \mathbb{C}_p & \longrightarrow & 0
\end{array} \tag{14}$$

by taking tensor products $- \otimes_{L,\sigma} E$, we deduce from the bottom exact sequence (recall $\mathbb{C}_p$ is viewed as an L-algebra via ι):

$$0 \to E(1) \to (tB_{e,L,\sigma}) \cap B^+_{\mathrm{dR},\sigma} \to \mathbb{C}_{p,\sigma} \to 0. \tag{15}$$

As in the proof of [5, Prop.3.8], we show that to prove (13) is perfect, it's sufficient to prove he following pairing is perfect:

$$H^1\big(G_L, \mathbb{C}_{p,\sigma} \otimes_E V\big) \times H^0\big(G_L, \mathbb{C}_{p,\sigma} \otimes_E V^\vee\big) \xrightarrow{\cup} H^1(G_L, \mathbb{C}_{p,\sigma}) \xrightarrow{\delta_\sigma} H^2(G_L, E(1)) \tag{16}$$

where δ_σ is induced by (15). Indeed, if we identify $D_{\mathrm{dR}}(V^\vee(1))_\sigma$ and $D_{\mathrm{dR}}(V^\vee)_\sigma$ as E-vector spaces, then the pairing (13) equals to the pairing

$$H^1(G_L, B_{\mathrm{dR},\sigma} \otimes_E V) \times D_{\mathrm{dR}}(V^\vee)_\sigma \xrightarrow{\cup} H^1(G_L, B_{\mathrm{dR},\sigma}) \longrightarrow H^2(G_L, E(1))$$

where the last map is induced by the top exact sequence in (14) (twisted via $- \otimes_{L,\sigma} E$); moreover by (14), one sees the restricted map

$$H^1(G_L, B^+_{\mathrm{dR},\sigma} \otimes_E V) \times D^+_{\mathrm{dR}}(V^\vee)_\sigma \big(\cong \mathrm{Fil}^{-1} D_{\mathrm{dR}}(V^\vee(1))_\sigma\big) \longrightarrow H^2(G_L, E(1))$$

factors through (16). Since V is σ-de Rham, we have

$$\begin{aligned}
H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E V) &\cong \mathrm{Ker}\big[H^1(G_L, B^+_{\mathrm{dR},\sigma} \otimes_E V) \twoheadrightarrow H^1(G_L, \mathbb{C}_{p,\sigma} \otimes_E V)\big], \\
D^+_{\mathrm{dR}}(V^\vee(1))_\sigma \cong \mathrm{Fil}^1 D^+_{\mathrm{dR}}(V^\vee)_\sigma &\cong \mathrm{Ker}\big[D^+_{\mathrm{dR}}(V^\vee)_\sigma \twoheadrightarrow H^0(G_L, \mathbb{C}_{p,\sigma})\big]
\end{aligned}$$

So if (16) is a perfect pairing, then the orthogonal complement of $H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E V)$ (via (13)) is exactly $D^+_{\mathrm{dR}}(V^\vee(1))_\sigma$, from which we deduce (where the second isomorphism follows from the above discussion applied to $V(n-1) := V \otimes_E \chi_{\mathrm{cyc}}^{n-1}$)

$$H^1(G_L, t^n B^+_{\mathrm{dR},\sigma} \otimes_E V)^\perp \cong H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E V(n-1))^\perp \cong D^+_{\mathrm{dR}}(V^\vee(2-n))_\sigma \cong \mathrm{Fil}^{1-n} D_{\mathrm{dR}}(V^\vee(1))$$

for $n \in \mathbb{Z}_{\geq 1}$. Thus to show (13) is perfect, it's sufficient to show $H^1(G_L, t^n B^+_{\mathrm{dR},\sigma} \otimes_E V) = 0$ when $n \gg 0$. Consider the Hodge-Tate decomposition (since V is σ-de Rham)

$$\mathbb{C}_{p,\sigma} \otimes_E V \xrightarrow{\sim} \oplus_{i \in \mathbb{Z}} \mathbb{C}_{p,\sigma}(i)^{\oplus n_i} \tag{17}$$

with $\mathbb{C}_{p,\sigma}(i) \cong \mathbb{C}_p(i) \otimes_{L,\sigma} E$ and $n_i \in \mathbb{Z}_{\geq 0}$, $n_i = 0$ for all but finitely many $i \in \mathbb{Z}$; this, together with the fact $H^1(G_L, \mathbb{C}_{p,\sigma}(i)) = 0$ for $i \neq 0$ shows the natural morphism

$$H^1(G_L, t^n B^+_{\mathrm{dR},\sigma} \otimes_E V) \longrightarrow H^1(G_L, t^{n-1} B^+_{\mathrm{dR},\sigma} \otimes_E V)$$

is an isomorphism when $n \gg 0$, and hence $H^1(G_L, t^n B^+_{\mathrm{dR},\sigma} \otimes_E V) = 0$ when $n \gg 0$.

In the following, we show (16) is perfect. By using the Hodge-Tate decomposition (17) of V, one reduces to the case $\mathbb{C}_{p,\sigma} \otimes_E V \cong \mathbb{C}_{p,\sigma}(n)$ for some $n \in \mathbb{Z}$. Since $H^i(G_L, \mathbb{C}_{p,\sigma}(n)) = 0$ for $i = 0, 1$, if $n \neq 0$, one reduces to the case $\mathbb{C}_{p,\sigma} \otimes_E V \cong \mathbb{C}_{p,\sigma}$; since $\dim_E H^i(G_L, \mathbb{C}_{p,\sigma}) = 1$ for $i = 0, 1$, one reduces to show the δ_σ in (16) is non-zero, and hence it's sufficient to show that the following map (induced by the bottom exact sequence of (14)),

$$\delta : H^1(G_L, \mathbb{C}_p) \longrightarrow H^2(G_L, L(1)) \tag{18}$$

is non-zero. Consider the following G_L-invariant exact sequence (induced by (2) by the same way as in (14) by replacing t by t_L)

$$0 \to L(\chi_{\mathcal{LT}}) \to (t_L B_{e,L}) \cap B^+_{\mathrm{dR}} \to \mathbb{C}_p \to 0. \tag{19}$$

Since $\big((t_L B_{e,L}) \cap B^+_{\mathrm{dR}}\big)^{G_L} = 0$, the induced map $\delta' : H^0(G_L, \mathbb{C}_p) \to H^1(G_L, L(\chi_{\mathcal{LT}}))$ is injective. Let $u_L := t/t_L \in (B^+_{\mathrm{dR}})^\times$, G_L acts on u_L (and also on $\theta(u_L) \in \mathbb{C}_p$) via the character $\chi' := \chi_{\mathrm{cyc}} \chi_{\mathcal{LT}}^{-1}$. One has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(\chi_{\mathcal{LT}}) & \longrightarrow & (t_L B_{e,L}) \cap B^+_{\mathrm{dR}} & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\ & & \times\chi' \downarrow & & \times u_L \downarrow & & \times\theta(u_L) \downarrow & & \\ 0 & \longrightarrow & L(1) & \longrightarrow & (t B_{e,L}) \cap B^+_{\mathrm{dR}} & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \end{array} \tag{20}$$

Thus the bottom exact sequence of (20) of G_L-representations is the just twist of (19) by the character χ'. One gets hence a commutative diagram

$$\begin{array}{ccc} H^0(G_L, \mathbb{C}_p) \times H^1(G_L, L(\chi')) & \xrightarrow{(\delta', \mathrm{id})} & H^1(G_L, L(\chi_{\mathcal{LT}})) \times H^1(G_L, L(\chi')) \\ \cup \downarrow & & \cup \downarrow \\ H^1(G_L, \mathbb{C}_p) & \xrightarrow{\delta} & H^2(G_L, L(1)) \end{array}, \tag{21}$$

from which one sees δ is non-zero since the top horizontal arrow is injective and the cup-product on the right side is a perfect pairing (by Tate duality). ☐

Corollary 2.3. *Let $J \subseteq \Sigma_L$, $J \neq \emptyset$, and assume that V is J-de Rham, then the Tate pairing induces a bijection of E-vector spaces: $H^1_{e,J}(G_L, V) \xrightarrow{\sim} H^1_{g,J}(G_L, V^\vee(1))^\perp$. In particular, $H^1_{g,J}(G_L, V^\vee(1)) = H^1(G_L, V^\vee(1))$ if $H^1_{e,\sigma}(G_L, V) = 0$ for any $\sigma \in J$.*

Corollary 2.4. *Assume that V is de Rham, then we have $H^1_{e,\Sigma_L}(G_L, V) \xrightarrow{\sim} H^1_e(G_L, V)$.*

Proof. We have the following isomorphisms

$$H^1_{e,\Sigma_L}(G_L, V)^\perp \cong H^1_{g,\Sigma_L}(G_L, V^\vee(1)) = H^1_g(G_L, V^\vee(1)) \cong H^1_e(G_L, V)^\perp,$$

(we refer to [5, Prop. 3.8] for the last isomorphism), and hence $H^1_{e,\Sigma_L}(G_L, V) \xrightarrow{\sim} H^1_e(G_L, V)$. ☐

Corollary 2.5. *Suppose V is de Rham and $V^{G_L} \xrightarrow{\sim} D_e(V)$, then the map*

$$\sum_{\sigma \in \Sigma_L} j_\sigma : \oplus_{\sigma \in \Sigma_L} H^1_{e,\sigma}(G_L, V) \longrightarrow H^1_e(G_L, V)$$

is bijective.

Proof. Since $V^{G_L} \xrightarrow{\sim} D_e(V)$, the injections $(V \otimes_{\mathbb{Q}_p} L)^{G_L} \hookrightarrow D'_{e,L}(V)$ and $(V \otimes_{\mathbb{Q}_p} L)^{G_L} \hookrightarrow D_{e,L}(V)$ are also bijective. By [5, Cor.3.8.4], $H^1_e(G_L, V) \cong D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V)$; by (10), one has $H^1_{e,\sigma}(G_L, V) \cong D_{\mathrm{dR}}(V)_\sigma/D^+_{\mathrm{dR}}(V)_\sigma$ for any $\sigma \in \Sigma_L$. In particular, we have

$$\dim_E H^1_e(G_L, V) = \sum_{\sigma\in\Sigma_L} \dim_E H^1_{e,\sigma}(G_L, V).$$

The corollary follows thus from Cor.2.4. □

Remark 2.6. *This corollary is not true in general. For example, if V is crystalline and $D^+_{\mathrm{dR}}(V) = 0$, $D_e(V) \neq 0$, by the lemma 1.3, we see $D_{e,L}(V) = 0$. So in this case, we have*

$$\sum_{\sigma\in\Sigma_L} \dim_E H^1_{e,\sigma}(G_L, V) = \sum_{\sigma\in\Sigma_L} \dim_E \big(D_{\mathrm{dR}}(V)_\sigma/D^+_{\mathrm{dR}}(V)_\sigma\big)$$
$$> \dim_E H^1_e(G_L, V) = \dim_E \big(D_{\mathrm{dR}}(V)/\big(D^+_{\mathrm{dR}}(V) + D_e(V)\big)\big).$$

Corollary 2.7. *Suppose V is de Rham and $V^{G_L} \xrightarrow{\sim} D_e(V)$, then the following map*

$$\sum_{\sigma\in J} j_\sigma : \oplus_{\sigma\in J} H^1_{e,\sigma}(G_L, V) \longrightarrow H^1_{e,J}(G_L, V) \tag{22}$$

is bijective for any $J \subseteq \Sigma_L$, $J \neq \emptyset$. Consequently, in this case,

$$\dim_E H^1_{g,J}(G_L, V^\vee(1)) = \dim_E H^1(G_L, V^\vee(1)) - \sum_{\sigma\in J} \dim_E H^1_{e,\sigma}(G_L, V)$$
$$= \dim_E H^1(G_L, V^\vee(1)) - \sum_{\sigma\in J} \dim_E \big(D_{\mathrm{dR}}(V)_\sigma/D^+_{\mathrm{dR}}(V)_\sigma\big).$$

Proof. The first part of the corollary follows from Cor.2.5. The second part follows from the isomorphism (22) together with Cor.2.3 and (10). □

Proof of Thm.0.1. By the corollary 2.7 applied to $V := \chi_2\chi_1^{-1}\chi_{\mathrm{cyc}}$, the theorem follows. □

3. B-pairs and cohomology of B-pairs

3.1. B-pairs.

Recall Berger's B-pairs ([3]).

