# Locally analytic $\mathrm{Ext}^1$ for $\mathrm{GL}_2(\mathbb{Q}_p)$ in de Rham non trianguline case

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**Abstract**

We prove Breuil's conjecture on locally analytic $\mathrm{Ext}^1$ for $\mathrm{GL}_2(\mathbb{Q}_p)$ in de Rham non-trianguline case.

## 1 Introduction

Let $E$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{R}_E$ be the Robba ring with $E$-coefficients. The (locally analytic) $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ associates to a $(\varphi,\Gamma)$-module $D$ of rank 2 over $\mathcal{R}_E$ a locally analytic representation $\pi(D)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E$ (see for example [8, § 0.1]). The representation $\pi(D)$ determines $D$ (and vice versa). Indeed, when $D$ is trianguline, this follows from the explicit structure of $\pi(D)$ and $D$. When $D$ is not trianguline, one can reduce to the case where $D$ is étale hence isomorphic to $D_{\mathrm{rig}}(\rho)$ for a certain 2-dimensional representation $\rho$ of the absolute Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$ over $E$. In this case, by [10, Thm. 0.2], the universal completion of $\pi(D)$ is exactly the Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\rho$, which determines $\rho$ (hence $D$) via Colmez's Montreal functor (see [7, Thm. 0.17 (iii)]).

The $p$-adic local Langlands correspondence is compatible with (and refines) the classical local Langlands correspondence. We recall the feature in more details. Suppose that $D$ is de Rham of Hodge-Tate weights $(0,k)$ with $k\geq 1$ (where we use the convention that the Hodge-Tate weight of the cyclotomic character is 1). We can associate to $D$ a smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation in the following way:

$$\underbrace{D \longleftrightarrow D_{\mathrm{pst}}(D) \rightsquigarrow \mathrm{DF}}_{p\text{-adic Hodge theory}} \longleftrightarrow \underbrace{\mathtt{r} \longleftrightarrow \pi_\infty(\mathtt{r})}_{\text{local Langlands}}$$

where

- $D_{\mathrm{pst}}(D)$ is the filtered $(\varphi, N, \mathrm{Gal}(L/\mathbb{Q}_p))$-module associated to $D$ (cf. [2, Thm. A]), where $L$ is a certain finite extension of $\mathbb{Q}_p$,
- DF is the underlying Deligne-Fontaine module (i.e. $(\varphi, N, \mathrm{Gal}(L/\mathbb{Q}_p))$-module) of $D_{\mathrm{pst}}(D)$ (by forgetting the Hodge filtration),
- $\mathtt{r}$ is the 2-dimensional Weil-Deligne representation associated to DF as in [6, § 4],
- $\pi_\infty(\mathtt{r}) := \mathrm{rec}^{-1}(\mathtt{r})$ is the smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated to $\mathtt{r}$ via the classical local Langlands correspondence (normalized as in [13], in particular, the central character $\omega_{\pi_\infty(\mathtt{r})}$ is $\wedge^2\mathtt{r}\otimes_E \mathrm{unr}(p)$, where we view the one-dimensional Weil representation $\wedge^2\mathtt{r}$ as a character of $\mathbb{Q}_p^\times$ via $W_{\mathbb{Q}_p}^{\mathrm{ab}}\cong \mathbb{Q}_p^\times$, normalized by sending geometric Frobenius to uniformizers, and where $\mathrm{unr}(p)$ is the unramified character of $\mathbb{Q}_p^\times$ sending uniformizers to $p$).

Put $\pi_{\mathrm{alg}}(\mathtt{r},k) := \mathrm{Sym}^{k-1} E^2 \otimes_E \pi_\infty(\mathtt{r})$, which is a locally algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (for the diagonal action). Then there is a natural injection ([12, Thm. 3.3.22]):

$$\pi_{\mathrm{alg}}(\mathtt{r},k) \hookrightarrow \pi(D).$$

It turns out that the quotient $\pi_c(\mathtt{r},k) := \pi(D)/\pi_{\mathrm{alg}}(\mathtt{r},k)$ (that is a locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as well) also depends only on and determines $\{\mathtt{r},k\}$ (see [9, § 0.2]). One may view the correspondence $\pi_c(\mathtt{r},k) \leftrightarrow \{\mathtt{r},k\}$ as a local Langlands correspondence for the simple reflection in the Weyl group $\mathscr{W} \cong S_2$ of $\mathrm{GL}_2$ (see [5, Remark 5.3.2 (iv)] for related discussions).

We let $\Delta$ be the $p$-adic differential equation associated to $D$, i.e. the $(\varphi,\Gamma)$-module associated to DF equipped with the trivial Hodge filtration via [2, Thm. A]. By *loc. cit.*, the category of $p$-adic differential equations is equivalent to the category of Deligne-Fontaine modules (that is equivalent to the category of Weil-Deligne representations). We have natural isomorphisms $D_{\mathrm{pst}}(\Delta) \xrightarrow{\sim} \mathrm{DF}$ (as Deligne-Fontaine module), and $D_{\mathrm{dR}}(\Delta) \xrightarrow{\sim} D_{\mathrm{dR}}(D)$ (as $E$-vector space). The Hodge filtration on $D_{\mathrm{dR}}(D)$ has the following form

$$\mathrm{Fil}^i D_{\mathrm{dR}}(D) = \begin{cases} D_{\mathrm{dR}}(\Delta) & i \leq -k \\ \mathcal{L}(D) & -k < i \leq 0 \\ 0 & i > 0 \end{cases}, \tag{1}$$