Definition 3.1 (cf. [3, §2])**.** *(1) A B-pair of G_L is a couple $W = (W_e, W^+_{\mathrm{dR}})$ where W_e is a finite free B_e-module equipped with a semi-linear continuous action of G_L, and W^+_{dR} is a G_L-stable B^+_{dR}-lattice of $W_{\mathrm{dR}} := W_e \otimes_{B_e} B^+_{\mathrm{dR}}$. Let $r \in \mathbb{Z}_{>0}$, we say that W is of dimension r if the rank of W_e over B_e equals to r.*

(2) Let W, W' be two B-pairs, a morphism $f : W \to W'$ is defined to be a G_L-invariant B_e-linear map $f_e : W_e \to W'_e$ such that the induced B_{dR}-linear map $f_{\mathrm{dR}} := f_e \otimes \mathrm{id} : W_{\mathrm{dR}} \to W'_{\mathrm{dR}}$ sends W^+_{dR} to $(W')^+_{\mathrm{dR}}$. Moreover, we say that f is strict if the B^+_{dR}-module $(W')^+_{\mathrm{dR}}/f^+_{\mathrm{dR}}(W^+_{\mathrm{dR}})$ is torsion free, where $f^+_{\mathrm{dR}} := f_{\mathrm{dR}}|_{W^+_{\mathrm{dR}}}$.

By [3, Thm. 2.2.7], there exists an equivalence of categories between the category of B-pairs and that of (φ, Γ)-modules over the Robba ring $B^\dagger_{\mathrm{rig},L'_0}$ (where L'_0 denotes the maximal unramified extension of $\mathbb{Q}_p$ in $L_\infty := \cup_n L(\zeta_{p^n})$, e.g. see [3, §1.1]).

Let $W = (W_e, W^+_{\mathrm{dR}})$ be a B-pair of dimension r, set $W'_{e,L} := W_e \otimes_{B_e} B'_{e,L} \cong W_e \otimes_{\mathbb{Q}_p} L$, which is a finite free $W'_{e,L}$-module of rank r equipped with a semi-linear action of G_L (induced by that on

W_e), and an action of $\mathrm{Gal}(L/\mathbb{Q}_p)$ given by $\mathrm{id}\otimes\sigma$ for $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$). These two actions commute, and $W_e \xrightarrow{\sim} (W'_{e,L})^{\mathrm{Gal}(L/\mathbb{Q}_p)}$. We define

$$W_{e,L} := \big\{w\in W'_{e,L} \mid \tau(w)\in W_{\mathrm{dR}}^+,\ \forall \tau\in \mathrm{Gal}(L/\mathbb{Q}_p),\ \tau\neq 1\big\},$$

which is hence a $B_{e,L}$-module by Lem.1.3. Note that $W_{e,L}$ is stable under the action of G_L.

Recall for $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\neq 1$, one has a valuation on $B'_{e,L}$: $\mathrm{ord}_{\tau(\infty_\iota)}: B'_{e,L}\to\mathbb{Z}\cup\{+\infty\}$ (cf. §1). Put $\mathrm{ord}_{\infty_\iota}(x):=\sup\{n\in\mathbb{Z}\mid x\in t_L^n B_{\mathrm{dR}}^+\}$, one sees $\mathrm{ord}_{\tau(\infty_\iota)}(x)=\mathrm{ord}_{\infty_\iota}(\tau^{-1}(x))$. For a B-pair $W=(W_e,W_{\mathrm{dR}}^+)$, $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, put

$$\mathrm{ord}_{\tau(\infty_\iota)}: W'_{e,L}\longrightarrow \mathbb{Z}\cup\{+\infty\},\ w\mapsto \sup\{n\in\mathbb{Z}\mid \tau^{-1}(w)\in t_L^n W_{\mathrm{dR}}^+\},$$

so $\mathrm{ord}_{\tau(\infty_\iota)}(aw)=\mathrm{ord}_{\tau(\infty_\iota)}(a)+\mathrm{ord}_{\tau(\infty_\iota)}(w)$ for any $a\in B'_{e,L}$ and $w\in W'_{e,L}$. We have thus

$$W_{e,L}=\big\{w\in W'_{e,L}\mid \mathrm{ord}_{\tau(\infty_\iota)}(w)\geq 0 \text{ for all } \tau\in\mathrm{Gal}(L/\mathbb{Q}_p),\, \tau\neq 1\big\}.$$

Proposition 3.2. *Let W be a B-pair of dimension r, then $W_{e,L}$ is a finite free $B_{e,L}$-module of rank r, and we have a natural isomorphism*

$$W_{e,L}\otimes_{B_{e,L}} B'_{e,L}\xrightarrow{\sim} W'_{e,L}. \tag{23}$$

Proof. Let $0\neq w\in W'_{e,L}$, and set $n_\tau:=\mathrm{ord}_{\tau(\infty_\iota)}(w)$ for $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$. Put

$$w':=\prod_{\substack{\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)\\ \tau\neq 1}}(t_L^\tau/t_L)^{-n_\tau}w.$$

By (3), one sees $\mathrm{ord}_{\tau(\infty_\iota)}(w')=0$ for any $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\neq 1$, and thus $w'\in W_{e,L}$. The surjectivity of (23) follows.

Let $\{e_1,\cdots,e_r\}$ be a basis of $W'_{e,L}$ over $B'_{e,L}$. For any $j=1,\cdots,r$, by multiplying e_j by an invertible element in $B'_{e,L}$, we can assume that $\mathrm{ord}_{\tau(\infty_\iota)}(e_j)=0$ for any $1\neq\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$ (see the above argument). So we have an inclusion

$$B_{e,L}e_1\oplus\cdots\oplus B_{e,L}e_r\subseteq W_{e,L}. \tag{24}$$

For $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, let $M_\tau\in\mathrm{GL}_r(B'_{e,L})$ such that

$$\tau(e_1,\cdots,e_r)=(e_1,\cdots,e_r)M_\tau.$$

Let $\{f_1,\cdots,f_r\}$ be a basis of W_{dR}^+ over B_{dR}^+, then there exists $N\in\mathrm{GL}_r(B_{\mathrm{dR}})$ such that

$$(f_1,\cdots,f_r)=(e_1,\cdots,e_r)N.$$

For $x=\sum_{j=1}^r\lambda_je_j\in W'_{e,L}$ (with $\lambda_j\in B'_{e,L}$ for all $1\leq j\leq r$), if $x\in W_{e,L}$, by definition, $\tau(x)\in W_{\mathrm{dR}}^+$ for all $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\neq 1$, from which one can deduce for all $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\neq 1$, there exist $\mu_{\tau,1},\cdots,\mu_{\tau,r}\in B_{\mathrm{dR}}^+$ such that

$$\begin{pmatrix}\tau(\lambda_1)\\ \vdots\\ \tau(\lambda_r)\end{pmatrix}=M_\tau^{-1}N\begin{pmatrix}\mu_{\tau,1}\\ \vdots\\ \mu_{\tau,r}\end{pmatrix}. \tag{25}$$

Let $n_\tau\in\mathbb{Z}_{\geq 0}$ such that $M_\tau^{-1}N\in t_L^{-n_\tau}\,\mathrm{M}_r(B_{\mathrm{dR}}^+)$. So $\tau(\lambda_j)\in t_L^{-n_\tau}B_{\mathrm{dR}}^+$ $\big($thus $\mathrm{ord}_{\tau^{-1}(\infty_\iota)}(\lambda_j)\geq -n_\tau\big)$ for any $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\neq 1$ and $1\leq j\leq r$. By Lem.1.1,

$$\prod_{\substack{\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)\\ \tau\neq 1}}(t_L^{\tau^{-1}}/t_L)^{n_\tau}\lambda_j\in B_{e,L}$$

for $1 \leq j \leq r$, and so $\big(\prod_{\substack{\tau\in \mathrm{Gal}(L/\mathbb{Q}_p)\\ \tau\neq 1}}(t_L^{\tau^{-1}}/t_L)^{n_\tau}\big)x\in \oplus_{j=1}^r B_{e,L}e_j$. Hence

$$W_{e,L}\subseteq \prod_{\substack{\tau\in \mathrm{Gal}(L/\mathbb{Q}_p)\\ \tau\neq 1}}(t_L^{\tau^{-1}}/t_L)^{-n_\tau}\Big(B_{e,L}e_1\oplus\cdots\oplus B_{e,L}e_r\Big). \tag{26}$$

Since $B_{e,L}$ is a PID, by (24) and (26), we see $W_{e,L}$ is a finite free $B_{e,L}$-module of rank r. The injectivity of (23) follows. □

Definition 3.3 (cf. [24, Def.1.2 and 1.4]). *(1) Let E be a finite extension of $\mathbb{Q}_p$ which contains all the p-adic embeddings of L in $\overline{\mathbb{Q}_p}$, an E-B-pair is a B-pair $W=(W_e,W_{\mathrm{dR}}^+)$ such that W_e is moreover a finite $B_e\otimes_{\mathbb{Q}_p}E$-module, and W_{dR}^+ is a G_L-stable $B_{\mathrm{dR}}^+\otimes_{\mathbb{Q}_p}E$-lattice of $W_{\mathrm{dR}}:=W_e\otimes_{B_e}B_{\mathrm{dR}}$.*

(2) Let W, W' be two E-B-pairs, a morphism $f:W\rightarrow W'$ is defined to be a morphism of B-pairs such that $f_e:W_e\rightarrow W_e'$ (cf. Def. 3.1 (2)) is moreover $B_e\otimes_{\mathbb{Q}_p}E$-linear.

Lemma 3.4 (cf. [24, Lem.1.7 and 1.8]). *Let $W=(W_e,W_{\mathrm{dR}}^+)$ be an E-B-pair, then W_e (resp. W_{dR}^+) is finite free over $B_e\otimes_{\mathbb{Q}_p}E$ (resp. $B_{\mathrm{dR}}^+\otimes_{\mathbb{Q}_p}E$). Moreover we have $\mathrm{rk}_{B_e\otimes_{\mathbb{Q}_p}E}W_e=\mathrm{rk}_{B_{\mathrm{dR}}^+\otimes_{\mathbb{Q}_p}E}W_{\mathrm{dR}}^+=:r$, and we call r the rank of the E-B-pair W.*

Note that there exists an equivalence of categories between the category of E-B-pairs and that of (φ,Γ)-modules over $B_{\mathrm{rig},L_0'}^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}E$ (cf. [24, Thm.1.36]).

Let $W=(W_e,W_{\mathrm{dR}}^+)$ be an E-B-pair of rank r. The following corollary follows easily from the lemma 3.4 and the proposition 3.2.

Corollary 3.5. *$W_{e,L}$ is a finite free $B_{e,L}\otimes_{\mathbb{Q}_p}E$-module of rank r and $W_{e,L}\otimes_{B_{e,L}}B_{e,L}'\xrightarrow{\sim}W_{e,L}'$.*

With the above notation, one has decompositions $W_{e,L}\xrightarrow{\sim}\oplus_{\sigma\in\Sigma_L}W_{e,L,\sigma}$, $W_{e,L}'\xrightarrow{\sim}\oplus_{\sigma\in\Sigma_L}W_{e,L,\sigma}'$, $W_{\mathrm{dR}}^+\xrightarrow{\sim}\oplus_{\sigma\in\Sigma_L}W_{\mathrm{dR},\sigma}^+$. The $\mathrm{Gal}(L/\mathbb{Q}_p)$-action on $W_{e,L}'$ induces isomorphisms

$$\tau: W_{e,L,\sigma}'\xrightarrow{\sim}W_{e,L,\sigma\circ\tau^{-1}}'$$

for $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\sigma\in\Sigma_L$. For $\sigma\in\Sigma_L$, one has

$$W_{e,L,\sigma}=\big\{w\in W_{e,L,\sigma}'\mid \tau(w)\in W_{\mathrm{dR},\sigma\circ\tau^{-1}}^+,\ \forall\tau\in\mathrm{Gal}(L/\mathbb{Q}_p),\ \tau\neq 1\big\}.$$

Definition 3.6. *(1) Let $\sigma: L\hookrightarrow E$, a B_σ-pair W_σ is a couple $(W_{e,L,\sigma},W_{\mathrm{dR},\sigma}^+)$ where $W_{e,L,\sigma}$ is a finite free $B_{e,L,\sigma}$-module equipped with a semi-linear G_L-action, and $W_{\mathrm{dR},\sigma}^+$ is a G_L-invariant $B_{\mathrm{dR},\sigma}^+$-lattice in $W_{\mathrm{dR},\sigma}:=W_{e,L,\sigma}\otimes_{B_{e,L,\sigma}}B_{\mathrm{dR},\sigma}$.*

(2) Let W_σ, W_σ' be two B_σ-pairs, a morphism $f:W_\sigma\rightarrow W_\sigma'$ is defined to be a G_L-invariant $B_{e,L,\sigma}$-linear map $f_{e,L,\sigma}:W_{e,L,\sigma}\rightarrow W_{e,L,\sigma}'$ such that the induced $B_{\mathrm{dR},\sigma}$-linear map $f_{\mathrm{dR},\sigma}:=f_{e,L,\sigma}\otimes\mathrm{id}: W_{\mathrm{dR},\sigma}\rightarrow W_{\mathrm{dR},\sigma}'$ sends $W_{\mathrm{dR},\sigma}^+$ to $(W')_{\mathrm{dR},\sigma}^+$.

Proposition 3.7. *Let $\sigma\in\Sigma_L$, the functor*

$$F_\sigma:\big\{E\text{-}B\text{-pairs}\big\}\longrightarrow\big\{B_\sigma\text{-pairs}\big\},\ W=(W_e,W_{\mathrm{dR}}^+)\mapsto W_\sigma:=(W_{e,L,\sigma},W_{\mathrm{dR},\sigma}^+) \tag{27}$$

induces an equivalence of categories.

We construct an inverse of F_σ:

Lemma 3.8. *Let $M_\sigma:=(M_{e,L,\sigma},M_{\mathrm{dR},\sigma})$ be a B_σ-pair, then there exists an E-B-pair $W=(W_e,W_{\mathrm{dR}}^+)$ such that $W_{e,L,\sigma}\cong M_{e,L,\sigma}$ and $W_{\mathrm{dR},\sigma}^+\cong M_{\mathrm{dR},\sigma}^+$.*

Proof. We construct W as follow. Let $\{e_{1,\sigma},\cdots,e_{r,\sigma}\}$ be a basis of $M_{e,L,\sigma}$ over $B_{e,L,\sigma}$, put $M'_{e,L,\sigma}:=M_{e,L,\sigma}\otimes_{B_{e,L}}B'_{e,L}$, which is thus free $B'_{e,L,\sigma}$-module of rank r equipped with a $B'_{e,L,\sigma}$-semi-linear action of G_L. For $g\in G_L$, let $A_g\in \mathrm{GL}_r(B_{e,L,\sigma})\subseteq \mathrm{GL}_r(B'_{e,L,\sigma})$ such that

$$g(e_{1,\sigma},\cdots,e_{r,\sigma})=(e_{1,\sigma},\cdots,e_{r,\sigma})A_g.$$

So $A_{g_1g_2}=A_{g_1}g_1(A_{g_1})$, for $g_1,g_2\in G_L$.

For $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, let $\sigma':=\sigma\circ\tau^{-1}\in\Sigma_L$, and $M'_{e,L,\sigma'}$ be a free $B'_{e,L,\sigma'}$-module of rank r equipped with a $B'_{e,L,\sigma'}$-semi-linear action of G_L such that the action on a basis $\{e_{1,\sigma'},\cdots,e_{r,\sigma'}\}$ is given by (recall one has an isomorphism $\tau: B'_{e,L,\sigma}\xrightarrow{\sim}B'_{e,L,\sigma'}$)

$$g(e_{1,\sigma'},\cdots,e_{r,\sigma'})=(e_{1,\sigma'},\cdots,e_{r,\sigma'})\tau(A_g) \tag{28}$$

for all $g\in G_L$. This action is well defined since $\tau(A_{g_1g_2})=\tau(A_{g_1})g_1(\tau(A_{g_2}))$ for $g_1,g_2\in G_L$.

Put $M'_{e,L}:=\oplus_{\sigma'\in\Sigma_L}M'_{e,L,\sigma'}$, on which we define an action of $\mathrm{Gal}(L/\mathbb{Q}_p)$ by

$$\tau\Big(\sum_{i=1}^r a_ie_{i,\sigma'}\Big):=\sum_{i=1}^r\tau(a_i)e_{i,\sigma'\circ\tau^{-1}},$$

for $\sum_{i=1}^r a_ie_{i,\sigma'}\in M_{e,L,\sigma'}$ and $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$. One can easily check $W_e:=(M'_{e,L})^{\mathrm{Gal}(L/\mathbb{Q}_p)}$ is a free $B_e\otimes_{\mathbb{Q}_p}E$-module of rank r (note $B_{e,L,\sigma'}\cong B_e\otimes_{\mathbb{Q}_p}E$ for all σ') equipped with a B_e-semi-linear and E-linear action of G_L.

For $1\neq\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, let $\sigma':=\sigma\circ\tau^{-1}\in\Sigma_L$, let $M_{\mathrm{dR},\sigma'}:=M'_{e,L,\sigma'}\otimes_{B'_{e,L,\sigma'}}B_{\mathrm{dR},\sigma'}$ which is a free $B_{\mathrm{dR},\sigma'}$-module of rank r equipped with a $B_{\mathrm{dR},\sigma'}$-semi-linear action of G_L. Let $M^+_{\mathrm{dR},\sigma'}$ be the $B^+_{\mathrm{dR},\sigma'}$-module generated by $\tau(M_{e,L,\sigma})\subseteq\tau(M'_{e,L,\sigma})\cong M'_{e,L,\sigma'}$. We claim

$$M^+_{\mathrm{dR},\sigma'}\xrightarrow{\sim}\oplus_{i=1}^rB^+_{\mathrm{dR},\sigma'}e_{i,\sigma'}. \tag{29}$$

Indeed, since $\tau(e_{i,\sigma})=e_{i,\sigma'}$ (by definition), the direction "$\supseteq$" is clear; since $\tau(B_{e,L,\sigma})\subseteq B^+_{\mathrm{dR},\sigma'}$, the other direction is also clear. Moreover, $M^+_{\mathrm{dR},\sigma'}$ is stable under the action of G_L (defined by (28)), because $\tau(A_g)\in\mathrm{GL}_r(B^+_{\mathrm{dR},\sigma'})$ for all $g\in G_L$. Put $W^+_{\mathrm{dR}}:=M^+_{\mathrm{dR},\sigma}\oplus\big(\oplus_{\sigma'\neq\sigma}M^+_{\mathrm{dR},\sigma}\big)$.

It's clear that $W:=(W_e,W^+_{\mathrm{dR}})$ is an E-B pair, $W'_{e,L}\cong M'_{e,L}$ and $W^+_{\mathrm{dR},\sigma}\cong M^+_{\mathrm{dR},\sigma}$, thus it's sufficient to prove $W_{e,L,\sigma}\cong M_{e,L,\sigma}$. For any $x_\sigma=\sum_{i=1}^r a_ie_{i,\sigma}\in W'_{e,L,\sigma}=M'_{e,L,\sigma}$, by (29), we see $x_\sigma\in W_{e,L,\sigma}$ if and only if $\tau(a_i)\in B^+_{\mathrm{dR},\sigma\circ\tau^{-1}}$ for all $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\neq 1$ and $1\leq i\leq r$, which implies, by Lem.1.2, that $a_i\in B_{e,L,\sigma}$ for all $1\leq i\leq r$. This concludes the proof. □

By this lemma, one gets a functor $G_\sigma:\{B_\sigma\text{-pairs}\}\to\{E\text{-}B\text{-pairs}\}$, $M_\sigma\mapsto W$, and $F_\sigma\circ G_\sigma=\mathrm{id}$. It's sufficient to show $G_\sigma\circ F_\sigma=\mathrm{id}$. For an E-B-pair W, let $\tilde{W}:=G_\sigma\circ F_\sigma(W)$, by Prop.3.2 (see also Cor.3.5) and the construction of G_σ, one has $\tilde{W}_e\cong W_e$ and $\tilde{W}^+_{\mathrm{dR},\sigma}\cong W^+_{\mathrm{dR},\sigma}$. Hence it's sufficient to prove $\tilde{W}^+_{\mathrm{dR},\sigma'}\cong W^+_{\mathrm{dR},\sigma'}$ for $\sigma'\neq\sigma$. By the construction of G_σ as in the proof of Lem.3.8, we only need to show

Lemma 3.9. *Let $W:=(W_e,W^+_{\mathrm{dR}})$ be an E-B-pair, let $\sigma\in\Sigma_L$, $\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau\in 1$, $\sigma':=\sigma\circ\tau^{-1}$, then $W^+_{\mathrm{dR},\sigma'}$ is generated by $\tau(W_{e,L,\sigma})\subseteq W_{\mathrm{dR},\sigma'}$ as a $B^+_{\mathrm{dR},\sigma'}$-module.*

Proof. By definition, one has $\tau(W_{e,L,\sigma})\subseteq W^+_{\mathrm{dR},\sigma'}$. Now let $\{e_1,\cdots,e_r\}$ be a basis of $W_{e,L,\sigma}$ over $B_{e,L,\sigma}$, thus $\{\tau(e_1),\cdots,\tau(e_r)\}$ is a basis of $W_{\mathrm{dR},\sigma'}$ over $B_{\mathrm{dR},\sigma'}$. Let $x\in W^+_{\mathrm{dR},\sigma'}$, so there exist $a_i\in B_{\mathrm{dR},\sigma'}$ for $1\leq i\leq r$ such that $x=\sum_{i=1}^r a_i\tau(e_i)$, it's sufficient to prove $a_i\in B^+_{\mathrm{dR},\sigma'}$ for all $1\leq i\leq r$.