where $\mathcal{L}(D)$ is a certain $E$-line in $D_{\mathrm{dR}}(\Delta)$. By [2, Thm. A], $D$ is equivalent to the data $\{\Delta, k, \mathcal{L}(D)\}$ (or equivalently $\{\mathtt{r}, k, \mathcal{L}(D)\}$). And we see when we pass from $D$ to $\{\mathtt{r},k\}$, we lose exactly the information on $\mathcal{L}(D)$. To make the notation more consistent, we write $\pi_{\mathrm{alg}}(\Delta,k) := \pi_{\mathrm{alg}}(\mathtt{r},k)$, and $\pi_c(\Delta,k) := \pi_c(\mathtt{r},k)$. As the whole locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\pi(D)$ can determine $D$ while the constituents $\pi_{\mathrm{alg}}(\Delta,k)$, $\pi_c(\Delta,k)$ only determine $\{\Delta,k\}$, this suggests the information on $\mathcal{L}(D)$ should be contained in the corresponding extension class (see [4, § 2.1] for the definition of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}$)

$$[\pi(D)] \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big).$$

In [4], Breuil formulated the following conjecture in this direction (see [4, Conj. 1.1] for general $\mathrm{GL}_n$-case):

**Conjecture 1.1.** *There is a natural $E$-linear bijection*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big) \xrightarrow{\sim} D_{\mathrm{dR}}(\Delta) \tag{2}$$

*such that for any de Rham $(\varphi,\Gamma)$-module $D$ of rank 2 over $\mathcal{R}_E$ of Hodge-Tate weights $(0,k)$ with the associated $p$-adic differential equation isomorphic to $\Delta$, the map sends the $E$-line $E[\pi(D)]$ to $\mathcal{L}(D)$.*

The conjecture was proved in the trianguline case (or equivalently, when $\Delta$ (or equivalently $\mathtt{r}$) is reducible) in [4, § 3.1]. The proof relied on a direct calculation of $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big)$. Indeed, when $D$ (or equivalently $\Delta$) is trianguline, the irreducible constituents of $\pi(D)$ are among those that appear in locally analytic principal series, so such a calculation can be carried out. In this note, we prove the conjecture in de Rham non-trianguline case hence complete all cases. In fact, we prove a refined version of the conjecture given in [5, Conj. 5.3.1] (see Corollary 2.4 and Theorem 2.5), which describes the bijection in Conjecture 1.1 in a functorial way.

**Remark 1.2.** *When $\Delta$ is de Rham non-trianguline, by [9, Thm. 0.6], there is an injective $E$-linear map*

$$D_{\mathrm{dR}}(\Delta) \longleftrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big) \tag{3}$$

*satisfying the same property as (the inverse) of (2). Hence* $\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big) \geq 2$*, and one may prove the conjecture by showing the equality holds. Indeed, by [9, Thm. 0.6 (iii)], one has an extension (which is the universal extension a posteriori, where $\pi(\Delta,k)$ is the representation $\Pi(M,k)$ of loc. cit)*

$$0 \to \pi_{\mathrm{alg}}(\Delta,k) \otimes_E D_{\mathrm{dR}}(\Delta) \to \pi(\Delta,k) \to \pi_c(\Delta,k) \to 0, \tag{4}$$

*satisfying that for any de Rham $(\varphi,\Gamma)$-module $D$ of rank $2$ over $\mathcal{R}_E$ of Hodge-Tate weights $(0,k)$ with the associated $p$-adic differential equation isomorphic to $\Delta$, $\pi(D) \cong \pi(\Delta,k)/(\pi_{\mathrm{alg}}(\Delta,k) \otimes_E \mathcal{L}(D))$. The extension class $[\pi(\Delta,k)]$ induces via the natural cup-product*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k) \otimes_E D_{\mathrm{dR}}(\Delta)\big) \times D_{\mathrm{dR}}(\Delta)^\vee \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big)$$

*an $E$-linear map*

$$D_{\mathrm{dR}}(\Delta)^\vee \longrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_c(\Delta,k), \pi_{\mathrm{alg}}(\Delta,k)\big) \tag{5}$$

*sending an $E$-line $\mathcal{L}(D)^\perp = (D_{\mathrm{dR}}(\Delta)/\mathcal{L}(D))^\vee \hookrightarrow D_{\mathrm{dR}}(\Delta)^\vee$ to $E[\pi(D)]$. Note that (5) is injective, as for different $E$-lines $\mathcal{L}(D_1) \neq \mathcal{L}(D_2)$, we have $D_1 \not\cong D_2$ hence $\pi(D_1) \not\cong \pi(D_2)$. Let $e_1, e_2$ be a basis of $D_{\mathrm{dR}}(\Delta)$, and $e_i^* \in D_{\mathrm{dR}}(\Delta)^\vee$ such that $e_i^*(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. We see the $E$-linear bijective map $D_{\mathrm{dR}}(\Delta) \xrightarrow{\sim} D_{\mathrm{dR}}(\Delta)^\vee$ given by $e_1 \mapsto e_2^*$, $e_2 \mapsto -e_1^*$ sends each $E$-line $\mathcal{L}$ to $\mathcal{L}^\perp$. This bijection pre-composed with (5) gives then the injection in (3). Finally, we remark that by [11, Thm. 1.4], the universal extension (4) can be realized in the de Rham complex of the coverings of Drinfeld's upper half-plane.*