Since $B_{e,L}+B_{\mathrm{dR}}^+=B_{\mathrm{dR}}$ (thus $B_{e,L,\sigma'}+B_{\mathrm{dR},\sigma'}^+=B_{\mathrm{dR},\sigma'}$), by subtracting x by elements in $\oplus_{i=1}^r B_{\mathrm{dR},\sigma'}^+\tau(e_i)$, one may assume $a_i\in B_{e,L,\sigma'}$ for all $1\leq i\leq r$ $\big($and thus $x\in W'_{e,L,\sigma'}\big)$. Since $\tau'(B_{e,d,\sigma'})\subseteq B^+_{\mathrm{dR},\sigma'\circ(\tau')^{-1}}$ and $\tau'(e_i)\in W_{\mathrm{dR}}^+$ for all $\tau'\in \mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau'\neq 1$, we see $\tau'(x)=\sum_{i=1}^r\tau'(a_i)\tau'\circ\tau(e_i)\in W_{\mathrm{dR}}^+$ for all $\tau'\in\mathrm{Gal}(L/\mathbb{Q}_p)$, $\tau'\neq\tau^{-1}$ (note $x\in W^+_{\mathrm{dR},\sigma'}$ by assumption). Thus $\tau^{-1}(x)\in W_{e,L,\sigma}$, this concludes the proof. □

Proof of Prop.3.7. By the above two lemmas, one has $F_\sigma\circ G_\sigma=\mathrm{id}$ and $G_\sigma\circ F_\sigma=\mathrm{id}$, the proposition follows. □

Let W be an E-B-pair of rank r, we put

$$D_{\mathrm{dR}}(W):=(W_e\otimes_{B_e}B_{\mathrm{dR}})^{G_L}\cong (W_{e,L}\otimes_{B_{e,L}}B_{\mathrm{dR}})^{G_L}\cong\oplus_{\sigma\in\Sigma_L}(W_{e,L,\sigma}\otimes_{B_{e,L,\sigma}}B_{\mathrm{dR},\sigma})^{G_L},$$

which is an $L\otimes_{\mathbb{Q}_p}E$-module, and we have $D_{\mathrm{dR}}(W)_\sigma\cong(W_{e,L,\sigma}\otimes_{B_{e,L,\sigma}}B_{\mathrm{dR},\sigma})^{G_L}$ for any $\sigma\in\Sigma_L$. It's known that $D_{\mathrm{dR}}(W)_\sigma$ is an E-vector space of dimension $\leq r$.

Definition 3.10. *With the above notation, let $J\subseteq\Sigma_L$, W is called J-de Rham if* $\dim_E(D_{\mathrm{dR}}(W)_\sigma)=r$ *for all $\sigma\in J$.*

Similarly, put $D_e(W):=W_e^{G_L}$, $D_{e,L}(W):=W_{e,L}^{G_L}\cong\oplus_{\sigma\in\Sigma_L}W_{e,L,\sigma}^{G_L}$, and $D'_{e,L}(W):=(W'_{e,L})^{G_L}\cong\oplus_{\sigma\in\Sigma_L}(W'_{e,L,\sigma})^{G_L}$. $\big($thus $D_{e,L}(W)_\sigma\cong W_{e,L,\sigma}^{G_L}$, $D'_{e,L}(W)_\sigma\cong(W'_{e,L,\sigma})^{G_L}$ for $\sigma\in\Sigma_L\big)$. Note the composition $W_e\hookrightarrow W'_{e,L}\twoheadrightarrow W'_{e,L,\sigma}$ is bijective, G_L-invariant and induces an isomorphism $D_e(W)\xrightarrow{\sim}D'_{e,L}(W)_\sigma$.

At last, note that one can naturally associate an E-B-pair $W(V)$ to a finite dimensional continuous representation V of G_L over E as follows:

$$W(V):=\big(W(V)_e:=B_e\otimes_{\mathbb{Q}_p}V, W(V)_{\mathrm{dR}}^+:=B_{\mathrm{dR}}^+\otimes_{\mathbb{Q}_p}V\big).$$

Denote by B_E the trivial E-B-pair $\big((B_E)_e:=B_e\otimes_{\mathbb{Q}_p}E,(B_E)^+_{\mathrm{dR}}:=B_{\mathrm{dR}}^+\otimes_{\mathbb{Q}_p}E\big)$, and $B_{E,\sigma}$ the trivial B_σ-pair $F_\sigma(B_E)$.

3.2. **Cohomology of B-pairs.** Let W be an E-B-pair, in [24, §2.1], Nakamura has defined the *Galois cohomology of* W, denoted by $H^i(G_L,W)$, as the G_L-cohomology of the following complex $C^\bullet(W)$:

$$C_0(W):=W_e\oplus W_{\mathrm{dR}}^+\xrightarrow{(x,y)\mapsto x-y}W_{\mathrm{dR}}=:C_1(W).$$

By definition, one has a long exact sequence

$$\begin{aligned}(30)\quad 0\to H^0(G_L,W)\to H^0(G_L,W_e)\oplus H^0(G_L,W_{\mathrm{dR}}^+)\to H^0(G_L,W_{\mathrm{dR}})\\ \xrightarrow{\delta}H^1(G_L,W)\to H^1(G_L,W_e)\oplus H^1(G_L,W^+_{\mathrm{dR}})\to H^1(G_L,W_{\mathrm{dR}}).\end{aligned}$$

One easily sees $H^0(G_L,W)\xrightarrow{\sim}H^0(G_L,W_e\cap W_{\mathrm{dR}}^+)$.

Proposition 3.11 ([24, Prop.2.2, Rem.2.3])**.** *(1) There exists a natural isomorphism $H^1(G_L,W)\xrightarrow{\sim}\mathrm{Ext}^1(B_E,W)$, where $\mathrm{Ext}^1(B_E,W)$ denotes the group of extensions of E-B-pairs of B_E by W.*

(2) Let V be a finite dimensional continuous G_L-representation over E, then we have natural isomorphisms $H^i(G_L,W(V))\cong H^i(G_L,V)$ for all $i\in\mathbb{Z}_{\geq 0}$.

Put (cf. [24, Def.2.4])

$$\begin{aligned} H^1_{g,\sigma}(G_L,W) &:= \mathrm{Ker}[H^1(G_L,W)\to H^1(G_L,W_{\mathrm{dR},\sigma})],\\ H^1_g(G_L,W) &:= \mathrm{Ker}[H^1(G_L,W)\to H^1(G_L,W_{\mathrm{dR}})],\\ H^1_e(G_L,W) &:= \mathrm{Ker}[H^1(G_L,W)\to H^1(G_L,W_e)],\end{aligned}$$

where the above maps are induced from the natural maps

$$C^\bullet(W)\to[W_e\to 0]\to[W_{\mathrm{dR}}\to 0]\to[W_{\mathrm{dR},\sigma}\to 0].$$

One has thus $H^1_g(G_L,W)\xrightarrow{\sim}\cap_{\sigma\in\Sigma_L}H^1_{g,\sigma}(G_L,W)$. If W is σ-de Rham, let $[X]\in H^1(G_L,W)\cong \mathrm{Ext}^1(B_E,W)$, then X is σ-de Rham if and only if $[X]\in H^1_{g,\sigma}(G_L,W)$. Put $H^1_{g,J}(G_L,W):=\cap_{\sigma\in J}H^1_{g,\sigma}(G_L,W)$, so if W is J-de Rham, $[X]\in H^1(G_L,W)\cong\mathrm{Ext}^1(B_E,W)$, then X is J-de Rham if and only if $[X]\in H^1_{g,J}(G_L,W)$.

For an E-B-pair W, denote by $W^\vee$ the dual of W:

$$W^\vee:=\Big(W_e^\vee:=\mathrm{Hom}_{B_e}(W_e,B_e),(W^\vee)^+_{\mathrm{dR}}:=\mathrm{Hom}_{B^+_{\mathrm{dR}}}(W^+_{\mathrm{dR}},B^+_{dR})\Big)$$

one can check $W^\vee$, equipped with the natural E-action and G_L-action, is also an E-B-pair (e.g. see [24, Def.1.9(3)]). Denote by $W(1)$ the twist of W by $W(\chi_{\mathrm{cyc}})$ where χ_{cyc} is the cyclotomic character of G_L over E (by base change):

$$W(1):=\Big(W(1)_e:=W_e\otimes_{B_e\otimes_{\mathbb{Q}_p}E}W(\chi_{\mathrm{cyc}})_e, W(1)^+_{\mathrm{dR}}:=W^+_{\mathrm{dR}}\otimes_{B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}W(\chi_{\mathrm{cyc}})^+_{\mathrm{dR}}\Big).$$

One can show the cup product (see [25, Thm.5.10], note the composition of the following cup product with the trace map $\mathrm{tr}_{E/\mathbb{Q}_p}$ equals that in *loc. cit.*)

$$\cup: H^1(G_L,W)\times H^1(G_L,W^\vee(1))\longrightarrow H^2(G_L,B_E(1))\cong H^2(G_L,\chi_{\mathrm{cyc}})\cong E \tag{31}$$

is a perfect pairing.

Proposition 3.12 ([24, Prop.2.10])**.** *If the E-B-pair W is de Rham, then the perfect pairing $\cup$ induces an isomorphism $H^1_g(G_L,W)\cong H^1_e(G_L,W^\vee(1))^\perp$.*

Now we move to the category of B_σ-pairs. Let $\sigma\in\Sigma_L$, for an E-B-pair W, put $W_\sigma:=F_\sigma(W)=(W_{e,L,\sigma},W^+_{\mathrm{dR},\sigma})$. Consider the following complex of G_L-modules, denoted by $C^\bullet(W_\sigma)$:

$$C_0(W_\sigma):=W_{e,L,\sigma}\oplus W^+_{\mathrm{dR},\sigma}\xrightarrow{(x,y)\mapsto x-y}W_{\mathrm{dR},\sigma}=:C_1(W_\sigma).$$

Put $H^i(G_L,W_\sigma):=H^i(G_L,C^\bullet(W_\sigma))$, so we have a long exact sequence

$$\begin{aligned} 0\to H^0(G_L,W_\sigma)\to H^0(G_L,W_{e,L,\sigma})\oplus H^0(G_L,W^+_{\mathrm{dR},\sigma})\to H^0(G_L,W_{\mathrm{dR},\sigma})\\ \xrightarrow{\delta_\sigma}H^1(G_L,W_\sigma)\to H^1(G_L,W_{e,L,\sigma})\oplus H^1(G_L,W^+_{\mathrm{dR},\sigma})\to H^1(G_L,W_{\mathrm{dR},\sigma}).\end{aligned}\tag{32}$$

One has $H^0(G_L,W_\sigma)\cong H^0(G_L,W_{e,L,\sigma}\cap W^+_{\mathrm{dR},\sigma})$.

Lemma 3.13. *(1) For a B-pair W, $W_{e,L}\cap W^+_{\mathrm{dR}}$ $\big($as an L-vector subspace of $W'_{e,L}\big)$ is stable under the $\mathrm{Gal}(L/\mathbb{Q}_p)$-action, and one has $W_e\cap W^+_{\mathrm{dR}}\cong(W_{e,L}\cap W^+_{\mathrm{dR}})^{\mathrm{Gal}(L/\mathbb{Q}_p)}$.*

(2) Let W be an E-B-pair, then the G_L-invariant composition $W_e\cap W^+_{\mathrm{dR}}\hookrightarrow W_{e,L}\cap W^+_{\mathrm{dR}}\twoheadrightarrow W_{e,L,\sigma}\cap W^+_{\mathrm{dR},\sigma}$ is an isomorphism.

Proof. One has $W_{e,L}\cap W^+_{\mathrm{dR}}=\big\{w\in W_e\otimes_{\mathbb{Q}_p}L\mid \tau(w)\in W^+_{\mathrm{dR}},\ \forall\tau\in\mathrm{Gal}(L/\mathbb{Q}_p)\big\}$. Part (1) follows. Part (2) follows easily from the isomorphism $W_e\cap W^+_{\mathrm{dR}}\cong(W_{e,L}\cap W^+_{\mathrm{dR}})^{\mathrm{Gal}(L/\mathbb{Q}_p)}$. ∎

Thus for an E-B-pair W (with W_σ the associated B_σ-pair), one has

$$H^0(G_L, W_\sigma) \cong H^0(G_L, W_{e,L,\sigma} \cap W^+_{\mathrm{dR},\sigma}) \cong H^0(G_L, W_e \cap W^+_{\mathrm{dR}}) \cong H^0(G_L, W).$$

Consider now $H^1(G_L, W_\sigma)$, by the same argument as in [24, §2.1] (see in particular the argument before [24, Prop.2.2]), one can show there exists a natural isomorphism between $H^1(G_L, W_\sigma)$ and the group of extensions of B_σ-pairs, i.e. G_L-extensions $(X_{e,L,\sigma}, X^+_{\mathrm{dR},\sigma})$:

$$(33)\qquad \begin{aligned} 0 \to W_{e,L,\sigma} \to X_{e,L,\sigma} \to B_{e,L,\sigma} \to 0, \\ 0 \to W^+_{\mathrm{dR},\sigma} \to X^+_{\mathrm{dR},\sigma} \to B^+_{\mathrm{dR},\sigma} \to 0, \end{aligned}$$

such that $X_{e,L,\sigma} \otimes_{B_{e,d}} B_{\mathrm{dR}} \cong X^+_{\mathrm{dR},\sigma} \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$. On the other hand, the functor G_σ induces an isomorphism $\mathrm{Ext}^1(B_{E,\sigma}, W_\sigma) \xrightarrow{\sim} \mathrm{Ext}^1(B_E, W)$. One gets thus

$$j_\sigma : H^1(G_L, W_\sigma) \xrightarrow{\sim} \mathrm{Ext}^1(B_{E,\sigma}, W_\sigma) \xrightarrow{\sim} \mathrm{Ext}^1(B_E, W) \xrightarrow{\sim} H^1(G_L, W).$$

Set

$$\begin{aligned} H^1_{g,\sigma}(G_L, W_\sigma) &:= \mathrm{Ker}[H^1(G_L, W_\sigma) \to H^1(G_L, W_{\mathrm{dR},\sigma})], \\ H^1_{e,\sigma}(G_L, W_\sigma) &:= \mathrm{Ker}[H^1(G_L, W_\sigma) \to H^1(G_L, W_{e,L,\sigma})], \end{aligned}$$

where the above maps are induced by the natural maps $C^\bullet(W_\sigma) \to [W_{e,L,\sigma} \to 0] \to [W_{\mathrm{dR},\sigma} \to 0]$. Suppose W is σ-de Rham, let $[X] \in H^1(G_L, W_\sigma) \cong \mathrm{Ext}^1(B_E, W)$, then X is σ-de Rham if and only if $[X] \in H^1_{g,\sigma}(G_L, W_\sigma)$: indeed, $[X] \in H^1_{g,\sigma}(G_L, W_\sigma)$ if and only if the following exact sequence induced by (33) is split

$$0 \to W_{e,L,\sigma} \otimes_{B_{e,d}} B_{\mathrm{dR}} \to X_{e,L,\sigma} \otimes_{B_{e,d}} B_{\mathrm{dR}} \to B_{e,L,\sigma} \otimes_{B_{e,d}} B_{\mathrm{dR}} \to 0.$$

Similarly, $[X] \in H^1_{e,\sigma}(G_L, W_\sigma)$ if and only if the corresponding extension (33) splits, if so, by the construction of X_e from $X_{e,L,\sigma}$ as in Lem.3.8, we can easily deduce $X_e \cong W_e \oplus (B_e \otimes_{\mathbb{Q}_p} E)$, so $[X]$ lies in $H^1_e(G_L, W)$. In summary, the isomorphism j_σ induces an isomorphism $j_\sigma : H^1_{g,\sigma}(G_L, W_\sigma) \xrightarrow{\sim} H^1_{g,\sigma}(G_L, W)$ and an injection

$$j_\sigma : H^1_{e,\sigma}(G_L, W_\sigma) \hookrightarrow H^1_e(G_L, W).$$

Denote by $H^1_{e,\sigma}(G_L, W)$ the image of the above map, and $H^1_{e,J}(G_L, W)$ the E-vector subspace of $H^1_e(G_L, W)$ generated by $H^1_{e,\sigma}(G_L, W)$ for $\sigma \in J$.

Remark 3.14 (Question)**.** *Since there is an equivalence of categories of E-B-pairs and (φ, Γ)-modules over $\mathcal{R}_E$, a natural question is how to describe the groups $H^1_{e,\sigma}(G_L, W)$ in terms of (φ, Γ)-modules.*

Let V be a finite dimensional continuous representation of G_L over E, we have $C^\bullet(W(V)) \cong V[0] \cong C^\bullet(W(V)_\sigma)$ (by the p-adic fundamental exact sequences), thus $H^i(G_L, W(V)_\sigma) \cong H^i(G_L, V) \cong H^i(G_L, W(V))$. Consider the cup product

$$(34)\qquad \cup_\sigma : H^1(G_L, W_\sigma) \times H^1(G_L, W^\vee(1)_\sigma) \longrightarrow H^2(G_L, B_E(1)_\sigma) \cong H^2(G_L, \chi_{\mathrm{cyc}}) \cong E,$$

by the same argument as in [24, Rem.2.9], one can show (34) is a perfect pairing and compatible with $\cup$ (cf. (31)) via j_σ. By the same method as in Prop.2.2, one can prove

Proposition 3.15. *Suppose W is σ-de Rham, then the perfect pairing $\cup_\sigma$ (34) induces a bijection*

$$H^1_{g,\sigma}(G_L, W_\sigma) \xrightarrow{\sim} H^1_{e,\sigma}(G_L, W^\vee(1)_\sigma)^\perp.$$

Corollary 3.16. *Suppose W is σ-de Rham, then the perfect pairing (31) induces a bijection*

$$H^1_{g,\sigma}(G_L, W) \xrightarrow{\sim} H^1_{e,\sigma}(G_L, W^\vee(1))^\perp;$$

more generally, let $J \subseteq \Sigma_L$, $J \neq \emptyset$, and assume W is J-de Rham, then the perfect pairing (31) induces a bijection

$$H^1_{g,J}(G_L, W) \xrightarrow{\sim} H^1_{e,J}(G_L, W^\vee(1))^\perp.$$

In particular, we have $H^1_{g,J}(G_L, W) = H^1(G_L, W)$ *if* $H^1_{e,\sigma}(G_L, W^\vee(1)) = 0$ *for any* $\sigma \in J$.

Proof. The first part follows from Prop.3.15, since the parings $\cup_\sigma$ and $\cup$ are compatible (see the discussion before Prop.3.15). The second part follows from the first part and the fact that $H^1_{g,J}(G_L, W) = \cap_{\sigma\in J} H^1_{g,\sigma}(G_L, W)$ and $H^1_{e,J}(G_L, W^\vee(1)) = \sum_{\sigma\in J} H^1_{e,\sigma}(G_L, W^\vee(1))$. □

This corollary combined with Prop.3.12 shows:

Corollary 3.17. *Assume that* W *is de Rham, then we have* $H^1_{e,\Sigma_L}(G_L, W) = H^1_e(G_L, W)$, *i.e. the natural map*

$$\oplus_{\sigma\in\Sigma_L} H^1_{e,\sigma}(G_L, W) \longrightarrow H^1_e(G_L, W),\ (a_\sigma)_{\sigma\in\Sigma_L} \mapsto \sum_{\sigma\in\Sigma_L} a_\sigma \tag{35}$$

is surjective.

By the same argument as in [5, Lem.3.8.1], one has (see also [24, Lem.2.6])

Lemma 3.18. *Suppose* W *is* σ*-de Rham, then* $H^1(G_L, W^+_{\mathrm{dR},\sigma}) \to H^1(G_L, W_{\mathrm{dR},\sigma})$ *is injective.*

Consequently, we have $\mathrm{Im}(\delta) = H^1_e(G_L, W)$ and $\mathrm{Im}(\delta_\sigma) = H^1_{e,\sigma}(G_L, W_\sigma)$ (cf. (30), (32)). One has thus

Lemma 3.19. *Suppose* W *is* σ*-de Rham, then* $H^1_{e,\sigma}(G_L, W) \cong D_{\mathrm{dR}}(W)_\sigma / \big(D^+_{\mathrm{dR}}(W)_\sigma + D_{e,L}(W)_\sigma\big)$.

Suppose the injection $H^0(G_L, W) \hookrightarrow H^0(G_L, W_e)$ is bijective, thus $H^0(G_L, W_\sigma) \hookrightarrow H^0(G_L, W_{e,L,\sigma})$ is also bijective: it's straightforward to see the composition

$$(W_e \cap W^+_{\mathrm{dR}})^{G_L} \xrightarrow{\sim} (W_{e,L,\sigma} \cap W^+_{\mathrm{dR},\sigma})^{G_L} \hookrightarrow D_{e,L}(W)_\sigma \hookrightarrow D'_{e,L}(W)_\sigma$$

equals to the composition $(W_e \cap W^+_{\mathrm{dR}})^{G_L} \to D_e(W) \xrightarrow{\sim} D'_{e,L}(W)_\sigma$, from which the claim follows. In this case, we have $H^1_e(G_L, W) \cong D_{\mathrm{dR}}(W)/D^+_{\mathrm{dR}}(W)$ and $H^1_{e,\sigma}(G_L, W) \cong H^1_{e,\sigma}(G_L, W_\sigma) \cong D_{\mathrm{dR}}(W)_\sigma / D^+_{\mathrm{dR}}(W)_\sigma$. By comparing the dimension, we see, in this case, the map (35) is bijective (in particular injective). Consequently, in this case, for $J \subseteq \Sigma_L$, $J \neq \emptyset$, the following map is also injective and hence bijective (the surjectivity follows from definition):

$$\oplus_{\sigma\in J} H^1_{e,\sigma}(G_L, W_\sigma) \longrightarrow H^1_{e,J}(G_L, W).$$

By Cor.3.16 and the above discussion, one obtains

Corollary 3.20. *Suppose* W *is* J*-de Rham and* $H^0(G_L, W) \hookrightarrow H^0(G_L, W_e)$ *is bijective, then*

$$\dim_E H^1_{g,J}(G_L, W^\vee(1)) = \dim_E H^1(G_L, W^\vee(1)) - \sum_{\sigma\in J} \dim_E \big(D_{\mathrm{dR}}(W)_\sigma / D^+_{\mathrm{dR}}(W)_\sigma\big). \tag{36}$$

3.3. **Trianguline representations.**

Definition 3.21 (cf.[11], [24])**.** *(1) An* E*-*B*-pair* W *is called triangulable if it's an successive extension of rank* 1 E*-*B*-pairs, i.e.* W *admits an increasing filtration of* E*-*B*-sub-pairs*

$$0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{r-1} \subsetneq W_r = W \tag{37}$$

such that W_i/W_{i-1} *is an* E*-*B*-pair of rank* 1 *for* $1 \leq i \leq r$. *The filtration (37) is called a triangulation of* W.