## 2 Main results

Before stating our main results, we quickly introduce some more notation. For $r \in \mathbb{Q}_{>0}$, let $\mathcal{R}_E^r$ be the Fréchet space of $E$-coefficient rigid analytic functions on the annulus $p^{-\frac{1}{r}} \leq |\cdot| < 1$ where $|\cdot|$ is the norm on $\mathbb{C}_p$ normalized such that $|p| = p^{-1}$. We have $\mathcal{R}_E \cong \varinjlim_r \mathcal{R}_E^r$. Let $\mathcal{R}_E^+$ be the Fréchet space of $E$-coefficient rigid analytic functions on the open unit disk $|\cdot| < 1$:

$$\mathcal{R}_E^+ = \Big\{ \sum_{i=0}^{+\infty} a_i X^i \mid a_i \in E \text{ for all } i, \text{ and } |a_i| r^i \to 0, i \to +\infty \text{ for all } 0 \leq r < 1 \Big\}.$$

We have $\mathcal{R}_E^+ \hookrightarrow \mathcal{R}_E^r$ for all $r$. The Robba ring $\mathcal{R}_E$ is equipped with a natural (standard) action of $\Gamma \cong \mathbb{Z}_p^\times$ and operators $\varphi$ and $\psi$. Recall that the $\Gamma$-action sends $\mathcal{R}_E^+$ (resp. $\mathcal{R}_E^r$) to $\mathcal{R}_E^+$ (resp. $\mathcal{R}_E^r$), and the $\psi$-operator sends $\mathcal{R}_E^+$ (resp. $\mathcal{R}_E^r$) to $\mathcal{R}_E^+$ (resp. $\mathcal{R}_E^r$ for $r \in \mathbb{Q}_{>p-1}$). Let $D$ be a generalized $(\varphi,\Gamma)$-module over $\mathcal{R}_E$ (cf. [14, § 4.1], noting $D$ is allowed to have $t$-torsions, $t = \log(1+X)$). Recall (see [5, Remark 2.2.2] and the discussion above it) there exist $r \in \mathbb{Q}_{>p-1}$, and a generalized $(\varphi,\Gamma)$-module $D_r$ over $\mathcal{R}_E^r$ (cf. *loc. cit.*) such that $f_r : D_r \otimes_{\mathcal{R}_E^r} \mathcal{R}_E \xrightarrow{\sim} D$. In fact, such $\{r, D_r, f_r\}$ form a filtered category $I(D)$, and $\varinjlim_{(r,f_r,D_r)\in I(D)} D_r \xrightarrow{\sim} D$ (see the discussion above [5, Remark 2.2.4]).

Recall (see for example [9, § 1.1.2]) that $\mathcal{R}_E^+$ is naturally isomorphic to the locally analytic distribution algebra $\mathcal{D}(\mathbb{Z}_p, E) = \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, E)^\vee$ of $\mathbb{Z}_p$. Under this isomorphism, the operators $\varphi$, $\psi$, and $\gamma \in \Gamma$ can be described as follows: for $\mu \in \mathcal{D}(\mathbb{Z}_p, E)$, $f \in \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, E)$,

$$\varphi(\mu)(f) = \mu([x \mapsto f(px)]),\ \psi(\mu)(f) = \mu([x \mapsto f(\frac{x}{p})]),$$

$$\gamma(\mu)(f) = \mu([x \mapsto f(\gamma x)]).$$

The element $t \in \mathcal{R}_E^+$ (resp. $X$) corresponds to the distribution $f \mapsto f'(0)$ (resp. $f \mapsto f(1) - f(0)$).

Let $\pi$ be an admissible locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E$. The continuous dual $\pi^\vee$ of $\pi$ (equipped with the strong topology) is then a Fréchet space over $E$. The action of $N(\mathbb{Z}_p) = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ on $\pi$ induces a (separately-continuous) $\mathcal{R}_E^+$-module structure on $\pi^\vee$. Note that $t \in \mathcal{R}_E^+$ acts on $\pi^\vee$ via the element $u_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the Lie algebra $\mathfrak{gl}_2$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, and we identify $u_+$ and $t$ frequently without further mention. Moreover, the $\mathcal{R}_E^+$-module $\pi^\vee$ is equipped with an operator $\psi$ given by the action of $\begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix}$, and with an action of $\Gamma \cong \mathbb{Z}_p^\times$ given by the action of $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$ satisfying

$$\psi(\varphi(x)v) = x\psi(v), \gamma(xv) = \gamma(x)\gamma(v)$$

for $x \in \mathcal{R}_E^+$, $v \in \pi^\vee$ and $\gamma \in \Gamma$. Recall in [5, § 2.3] (see in particular [5, Ex. 2.3.3]), we associated to $\pi$ a (covariant) functor $F(\pi)$ from the category of generalized $(\varphi, \Gamma)$-modules to the category of $E$-vector spaces:

$$F(\pi)(D) = \varinjlim_{(r, f_r, D_r) \in I(D)} \mathrm{Hom}_{(\psi,\Gamma)}(\pi^\vee, D_r),$$

where $\mathrm{Hom}_{(\psi,\Gamma)}$ consists of continuous $\mathcal{R}_E^+$-linear morphisms that are $(\psi, \Gamma)$-equivariant. Note that if $D$ has no $t$-torsion, then by [?, Cor. 8.9], we have $F(\pi)(D) = \mathrm{Hom}_{(\psi,\Gamma)}(\pi^\vee, D)$ (where $D$ is equipped with the inductive limit topology). Let $M(\mathbb{Q}_p) := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}_p^\times,\ b \in \mathbb{Q}_p \right\}$. By definition, the functor $F(\pi)$ only depends on $\pi|_{M(\mathbb{Q}_p)}$.