(2) A finite dimensional continuous G_L*-representation* V *over* E *is called trianguline if the associated* E*-*B*-pair* $W(V)$ *is triangulable.*

Let χ be a continuous character of $L^\times$ over E, as in [24, §1.4], we can associate to χ an E-B-pair $B_E(\chi)$ of rank 1 (where we refer to *loc. cit.* for details). Conversely, given an E-B-pair of rank 1, by [24, Thm.1.45], there exists a continuous character χ of $L^\times$ over E such that $W \cong B_E(\chi)$. A continuous character $\chi : L^\times \to E^\times$ induces an E-linear map

$$d_\chi : L \otimes_{\mathbb{Q}_p} E \longrightarrow E, \ a \mapsto \frac{d}{dt}\chi(\exp(at))|_{t=0}.$$

So there exist $\mathrm{wt}(\chi)_\sigma \in E$ for all $\sigma \in \Sigma_L$ such that $d_\chi\big((a_\sigma)_{\sigma\in\Sigma_L}\big) = \sum_{\sigma\in\Sigma_L} a_\sigma \,\mathrm{wt}(\chi)_\sigma$ for any $(a_\sigma)_{\sigma\in\Sigma_L} \in L\otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma\in\Sigma_L} E$. We call $(\mathrm{wt}(\chi)_\sigma)_{\sigma\in\Sigma_L}$ *the weights of* χ. In fact, $(-\mathrm{wt}(\chi)_\sigma)_{\sigma\in\Sigma_L}$ equal to the *generalized Hodge-Tate weights* of the E-B-pair $B_E(\chi)$ (cf. [24, Def. 1.47]).

Lemma 3.22. *Let χ be a continuous character of $L^\times$ over E, for $\sigma \in \Sigma_L$, $B_E(\chi)$ is σ-de Rham if and only if* $\mathrm{wt}(\chi)_\sigma \in \mathbb{Z}$.

Proof. The "only if" part is clear. Suppose now $\mathrm{wt}(\chi)_\sigma \in \mathbb{Z}$, by multiplying χ by $\sigma^{-\mathrm{wt}(\chi)_\sigma}$ and then an unramified character of $L^\times$, one can assume that χ corresponds to a Galois character $\chi : G_L \to E^\times$ and $\mathrm{wt}(\chi)_\sigma = 0$. In this case, by Sen's theory, one has $\mathbb{C}_{p,\sigma} \otimes_E \chi \cong \mathbb{C}_{p,\sigma}$ (since χ is of Hodge-Tate weight 0 at σ). Consider the exact sequence

$$0 \to (tB^+_{\mathrm{dR},\sigma} \otimes_E \chi)^{G_L} \to (B^+_{\mathrm{dR},\sigma} \otimes_E \chi)^{G_L} \to (\mathbb{C}_{p,\sigma} \otimes_E \chi)^{G_L} \to H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E \chi),$$

it's sufficient to prove $H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E \chi) = 0$. For $i \in \mathbb{Z}_{>0}$, we claim $H^1(G_L, t^{i+1}B^+_{\mathrm{dR},\sigma} \otimes_E \chi) \to H^1(G_L, t^iB^+_{\mathrm{dR},\sigma} \otimes_E \chi)$ is an isomorphism: indeed, one has an exact sequence

$$(\mathbb{C}_{p,\sigma}(i) \otimes_E \chi)^{G_L} \to H^1(G_L, t^{i+1}B^+_{\mathrm{dR},\sigma} \otimes_E \chi) \to H^1(G_L, t^iB^+_{\mathrm{dR},\sigma} \otimes_E \chi) \to H^1(G_L, \mathbb{C}_{p,\sigma}(i) \otimes_E \chi),$$

since $\mathbb{C}_{p,\sigma} \otimes_E \chi \cong \mathbb{C}_{p,\sigma}$, the first and fourth terms vanish. We get an isomorphism $H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E \chi) \xrightarrow{\sim} H^1(G_L, t^nB^+_{\mathrm{dR},\sigma} \otimes_E \chi)$ for $n \gg 0$, from which we deduce $H^1(G_L, tB^+_{\mathrm{dR},\sigma} \otimes_E \chi) = 0$. □

Definition 3.23 (cf. [22, Def.4.3.1])**.** *Let W be a triangulable E-B-pair of rank r with a triangulation given as in (37), let $\chi_i : L^\times \to E^\times$ such that $W_i/W_{i-1} \cong B_E(\chi_i)$, for $\sigma \in \Sigma_L$, suppose* $\mathrm{wt}(\chi_i)_\sigma \in \mathbb{Z}$ *for all $1 \leq i \leq r$, W is called non σ-critical if* (*note the generalized Hodge-Tate weight of $B_E(\chi_i)$ at σ is* $-\mathrm{wt}(\chi_i)_\sigma$)

$$\mathrm{wt}(\chi_1)_\sigma > \mathrm{wt}(\chi_2)_\sigma > \cdots > \mathrm{wt}(\chi_r)_\sigma;$$

for $\emptyset \neq J \subseteq \Sigma_L$, suppose $\mathrm{wt}(\chi_i)_\sigma \in \mathbb{Z}$ *for $1 \leq i \leq r$, $\sigma \in J$, then W is called non J-critical if W is non σ-critical for all $\sigma \in J$.*

Proposition 3.24. *Keep the notation in Def.3.23, let $\emptyset \neq J \subseteq \Sigma_L$, suppose W is non J-critical, then W is J-de Rham.*

Proof. It's sufficient to prove if W is non-σ-critical, then W is σ-de Rham for $\sigma \in J$. We use induction on $0 \leq i \leq r$, suppose W_i is σ-de Rham, we show W_{i+1} is also σ-de Rham. Note $[W_{i+1}] \in \mathrm{Ext}^1\big(W_i, B_E(\chi_{i+1})\big)$, let $W_i' := W_i \otimes B_E(\chi_{i+1}^{-1})$, $W_{i+1}' := W_{i+1} \otimes B_E(\chi_{i+1}^{-1})$, by Lem.3.22, W_{i+1} is σ-de Rham if and only if W_{i+1}' is σ-de Rham. And one has $[W_{i+1}'] \in H^1(G_L, W_i')$. It's sufficient to prove $H^1(G_L, W_i') = H^1_{g,\sigma}(G_L, W_i')$, and thus sufficient to prove $H^1_{e,\sigma}\big(G_L, (W_i')^\vee(1)\big) = 0$. Since $\mathrm{wt}(\chi_j)_\sigma > \mathrm{wt}(\chi_{i+1})_\sigma$ for $1 \leq j \leq i$, we see the generalized Hodge-Tate weights of W_i' at σ are negative integers. Thus the generalized Hodge-Tate weights of $(W_i')^\vee(1)$ at σ are positive integers, so one has $D_{\mathrm{dR}}\big((W_i')^\vee(1)\big)_\sigma \cong D^+_{\mathrm{dR}}\big((W_i')^\vee(1)\big)_\sigma$. By Lem.3.19, $H^1_{e,\sigma}\big(G_L, (W_i')^\vee(1)\big) = 0$, which concludes the proof. □

4. Some applications

We give some applications of the above results in p-adic arithmetic.

4.1. **Overconvergent Hilbert modular forms.** Let F be a totally real number field of degree d_F, Σ_F the set of embeddings of F in $\overline{\mathbb{Q}}$, $w \in \mathbb{Z}$, and $k_\sigma \in \mathbb{Z}_{\geq 2}$, $k_\sigma \equiv w \pmod 2$ for all $\sigma \in \Sigma_F$. Let $\mathfrak{c}$ be a fractional ideal of F. Let h be an overconvergent Hilbert eigenform of weights $(\underline{k}, w)$ (where we adopt Carayol's convention of weights as in [10])), of tame level N ($N \geq 4$, $p \nmid N$), of polarization $\mathfrak{c}$, with Hecke eigenvalues in E (see [1, Def.1.1]). For a place $\wp$ of F above p, let $a_\wp$ denote the $U_\wp$-eigenvalue of h, and suppose $a_\wp \neq 0$ for all $\wp | p$. Denote by $\rho_h : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(E)$ the associated (semi-simple) Galois representation (enlarge E if necessary) (e.g. see [1, Thm.5.1]). For $\wp | p$, denote by $\rho_{h,\wp}$ the restriction of ρ_h to the decomposition group at $\wp$, which is thus a continuous representation of $G_{F_\wp}$ over E, where $F_\wp$ denotes the completion of F at $\wp$. Let $v_\wp : \overline{\mathbb{Q}_p} \to \mathbb{Q} \cup \{+\infty\}$ be an additive valuation normalized by $v_\wp(F_\wp) = \mathbb{Z} \cup \{+\infty\}$. Denote by $\Sigma_{F_\wp}$ the set of embeddings of $F_\wp$ in $\overline{\mathbb{Q}_p}$. This section is devoted to prove

Theorem 4.1. *With the above notation, and let $\emptyset \neq J \subseteq \Sigma_{F_\wp}$.*

(1) If $v_\wp(a_\wp) < \inf_{\sigma \in J}\{k_\sigma - 1\} + \sum_{\sigma \in \Sigma_{F_\wp}} \frac{w - k_\sigma + 2}{2}$, then $\rho_{h,\wp}$ is J-de Rham.

(2) If $v_\wp(a_\wp) < \sum_{\sigma \in S}(k_\sigma - 1) + \sum_{\sigma \in \Sigma_{F_\wp}} \frac{w - k_\sigma + 2}{2}$, then there exists $\sigma \in S$ such that $\rho_{h,\wp}$ is σ-de Rham.

Remark 4.2. *This theorem gives evidence to Breuil's conjectures in [7] (but in terms of Galois representations) (see in particular [7, Prop.4.3]). When $J = \Sigma_{F_\wp}$ (and $F_\wp$ unramified), the part (1) follows directly from the known classicality result in [30].*

Proposition 4.3. *For $\wp | p$, $\rho_{h,\wp}$ is trianguline with a triangulation given by*

$$0 \to B_E(\delta_1) \to W(\rho_{h,\wp}) \to B_E(\delta_2) \to 0,$$

with

$$\begin{cases} \delta_1 = \mathrm{unr}_\wp(a_\wp) \prod_{\sigma \in \Sigma_\wp} \sigma^{-\frac{w - k_\sigma + 2}{2}} \prod_{\sigma \in \Sigma_h} \sigma^{1 - k_\sigma}, \\ \delta_2 = \mathrm{unr}_\wp(q_\wp b_\wp / a_\wp) \prod_{\sigma \in \Sigma_\wp} \sigma^{-\frac{w + k_\sigma}{2}} \prod_{\sigma \in \Sigma_h} \sigma^{k_\sigma - 1}, \end{cases}$$

where $\mathrm{unr}_\wp(z)$ denotes the unramified character of $F_\wp^\times$ sending uniformizers to z, $q_\wp := p^{f_\wp}$ with $f_\wp$ the degree of the maximal unramified extension inside $F_\wp$ (thus $v_\wp(q_\wp) = d_\wp$, the degree of $F_\wp$ over $\mathbb{Q}_p$), and Σ_h is a certain subset of $\Sigma_\wp$.

Proof. Consider the eigenvariety $\mathcal{E}$ constructed in [1, Thm.5.1], one can associate to h a closed point z_h in $\mathcal{E}$. For classical Hilbert eigenforms, the result is known by Saito's results in [26] and Nakamura's results on triangulations of 2-dimensional potentially semi-stable Galois representations (cf. [24, §4]). Since the classical points are Zariski-dense in $\mathcal{E}$ and accumulate over the point z_h (here we use the classicality results, e.g. see [4]), the proposition follows from the global triangulation theory [20, Thm.6.3.13] [22, Thm.4.4.2]. □

Since $W(\rho_\wp)$ is étale (purely of slope zero), by Kedlaya's slope filtration theroy ([19, Thm.1.7.1]), one has (see also [24, Lem.3.1])

Lemma 4.4. *Let $\varpi_\wp$ be a uniformizer of $F_\wp$, then $v_\wp(\delta_1(\varpi_\wp)) \geq 0$.*

Proof of Thm.4.1. By the above lemma, one has $v_\wp(a_\wp) \geq \sum_{\sigma \in \Sigma_h}(k_\sigma - 1) + \sum_{\sigma \in \Sigma_\wp} \frac{w - k_\sigma + 2}{2}$. Thus for $\emptyset \neq J \subseteq \Sigma_{F_\wp}$, if $v_\wp(a_\wp) < \inf_{\sigma \in J}\{k_\sigma - 1\} + \sum_{\sigma \in \Sigma_{F_\wp}} \frac{w - k_\sigma + 2}{2}$ $\Big($resp. $v_\wp(a_\wp) < \sum_{\sigma \in S}(k_\sigma - 1) + \sum_{\sigma \in \Sigma_{F_\wp}} \frac{w - k_\sigma + 2}{2}\Big)$, then $J \cap \Sigma_h = \emptyset$ $\big($resp. $J \nsubseteq \Sigma_h\big)$ and thus $\rho_{h,\wp}$ is non-J-critical $\big($resp. there exists $\sigma \in J$ such that $\rho_{h,\wp}$ is non-σ-critical$\big)$ (Note $\Sigma_{F_\wp} \setminus \Sigma_h$ is exactly the set of embeddings where $\rho_{h,\wp}$ is non-critical). The theorem then follows from Prop.3.24. □

4.2. **Locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(L)$ in the trianguline case.** As another application, we associate a (semi-simple) locally $\mathbb{Q}_p$-analytic representation $\Pi(V)$ of $\mathrm{GL}_2(L)$ to a 2-dimensional trianguline representation V of G_L, and we expect it to be the socle of the "right" representation associated to V in the p-adic Langlands program (cf. Conj.4.9).

Firstly recall some notions on locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_2(L)$. We denote by $\mathfrak{g}$ the Lie algebra of $\mathrm{GL}_2(L)$, and $\mathfrak{g}_\sigma := \mathfrak{g} \otimes_{L,\sigma} E$ for $\sigma \in \Sigma_L$. We have a natural decomposition

$$\mathfrak{g} \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \prod_{\sigma \in \Sigma_L} \mathfrak{g}_\sigma.$$

For $J \subseteq \Sigma_L$, we put $\mathfrak{g}_J := \prod_{\sigma \in J} \mathfrak{g}_\sigma$ (and $\mathfrak{g}_\emptyset := \{0\}$).

Let V be a locally $\mathbb{Q}_p$-analytic representation of $\mathrm{GL}_2(L)$ over E, which is thus equipped with a natural E-linear action of $\mathfrak{g}$ (hence of $\mathfrak{g} \otimes_{\mathbb{Q}_p} E$) given by

$$\mathfrak{x} \cdot v := \frac{d}{dt} \exp(t\mathfrak{x})(v)|_{t=0}.$$

For $J \subseteq \Sigma_L$, a vector $v \in V$ is called *locally J-analytic* if the action of $\mathfrak{g}_{\Sigma_L}$ on v factors through $\mathfrak{g}_J$ (cf. [28, Def.2.4]); v is called *quasi-J-classical* if there exist a finite dimensional representation U of $\mathfrak{g}_J$ and a $\mathfrak{g}_J$-invariant map

$$U^{\oplus n} \longrightarrow V \tag{38}$$

(with $n \in \mathbb{Z}_{>0}$) whose image contains v, if the $\mathfrak{g}_J$-representation U can moreover give rise to an algebraic representation of $\mathrm{GL}_2(L)$, then we say that v is *J-classical*. In particular, v is $\Sigma_L \setminus J$-classical if v is locally J-analytic.

Let $\chi : L^\times \to E^\times$ be a continuous character such that

$$\begin{cases} \mathrm{wt}(\chi)_\sigma \neq 0, \text{ for all } \sigma \in \Sigma_L, \\ \chi \neq \prod_{\sigma \in \Sigma_L} \sigma^{k_\sigma}, \text{ for all } (k_\sigma)_{\sigma \in \Sigma_L} \in \mathbb{Z}^{|d|}. \end{cases} \tag{39}$$

Put $S(\chi) := \{\sigma \in \Sigma_L \mid \mathrm{wt}(\chi)_\sigma \in \mathbb{Z}\}$, $N(\chi) := \{\sigma \in S(\chi) \mid \mathrm{wt}(\chi)_\sigma > 0\}$. Consider $\Pi(\chi) := \big(\mathrm{Ind}_{B(L)}^{\mathrm{GL}_2(L)} 1 \otimes \big(\chi\, \mathrm{unr}(q) \prod_{\sigma \in \Sigma_L} \sigma^{-1}\big)\big)^{\mathbb{Q}_p-\mathrm{an}}$ where $B(L)$ denotes the Borel subgroup of upper triangular matrices, and we refer to [28, §2.3] for locally $\mathbb{Q}_p$-analytic (and locally J-analytic) parabolic inductions. We begin by an observation (which follows directly from results in [28, §2]):

Lemma 4.5. *One has*

$$\pi(\chi, N(\chi)) := \mathrm{soc}_{\mathrm{GL}_2(L)} \Pi(\chi) \xrightarrow{\sim} \bigotimes_{\sigma \in N(\chi)} (\mathrm{Sym}^{\mathrm{wt}(\chi)_\sigma - 1} E^2)^\sigma$$
$$\otimes_E \Big(\mathrm{Ind}_{B(L)}^{\mathrm{GL}_2(L)} 1 \otimes \Big(\chi\, \mathrm{unr}(q) \prod_{\sigma \in \Sigma_L} \sigma^{-1} \prod_{\sigma \in N(\chi)} \sigma^{1-\mathrm{wt}(\chi)_\sigma}\Big)\Big)^{\Sigma_L \setminus N(\chi)-\mathrm{an}},$$

where the action of $\mathrm{GL}_2(L)$ on $(\mathrm{Sym}^{n_\sigma} E^2)^\sigma$, for $n_\sigma \in \mathbb{Z}_{\geq 0}$, is induced by the standard action of $\mathrm{GL}_2(E)$ on $\mathrm{Sym}^{n_\sigma - 1} E^2$ via the embedding $\sigma : \mathrm{GL}_2(L) \hookrightarrow \mathrm{GL}_2(E)$. In particular, $\Pi(\chi)$ has non-zero $N(\chi)$-classical vectors.

For $N(\chi) \subseteq J \subseteq S(\chi)$, we put

$$I(\chi, J) := \Big(\mathrm{Ind}_{B(L)}^{\mathrm{GL}_2(L)} 1 \otimes \Big(\chi\, \mathrm{unr}(q) \prod_{\sigma \in \Sigma_L} \sigma^{-1} \prod_{\sigma \in J} \sigma^{1-\mathrm{wt}(\chi)_\sigma}\Big)\Big)^{\Sigma_L \setminus J-\mathrm{an}},$$
$$\pi(\chi, J) := \bigotimes_{\sigma \in J} \big(\mathrm{Sym}^{|\mathrm{wt}(\chi)_\sigma| - 1} E^2\big)^\sigma \otimes_E \Big(\prod_{\sigma \in J \setminus N(\chi)} \sigma^{\mathrm{wt}(\chi)_\sigma} \circ \det\Big) \otimes_E I(\chi, J).$$

The locally analytic representations $I(\chi, J)$, $\pi(\chi, J)$ are J-classical, and (topologically) irreducible if $J \neq \Sigma_L$ or $\chi \prod_{\sigma\in\Sigma_L} \sigma^{k_\sigma} \neq \mathrm{unr}(q^{\pm 1})$ for all $(k_\sigma)_{\sigma\in\Sigma_L} \in \mathbb{Z}^d$.

Now let W be a trianguline E-B-pair with a triangulation given by

$$0 \to W(\chi_1) \to W \to W(\chi_2) \to 0, \tag{40}$$

suppose $\chi := \chi_1\chi_2^{-1}$ satisfies the hypothesis in (39). Denote by $C(W)$ the set of embeddings σ such that $W \otimes B_E(\chi_2^{-1})$ is σ-de Rham. By Prop.3.24, $N(\chi) \subseteq C(W)$.

Assume firstly that W has a unique triangulation as in (40). If $C(W) \neq \Sigma_L$ or $\chi \prod_{\sigma\in\Sigma_L} \sigma^{k_\sigma} \neq \mathrm{unr}(q^{-1})$ for all $(k_\sigma)_{\sigma\in\Sigma_L} \in \mathbb{Z}^d$, set

$$\Pi(W) := (\chi_2 \circ \det) \otimes_E \Big(\oplus_{N(\chi)\subseteq J\subseteq C(W)} \pi(\chi, J)\Big); \tag{41}$$

if $C(W) = \Sigma_L$ and $\chi = \mathrm{unr}(q^{-1}) \prod_{\sigma\in\Sigma_L} \sigma^{\mathrm{wt}(\chi)_\sigma}$, we put

$$\Pi(W) := (\chi_2 \circ \det) \otimes_E \Big(\pi(\chi, \Sigma_L)/F(\chi) \oplus \big(\oplus_{N(\chi)\subseteq J\subsetneq \Sigma_L} \pi(\chi, J)\big)\Big), \tag{42}$$

where $F(\chi) := \bigotimes_{\sigma\in\Sigma_L} \big(\mathrm{Sym}^{|\mathrm{wt}(\chi)_\sigma|-1} E^2\big)^\sigma$ is the unique non-zero finite dimensional subrepresentation of $\pi(\chi, \Sigma_L)$ in this case.