Our first result is on the representability of $F(\pi)$ in de Rham non-trianguline case. Namely, let $\Delta$ be an irreducible $(\varphi, \Gamma)$-module free of rank 2 over $\mathcal{R}_E$, de Rham of constant Hodge-Tate weight 0. Let $\pi(\Delta)$ be the locally analytic representation associated to $\Delta$ (cf. [9, § 2.1]) normalized such that the central character $\omega_\Delta$ of $\pi(\Delta)$ satisfies $\mathcal{R}_E(\omega_\Delta) \cong \wedge^2 \Delta \otimes_{\mathcal{R}_E} \mathcal{R}_E(\varepsilon^{-1})$, where $\varepsilon = z|z|^{-1} : \mathbb{Q}_p^\times \to E^\times$ and for a continuous character $\delta : \mathbb{Q}_p^\times \to E^\times$, we denote by $\mathcal{R}_E(\delta)$ the associated rank one $(\varphi, \Gamma)$-module. Let $\check{\Delta} := \Delta^\vee \otimes_{\mathcal{R}_E} \mathcal{R}_E(\varepsilon) \cong \Delta \otimes_{\mathcal{R}_E} \mathcal{R}_E(\omega_\Delta^{-1})$ be the Cartier dual of $\Delta$.

**Theorem 2.1.** *The functor $F(\pi(\Delta))$ is representable by $\check{\Delta}$, i.e. for any generalized $(\varphi, \Gamma)$-module $D$, $F(\pi(\Delta))(D) = \mathrm{Hom}_{(\varphi,\Gamma)}(\check{\Delta}, D)$.*

**Remark 2.2.** *The same statement in the trianguline case was obtained in [5, Thm. 5.4.2 (i)] (see Step 2 & 3 of the proof), where a key ingredient is the representability of $F(\pi)$ for locally analytic principal series $\pi$. While, our proof of Theorem 2.1 is based on Colmez's results in [9] and the representability of $F(\pi)$ for locally algebraic representations $\pi$.*

For $k \in \mathbb{Z}$, let $\pi(\Delta, k)$ be the locally analytic representation $\Pi(M, k)$ in [9, Thm. 0.8 (iii)] (for $M = \mathrm{DF}$, the irreducible Deligne-Fontaine module associated to $\Delta$). Recall by *loc. cit.*, there exists an isomorphism of topological $E$-vector spaces: $\partial : \pi(\Delta) \to \pi(\Delta)$ such that the following maps

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \pi(\Delta) \to \pi(\Delta),\ v \mapsto (-c\partial + a)^k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (v)$$

define a(nother) locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-action and the resulting representation is isomorphic to $\pi(\Delta, k)$. As $B(\mathbb{Q}_p)$-representation, we have $\pi(\Delta, k) \cong \pi(\Delta) \otimes_E (x^k \otimes 1)$ hence:

$$F(\pi(\Delta))(D) \cong F(\pi(\Delta, k))(D \otimes_{\mathcal{R}_E} \mathcal{R}_E(x^{-k}))$$

for all generalized $(\varphi, \Gamma)$-modules $D$. We then deduce from Theorem 1.1:

**Corollary 2.3.** *The functor $F(\pi(\Delta, k))$ is representable by $t^{-k}\check{\Delta}$.*

As in § 1, let $\pi_\infty(\Delta)$ be the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\Delta$, and for $k \in \mathbb{Z}_{\geq 1}$, let $\pi_{\mathrm{alg}}(\Delta, k) := \mathrm{Sym}^{k-1} E^2 \otimes_E \pi_\infty(\Delta)$ and $\pi_c(\Delta, k) := \pi(\Delta, -k) \otimes_E (x^k \circ \det)$. Recall by [5, Thm. 3.3.1], $F(\pi_{\mathrm{alg}}(\Delta, k))$ is representable by $\mathcal{R}_E(x^{1-k})/t^k$. By Theorem 2.1 and $\pi_c(\Delta, k)|_{M(\mathbb{Q}_p)} \cong \pi(\Delta)|_{M(\mathbb{Q}_p)}$, we see $F(\pi_c(\Delta, k)) = F(\pi(\Delta))$ is representable by $\check{\Delta}$. By [5, Thm. 4.1.5], we then obtain:

**Corollary 2.4.** *There exists a natural $E$-linear map*

$$\mathcal{E} : \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)} \big(\pi_c(\Delta, k), \pi_{\mathrm{alg}}(\Delta, k)\big) \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)} \big(\mathcal{R}_E(x^{1-k})/t^k, \check{\Delta}\big) \tag{6}$$

*satisfying that for $[\pi] \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)} \big(\pi_c(\Delta, k), \pi_{\mathrm{alg}}(\Delta, k)\big)$, the functor $F(\pi)$ is representable by the extension of class $\mathcal{E}([\pi])$.*

By [5, Prop. 5.1.2], there is a natural isomorphism of $E$-vector spaces:

$$\mathrm{Ext}^1_{(\varphi,\Gamma)} \big(\mathcal{R}_E(x^{1-k})/t^k, \check{\Delta}\big) \xrightarrow{\sim} D_{\mathrm{dR}}(\check{\Delta}) \tag{7}$$

satisfying that for each non-split $[D] \in \mathrm{Ext}^1_{(\varphi,\Gamma)} \big(\mathcal{R}_E(x^{1-k})/t^k, \check{\Delta}\big)$, the map sends the line $E[D]$ to $\mathcal{L}(D) \hookrightarrow D_{\mathrm{dR}}(D) \cong D_{\mathrm{dR}}(\check{\Delta})$, where $\mathcal{L}(D)$ is defined in a similar way as in (1): $\mathcal{L}(D) := \mathrm{Fil}^{\max} D_{\mathrm{dR}}(D) = \mathrm{Fil}^i D_{\mathrm{dR}}(D)$ for $i = 0, \cdots, k-1$ (noting such $D$ has Hodge-Tate weights $(1-k, 1)$). Using the isomorphism $D_{\mathrm{dR}}(\check{\Delta}) \cong D_{\mathrm{dR}}(\Delta)$ (with a shift of the Hodge filtration) induced by $\check{\Delta} \cong \Delta \otimes_{\mathcal{R}_E} \mathcal{R}_E(\omega_\Delta^{-1})$, the composition of (6) and (7) gives a map as in (2) (satisfying the properties below (2)). Conjecture 1.1 (in de Rham non-trianguline case) then follows from the following theorem.

**Theorem 2.5.** *The map $\mathcal{E}$ is bijective, in particular,*

$$\dim_E \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)} \big(\pi_c(\Delta, k), \pi_{\mathrm{alg}}(\Delta, k)\big) = 2$$

*and any non-split extension of $\pi_c(\Delta, k)$ by $\pi_{\mathrm{alg}}(\Delta, k)$ is associated to a $(\varphi, \Gamma)$-module of rank 2 over $\mathcal{R}_E$.*

By an easy variation of the proof of Theorem 2.5, we also obtain the following result on locally analytic $\mathrm{Ext}^1$:

**Corollary 2.6.** *Let $\pi_\infty$ be a generic irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E$, and $W$ be an irreducible algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E$. Then $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_c(\Delta, k), \pi_\infty \otimes_E W) \neq 0$ if and only if $\pi_\infty \otimes_E W \cong \pi_{\mathrm{alg}}(\Delta, k)$.*

## 3 Proofs

We keep the notation in § 2. Let $D_r$ be a generalized $(\varphi,\Gamma)$-module over $\mathcal{R}_E^r$ (cf. [5, § 2.2]). We call $D_r$ is good if $D_r \cong (\mathcal{R}_E^r)^{m_1} \oplus \oplus_{i=1}^{m_2} \mathcal{R}_E^r/t^{s_i}$ as $\mathcal{R}_E^r$-module for some integers $m_1 \geq 0$, $m_2 \geq 0$, and $s_i \geq 1$. And if so, we call $m = m_1 + m_2$ the rank of $D_r$. Note for a general $D_r$, $D_r \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^{r'}$ is good for $r' \gg r$ (cf. [5, (22)]). We begin with a key lemma.

**Lemma 3.1.** *Let $D_r$ be a good generalized $(\varphi,\Gamma)$-module over $\mathcal{R}_E^r$, and $f \in \mathrm{Hom}_{(\psi,\Gamma)}(\pi(\Delta)^\vee, D_r)$. Suppose the induced morphism*

$$\pi(\Delta)^\vee \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r \longrightarrow D_r$$

*has dense image. Then the rank of $D_r$ is at most 2.*

*Proof.* As $\pi(\Delta,1) \cong \pi(\Delta) \otimes (x \otimes 1)$ as $B(\mathbb{Q}_p)$-representation, we have

$$\mathrm{Hom}_{(\psi,\Gamma)}\big(\pi(\Delta,1)^\vee, D_r \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^r(x^{-1})\big) = \mathrm{Hom}_{(\psi,\Gamma)}(\pi(\Delta)^\vee, D_r).$$

In particular, $f$ induces a morphism

$$\pi(\Delta,1)^\vee \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r \longrightarrow D_r \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^r(x^{-1}) =: D_r(x^{-1}),$$

which has dense image. This morphism further induces a morphism with dense image:

$$\big(\pi(\Delta,1)^\vee/u_+\pi(\Delta,1)\big) \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r \cong \pi(\Delta,1)^\vee/t \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r \longrightarrow D_r(x^{-1})/t.$$

Using [11, Cor. 9.3], we see $\pi(\Delta,1)^\vee/u_+\pi(\Delta,1)^\vee \cong (\pi(\Delta,1)[u_+])^\vee$, where $(-)[u_+]$ denotes the subspace annihilated by $u_+$. By [9, Lemma 3.24, Thm. 3.31], $\pi(\Delta,1)[u_+] \subset \pi(\Delta,1)$ is stabilized by $\mathrm{GL}_2(\mathbb{Q}_p)$, and is isomorphic to $\pi_{\mathrm{alg}}(\Delta,1)^{\oplus 2} \cong \pi_\infty(\Delta)^{\oplus 2}$ as $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. By [5, Thm. 3.3.1], the induced morphism $\pi(\Delta,1)^\vee/t \to D_r(x^{-1})/t \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^{r'}$ for $r' \gg r$ factors through $(\mathcal{R}_E^{r'}/t)^{\oplus 2}$. For such $r'$, we obtain thus a (continuous $\mathcal{R}_E^{r'}$-linear) morphism with dense image $(\mathcal{R}_E^{r'}/t)^{\oplus 2} \to D_r(x^{-1})/t \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^{r'}$. As $\mathcal{R}_E^{r'}$ is Bézout (see for example [1, Prop. 4.12]), it is not difficult to see the rank of $D_r(x^{-1})/t \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^{r'}$ is at most 2 (= the rank of $(\mathcal{R}_E^{r'}/t)^{\oplus 2}$). Since the rank of $D_r(x^{-1})/t \otimes_{\mathcal{R}_E^r} \mathcal{R}_E^{r'}$ over $\mathcal{R}_E^{r'}/t$ is the same as the rank of $D_r$, the lemma follows. □