Assume now that W has a triangulation other than (40) (this is the case potentially cristallin up to twists of characters, cf. [24]):

$$0 \to W(\chi_3) \to W \to W(\chi_4) \to 0. \tag{43}$$

By [24, Thm.3.7], one has $C(W) = S(\chi) = \Sigma_L$, and there exists $S \subseteq N(\chi)$ such that

$$\chi_3 = \chi_2 \prod_{\sigma\in S} \sigma^{\mathrm{wt}(\chi)_\sigma}, \chi_4 = \chi_1 \prod_{\sigma\in S} \sigma^{-\mathrm{wt}(\chi)_\sigma}. \tag{44}$$

Let $\chi' := \chi_3\chi_4^{-1}$, thus $S(\chi') = S(\chi) = \Sigma_L$ $|\mathrm{wt}(\chi')_\sigma| = |\mathrm{wt}(\chi)_\sigma|$ for all $\sigma \in \Sigma_L$, and $N(\chi') = S \cup (\Sigma_L \setminus N(\chi))$. Put $\chi_0^* := \chi^* \prod_{\sigma\in\Sigma_L} \sigma^{\mathrm{wt}(\chi^*)_\sigma}$ for $* \in \{', \emptyset\}$. Thus $\chi_0' = \chi_0^{-1}$. If $\chi_0 \neq \mathrm{unr}(q^{\pm 1})$, by the intertwining relation of smooth parabolic inductions of $\mathrm{GL}_2(L)$ (and (44)), one can easily deduce $(\chi_2 \circ \det) \otimes_E \pi(\chi, \Sigma_L) \cong (\chi_4 \circ \det) \otimes_E \pi(\chi', \Sigma_L) =: \pi(W)$; if $\chi_0 = \mathrm{unr}(q^{-1})$ $\big($resp. $\chi_0 = \mathrm{unr}(q)\big)$, then $\chi_0' = \mathrm{unr}(q)$ $\big($resp. $\chi_0' = \mathrm{unr}(q^{-1})\big)$, one has $\pi(W) := (\chi_2 \circ \det) \otimes_E \big(\pi(\chi, \Sigma_L)/F(\chi)\big) \cong (\chi_4 \circ \det) \otimes_E \mathrm{soc}_{\mathrm{GL}_2(L)} \pi(\chi', \Sigma_L)$ $\big($resp. $\pi(W) := (\chi_2 \circ \det) \otimes_E \mathrm{soc}_{\mathrm{GL}_2(L)} \pi(\chi, \Sigma_L) \cong (\chi_4 \circ \det) \otimes_E \big(\pi(\chi', \Sigma_L)/F(\chi')\big)\big)$. In this case, put

$$\Pi(W) := \pi(W) \oplus \big((\chi_2\circ\det) \otimes_E \big(\oplus_{N(\chi)\subseteq J\subsetneq\Sigma_L} \pi(\chi, J)\big)\big) \oplus \big((\chi_4\circ\det) \otimes_E \big(\oplus_{N(\chi')\subseteq J\subsetneq\Sigma_L} \pi(\chi', J)\big)\big). \tag{45}$$

Remark 4.6. *(1) The representation $\Pi(W)$ is semi-simple, indeed, in this note we only consider the socle of the "right" representation associated to W in p-adic Langlands (cf. Conj.4.9 below). By definition and [24, Th.1.18 (3)], we see that $\Pi(W)$ is no other than $\mathrm{soc}_{\mathrm{GL}_2(L)} \Pi(D)$, with $\Pi(D)$ defined by Breuil in [8], when W is crystalline and $D := D_{\mathrm{cris}}(W) \cong (B_{\mathrm{cris}} \otimes_{B_e} W_e)^{G_L}$ $\big($which is hence a filtered φ-module over $L \otimes_{\mathbb{Q}_p} E$ as in [8, §3]$\big)$.*

(2) By the very construction of $\Pi(W)$, we see that if W is J-de Rham up to twist of characters, then $\Pi(W)$ has non-zero quasi-J-classical vectors. Note also that our definition of $\Pi(W)$ highly relies on Prop.3.24, since if we don't know $N(\chi) \subseteq C(W)$ a priori, this construction does not make any sense (see (41)).

As in [8, Cor.5.2], one can prove

Proposition 4.7. *Keep the above notation, if* $\Pi(W)$ *has a* $\mathrm{GL}_2(L)$*-invariant* $\mathcal{O}_E$*-lattice* $\big($*in other words,* $\Pi(W)$ *is contained in a unitary Banach representation of* $\mathrm{GL}_2(L)\big)$*, then the* E*-*B*-pair* W *is étale, i.e. there exists a* 2*-dimensional continuous* G_L*-representation* V *over* E *such that* $W \cong W(V)$*.*

Proof. We use the notation of [24, §3].

If $\Pi(W)$ is contained in a unitary Banach representation of $\mathrm{GL}_2(L)$, by [8, Prop.5.1] applied to $(\chi_2 \circ \det) \otimes_E \pi(\chi, N(\chi))$, we have

$$\upsilon(\chi_1(\varpi_L)) + \upsilon(\chi_2(\varpi_L)) = 0, \ \upsilon(\chi_1(\varpi_L)) \geq 0,$$

and hence $(\chi_1, \chi_2) \in S^+$ (cf. [24, §3.1]).

If W has a unique triangulation as in (40), by [24, Thm.3.7 (1)] $\big($although this theorem is in terms of trianguline representations, we can get similar results for rank 2 triangulable E-B-pairs by the same argument$\big)$, we have $[W] \notin S'(\chi_1, \chi_2)$. Then by [24, Thm.3.4], we see W is étale.

In the case where W have two different triangulations given by (40) and (43), by the same argument as in the proof of [24, Thm.3.7], we see

$$[W] \in S'(\chi_1, \chi_2) \cong \mathbb{P}_E\big(\oplus_{\sigma \in N(\chi)} E e_\sigma\big).$$

There exist hence $a_\sigma \in E$ for all $\sigma \in N(\chi)$ such that $[W] = \big[\sum_{\sigma \in N(\chi)} a_\sigma e_\sigma\big] \in \mathbb{P}_E\big(\oplus_{\sigma \in N(\chi)} E e_\sigma\big)$. Moreover, $a_\sigma \neq 0$ if and only if $\sigma \in S$ (cf. (44)). By [8, Prop.5.1] applied to $(\chi_4 \circ \det) \otimes_E \pi(\chi', N(\chi'))$, we have $\upsilon(\chi_3(\varpi_L)) \geq 0$, and hence (by (44))

$$\upsilon(\chi_2(\varpi_L)) + \sum_{\sigma \in S} \mathrm{wt}(\chi)_\sigma \geq 0.$$

So $[W] \in S'^{\text{ét}}(\chi_1, \chi_2)$ (cf. [24, §3.1]) and W is étale by [24, Thm.3.4]. □

Remark 4.8. *The proposition in the case where* W *is cristallin (so* W *have two different triangulations) can also be deduced directly from* [8, Cor.5.2] *(see Rem.4.6 (1)), since in this case,* W *is étale if and only if the associated filtered* φ*-module* $D_{\mathrm{cris}}(W)$ *is weakly admissible (cf.* [2, Thm.B] *and* [24, Thm.1.18 (3)]*).*

4.2.1. *Some local-global consideration.* Let F be a totally real field of degree $d \in \mathbb{Z}_{\geq 1}$ over $\mathbb{Q}$, assume that p is inert in F and the completion F_p of F at p is isomorphic to L. Let B be a quaternion algebra of center F, denote by $S(B)$ the set of places of F where B is ramified. We assume $|S(B) \cap \Sigma_\infty| = d - 1$ and $p \notin S(B)$. In this case, we can associate to B a projective system of algebraic curves $\{S(K)\}_{K \subset (B \otimes_F \mathbb{A}_F^\infty)^\times}$ over F, indexed by compact open subgroups of $(B \otimes_F \mathbb{A}_F^\infty)^\times$ (where $\mathbb{A}_F^\infty := \prod'_{v \nmid \infty} F_v$ denotes the ring of finite adeles of F). Let K^p be a compact open subgroup of $(B \otimes_F \mathbb{A}_F^{\infty,p})^\times$ (where $\mathbb{A}_F^{\infty,p} := \prod_{v \nmid \infty, p} F_v$), following Emerton ([14, §2]), we put

$$\widehat{H}^1(K^p) := \Big(\varprojlim_n \varinjlim_{K_p} H^1_{\text{ét}}\big(S(K^p K_p) \times_F \overline{\mathbb{Q}}, \mathcal{O}_E/p^n \mathcal{O}_E\big)\Big) \otimes_{\mathcal{O}_E} E$$

where K_p ranges over all compact open subgroups of $(B \otimes_F F_p)^\times \cong \mathrm{GL}_2(L)$. By *loc. cit.*, $\widehat{H}^1(K^p)$ is a Banach space over E equipped with a continuous unitary action of $\mathrm{GL}_2(L) \times \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Let ρ be a 2-dimensional continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-representation over E, we put

$$\widehat{\Pi}(\rho) := \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}\big(\rho, \widehat{H}^1(K^p)\big).$$

Conjecture 4.9 (See [8, Cor.8.1] for the crystalline case)**.** *Suppose* $\widehat{\Pi}(\rho) \neq 0$ *and* $\rho_p := \rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F_p)}$ *is trianguline, then there exist* $r \in \mathbb{Z}_{>0}$ *and an isomorphism of* $\mathrm{GL}_2(L)$*-representations*

$$\Pi(W(\rho_p))^{\oplus r} \xrightarrow{\sim} \mathrm{soc}_{\mathrm{GL}_2(F_p)} \widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}},$$

where $\widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}$ *denotes the locally* $\mathbb{Q}_p$*-analytic vectors of* $\widehat{\Pi}(\rho)$ $\big($*for the action of* $\mathrm{GL}_2(L)\big)$.

Remark 4.10. *(1) When* ρ *is modular (i.e.* ρ *appears in the classical étale cohomology of quaternion Shimura curves, thus* $C(W(\rho_p)) = \Sigma_L$*), then by local-global compatibility in local Langlands correspondence for* $\ell = p$, $\pi(W(\rho_p))$ *is a subrepresentation of* $\widehat{\Pi}(\rho)$ *(cf. (45)).*

(2) This conjecture implies that if ρ_p *is* J*-de Rham, then* $\widehat{\Pi}(\rho)$ *has non-zero* J*-classical vectors. In particular, if* ρ_p *is de Rham, then* $\widehat{\Pi}(\rho)$ *would have non-zero classical vectors, which is more or less an equivalent formulation, in terms of locally analytic representations, of the Fontaine-Mazur conjecture for finite slope overconvergent Hilbert modular forms.*

(3) As in [12, Prop.6.2.40] [13, Prop.4.14]*, one can actually show if* $\Pi(W(\rho))$ *is a subrepresentation of* $\widehat{\Pi}(\rho)$*, then* ρ_p *is* $C(W(\rho_\wp))$*-de Rham.*

(4) This conjecture also implies that if $\widehat{\Pi}(\rho) \neq 0$ *and* ρ_p *trianguline, then* $J_B(\widehat{\Pi}(\rho)_{\mathbb{Q}_p-\mathrm{an}}) \neq 0$ *where* $J_B(\cdot)$ *denotes the Jacquet-Emerton functor* [15].

(5) If we suppose $J_B(\widehat{\Pi}(\rho)) \neq 0$*, one can then associate to* ρ *certain points in the eigenvariety (constructed by Emerton's theory). Moreover, as in* [12, §6.2.2]*, by companion points theory and adjunction formula for Jacquet-Emerton functor (cf.* [16]*), one can actually prove there exists certain non-zero subrepresentation* $\Pi'(W(\rho_p)) \subseteq \Pi(W(\rho_p))$*, such that* $\Pi'(W(\rho_p)) \hookrightarrow \widehat{\Pi}(\rho)$.

(6) In [9]*, Breuil explains to the author his beautiful idea on how to see a larger locally analytic representation, say* $\tilde{\Pi}(W(\rho_\wp))$*, inside* $\widehat{\Pi}(\rho)$*, with* $\mathrm{soc}_{\mathrm{GL}_2(L)}\tilde{\Pi}(W(\rho_\wp)) \cong \Pi(W(\rho_\wp))$*. Roughly speaking, Breuil associates to each irreducible constituent of* $\Pi(W(\rho_\wp))$ *a "hypercube" (which is kind of similar as that in* [8, §4]*, but involves some extensions in "opposite" direction). This representation still carries the same information on* $\rho_\wp$ *as* $\Pi(W(\rho_\wp))$*. We would not include it in the note, since this requires much more locally analytic representation theory.*

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