*Proof of Theorem 2.1.* Recall (e.g. see [9, § 2.1]) $\Delta$ extends uniquely to a $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf over $\mathbb{P}^1(\mathbb{Q}_p)$ of central character $\omega_\Delta$, and the space $\Delta \boxtimes \mathbb{P}^1$ of global sections sit in a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant exact sequence

$$0 \to \pi(\Delta)^\vee \otimes_E (\omega_\Delta \circ \det) \to \Delta \boxtimes \mathbb{P}^1 \to \pi(\Delta) \to 0.$$

The space of sections of the $\mathrm{GL}_2(\mathbb{Q}_p)$-sheaf on the open set $\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \hookrightarrow \mathbb{P}^1(\mathbb{Q}_p)$ is isomorphic to $\Delta$, and the composition $\iota : \pi(\Delta)^\vee \otimes_E (\omega_\Delta \circ \det) \hookrightarrow \Delta \boxtimes \mathbb{P}^1 \xrightarrow{\mathrm{Res}_{\mathbb{Z}_p}} \Delta$ is $\mathcal{R}_E^+$-linear, continuous and $(\psi,\Gamma)$-equivariant. By the same argument as in Step 1 of the proof of [5, Thm. 5.4.2] (noting since $\Delta$ is irreducible, $\Delta$ is étale up to twist by characters), $\iota$ has image in $\Delta_r$ for $r$ sufficiently large (where $\Delta_r$ is a $(\varphi,\Gamma)$-module over $\mathcal{R}_E^r$ such that $\Delta_r \otimes_{\mathcal{R}_E^r} \mathcal{R}_E \cong \Delta$), and induces a surjective morphism $\iota : \pi(\Delta)^\vee \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r \twoheadrightarrow \Delta_r$.

Let $D$ be a generalized $(\varphi,\Gamma)$-module over $\mathcal{R}_E$, and let $\mu \in F\big(\pi(\Delta) \otimes_E (\omega_\Delta^{-1} \circ \det)\big)(D)$. Let $(r, D_r, f_r) \in I(D)$ such that $\mu \in \mathrm{Hom}_{(\psi,\Gamma)}\big(\pi(\Delta) \otimes_E (\omega_\Delta^{-1} \circ \det), D_r\big)$. It is sufficient to show that, enlarging $r$ if needed, $\mu$ factors through $\iota$. Indeed, if so, the following map induced by $\iota$ (see [5, Lemma 2.2.3 (iii), Remark 2.3.1 (iv)]):

$$\mathrm{Hom}_{(\varphi,\Gamma)}(\Delta, D) \longrightarrow F\big(\pi(\Delta)\otimes_E (\omega_{\Delta}^{-1}\circ \det)\big)(D)$$

is surjective hence bijective (as $\iota$ is surjective after tensoring the source by $\mathcal{R}_E^r$). The theorem then follows using $\check{\Delta}\cong \Delta\otimes_{\mathcal{R}_E}\mathcal{R}_E(\omega_{\Delta}^{-1})$.

Replacing $r$ by $r'\gg r$ (and $D_r$ by $D_r\otimes_{\mathcal{R}_E^r}\mathcal{R}_E^{r'}$), we can and do assume that $\iota$ factors through $\Delta_r$, and $D_r$ is good. Consider

$$\tilde{\mu}: \pi(\Delta)^{\vee}\otimes_E(\omega_{\Delta}\circ\det)\xrightarrow{(\iota,\mu)}\Delta_r\oplus D_r.$$

Denote by $M_r$ the closed $\mathcal{R}_E^r$-submodule of $\Delta_r\oplus D_r$ generated by $\mathrm{Im}(\tilde{\mu})$. As $\tilde{\mu}$ is $(\psi,\Gamma)$-equivariant, we see $M_r\subset \Delta_r\oplus D_r$ is stabilized by $\psi$ and $\Gamma$. By the discussion in the end of [5, § 2.2] (see in particular [5, (24)]), $M_r$ is stabilized by $\varphi$ and $\Gamma$, hence is a generalized $(\varphi,\Gamma)$-module over $\mathcal{R}_E^r$. For $r'\geq r$, $M_{r'}:=M_r\otimes_{\mathcal{R}_E^r}\mathcal{R}_E^{r'}$ is the closed $\mathcal{R}_E^{r'}$-submodule of $\Delta_{r'}\oplus D_{r'}:=(\Delta_r\otimes_{\mathcal{R}_E^r}\mathcal{R}_E^{r'})\oplus(D_r\otimes_{\mathcal{R}_E^r}\mathcal{R}_E^{r'})$ generated by the image of $\tilde{\mu}:\pi(\Delta)^{\vee}\otimes_E(\omega_{\Delta}\circ\det)\xrightarrow{(\iota,\mu)}\Delta_{r'}\oplus D_{r'}$. Let $r'$ be sufficiently large such that $M_{r'}$ is good. Then by Lemma 3.1, the rank $M_{r'}$ is at most 2.

The following composition

$$(\pi(\Delta)^{\vee}\otimes_E(\omega_{\Delta}\circ\det))\otimes_{\mathcal{R}_E^+}\mathcal{R}_E^{r'}\to M_{r'}\hookrightarrow \Delta_{r'}\oplus D_{r'}\xrightarrow{\mathrm{pr}_1}\Delta_{r'}$$

is equal to $\iota$ hence surjective. We see the induced morphism $\kappa: M_{r'}\to\Delta_{r'}$ is surjective. It is clear that $\kappa$ is continuous $\mathcal{R}_E^{r'}$-linear and $(\psi,\Gamma)$-equivariant. By [5, Remark 2.3.1 (iv)], we see $\kappa$ is $(\varphi,\Gamma)$-equivariant (hence is a morphism of generalized $(\varphi,\Gamma)$-modules). Since the rank of $M_{r'}$ is at most the rank of $\Delta_{r'}$, and $\Delta_{r'}$ has no $t$-torsion, we deduce using [1, Prop. 4.12] that $\kappa: M_{r'}\xrightarrow{\sim}\Delta_{r'}$ (as $(\varphi,\Gamma)$-module over $\mathcal{R}_E^{r'}$) and $M_{r'}$ is actually the $\mathcal{R}_E^{r'}$-submodule of $\Delta_{r'}\oplus D_{r'}$ generated by $\mathrm{Im}(\tilde{\mu})$ (i.e. there is no need to take closure). Thus $\tilde{\mu}=\kappa^{-1}\circ\iota$ and $\mu=\mathrm{pr}_2\circ\tilde{\mu}=(\mathrm{pr}_2\circ\kappa^{-1})\circ\iota$, in particular, $\mu$ factors though $\pi(\Delta)^{\vee}\otimes_E(\omega_{\Delta}\circ\det)\xrightarrow{\iota}\Delta_{r'}\to D_{r'}$. This concludes the proof. □

**Remark 3.2.** *The proof of Lemma 3.1 (hence of Theorem 2.1) is crucially based on the fact that $\pi(\Delta)[u_+]|_{M(\mathbb{Q}_p)}$ is isomorphic, up to finite dimensional subquotients and up to twist by characters, to **two** copies of (the $E$-model of) the standard Kirillov model of generic irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (i.e. $W_E$ in the proof of Theorem 2.5 below). One may expect this holds in general (see [9, Remark 2.14]). If so, one may deduce by the same argument that $F(\pi(D))$ is representable by $\check{D}$ for any $(\varphi,\Gamma)$-module $D$ free of rank 2 over $\mathcal{R}_E$.*

For any non-split $[D]\in\mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E/t^k, t^k\Delta)$, one can associate (e.g. see [9, Thm. 0.6 (iii)]) a locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\pi(D)$ that is isomorphic to an extension of $\pi_c(\Delta,k)$ by $\pi_{\mathrm{alg}}(\Delta,k)$. By [5, Thm. 5.4.2 (ii)], we have:

**Corollary 3.3.** *The functor $F(\pi(D))$ is representable by $\check{D}$.*

*Proof of Theorem 2.5.* By Corollary 3.3, the map (6) is surjective. We prove it is injective. Let $[\pi]\in\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_c(\Delta,k),\pi_{\mathrm{alg}}(\Delta,k))$ be non-split. Suppose $\mathcal{E}([\pi])=0$, i.e. $F(\pi)$ is representable by $\check{\Delta}\oplus\mathcal{R}_E(x^{1-k})/t^k$. We will use this property to construct a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant subspace $M\subset\pi^{\vee}$ giving a splitting of $\pi^{\vee}\to\pi_{\mathrm{alg}}(\Delta,k)^{\vee}$ (which leads to a contradiction). The proof is organized as follows: we first construct $M$ as an $\mathcal{R}_E^+$-submodule of $\pi^\vee$ preserved by $\psi$ and $\Gamma$, then we show $M \neq 0$, and $M$ is stabilized by $\mathrm{GL}_2(\mathbb{Q}_p)$ and isomorphic to $\pi_{\mathrm{alg}}(\Delta, k)^\vee$.

For $r \in \mathbb{Q}_{>0}$ sufficiently large, we have a natural $\mathcal{R}_E^+$-linear continuous $(\psi, \Gamma)$-equivariant morphism $\jmath : \pi^\vee \to \check{\Delta}_r \oplus \mathcal{R}_E^r(x^{1-k})/t^k$ such that the induced morphism

$$\pi^\vee \otimes_{\mathcal{R}_E} \mathcal{R}_E^r \longrightarrow \check{\Delta}_r \oplus \mathcal{R}_E^r(x^{1-k})/t^k \tag{8}$$

is surjective. Indeed, we have by [5, Thm. 4.1.5] a natural commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_c(\Delta,k)^\vee \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r & \longrightarrow & \pi^\vee \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r & \longrightarrow & \pi_{\mathrm{alg}}(\Delta,k)^\vee \otimes_{\mathcal{R}_E^+} \mathcal{R}_E^r & \longrightarrow & 0 \\ & & \downarrow & & \downarrow {\scriptstyle (8)} & & \downarrow & & . \\ 0 & \longrightarrow & \check{\Delta}_r & \longrightarrow & \check{\Delta}_r \oplus \mathcal{R}_E^r(x^{1-k})/t^k & \longrightarrow & \mathcal{R}_E^r(x^{1-k})/t^k & \longrightarrow & 0 \end{array} \tag{9}$$

The left vertical map is surjective as it is induced from:

$$\iota : \pi_c(\Delta,k)^\vee \cong \pi(\Delta)^\vee \xrightarrow{\iota} \check{\Delta}_r$$

where the first isomorphism is $M(\mathbb{Q}_p)$-equivariant, and the second map is given as in the proof of Theorem 2.1. By [5, Lemma 3.3.5 (ii)] and its proof, the right vertical map is also surjective, hence so is the middle vertical map. Let $M := \mathrm{Ker}(\mathrm{pr}_1 \circ \jmath : \pi^\vee \to \check{\Delta}_r)$. As the composition $\pi_c(\Delta,k)^\vee \hookrightarrow \pi^\vee \xrightarrow{\mathrm{pr}_1 \circ \jmath} \check{\Delta}_r$ is equal to $\iota$ and hence is injective by [9, Prop. 2.20], we deduce $M \cap \pi_c(\Delta,k)^\vee = 0$. So the following composition (continuous $\mathcal{R}_E^+$-linear and $(\psi,\Gamma)$-equivariant)

$$M \hookrightarrow \pi^\vee \twoheadrightarrow \pi_{\mathrm{alg}}(\Delta,k)^\vee$$

is injective.

(1) We first prove $M \neq 0$. Suppose $M = 0$ hence $\pi^\vee \hookrightarrow \check{\Delta}_r$. As $\check{\Delta}_r$ is $t$-torsion free, so is $\pi^\vee$. From the commutative diagram (recalling $u_+ = t$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_c(\Delta,k)^\vee & \longrightarrow & \pi^\vee & \longrightarrow & \pi_{\mathrm{alg}}(\Delta,k)^\vee & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle u_+} & & \downarrow {\scriptstyle u_+} & & \downarrow {\scriptstyle u_+} & & \\ 0 & \longrightarrow & \pi_c(\Delta,k)^\vee & \longrightarrow & \pi^\vee & \longrightarrow & \pi_{\mathrm{alg}}(\Delta,k)^\vee & \longrightarrow & 0 \end{array}$$

we deduce an exact sequence (consisting of continuous maps)

$$0 \to \pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+] \xrightarrow{\delta} \pi_c(\Delta,k)^\vee/u_+\pi_c(\Delta,k)^\vee \to \pi^\vee/u_+\pi^\vee \to \pi_{\mathrm{alg}}(\Delta,k)^\vee/u_+\pi_{\mathrm{alg}}(\Delta,k)^\vee \to 0. \tag{10}$$

Roughly speaking, we will show a contradiction by considering the multiplicities of the Kirillov model in the dual of each term of (10). By the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism $\pi_{\mathrm{alg}}(\Delta,k)^\vee \cong \pi_\infty(\Delta)^\vee \otimes_E (\mathrm{Sym}^{k-1} E^2)^\vee$, we get a $B(\mathbb{Q}_p)$-equivariant isomorphism (of reflexive Fréchet $E$-spaces):

$$\pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+] \cong \pi_\infty(\Delta)^\vee \otimes_E (1 \otimes x^{1-k}). \tag{11}$$

Using the isomorphisms of $B(\mathbb{Q}_p)$-representations

$$\pi_c(\Delta,k) \cong \pi(\Delta,1) \otimes_E (x^{-1} \otimes x^{k-1}),\ \pi(\Delta,1)[u_+] \cong \pi_\infty(\Delta)^{\oplus 2},$$

we deduce a $B(\mathbb{Q}_p)$-equivariant isomorphism of reflexive Fréchet $E$-spaces (similarly as in the proof of Lemma 3.1, the first isomorphism following from [11, Cor. 9.3]):

$$\pi_c(\Delta,k)^\vee/u_+\pi_c(\Delta,k)^\vee \cong \pi_c(\Delta,k)[u_+]^\vee \cong \big(\pi_\infty(\Delta)^\vee \otimes_E (x\otimes x^{-k})\big)^{\oplus 2}. \tag{12}$$

By similar arguments of [5, Lemma 2.1.5], the injection $\delta$ induces a continuous map of spaces of compact type with dense image $\delta^\vee : \pi_c(\Delta,k)[u_+] \to \pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+]^\vee \cong \pi_\infty(\Delta)\otimes_E (1\otimes x^{k-1})$. As $\pi_\infty(\Delta)$ is equipped with the finest locally convex topology, $\delta^\vee$ is surjective (see for example [**?**, § 5.C]). We have hence an exact sequence of spaces of compact type (all equipped with the finest locally convex topology):

$$0 \to \mathrm{Ker}(\delta^\vee) \to \pi_c(\Delta,k)[u_+] \to \pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+]^\vee \to 0. \tag{13}$$

One directly checks (by diagram chasing) that for $b\in B(\mathbb{Q}_p)$ and $v\in \pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+]$, $\delta(bv) = (x^{-1}\otimes x)(b)b(\delta(v))$. We see $\mathrm{Ker}(\delta^\vee)$ is stabilized by $B(\mathbb{Q}_p)$, and the exact sequence in (13) becomes $B(\mathbb{Q}_p)$-equivariant if we twist $\pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+]^\vee$ by the character $x^{-1}\otimes x$ of $B(\mathbb{Q}_p)$.

Let $\eta : \mathbb{Q}_p \to \mathbb{C}_p$ be a non-trivial locally constant (additive) character. Let $W := \mathcal{C}_c^\infty(\mathbb{Q}_p^\times, \mathbb{C}_p)$ be the space of locally constant $\mathbb{C}_p$-valued functions on $\mathbb{Q}_p^\times$, which is equipped with a natural $M(\mathbb{Q}_p)$-action given by

$$\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} f\right)(x) = f(ax),\ \left(\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix} f\right)(x) = \eta(bx)f(x).$$

Recall $W$ is irreducible and admits an $E$-model $W_E$, that is unique up to scalars in $\mathbb{C}_p^\times$ (see [5, Lemma 3.3.2]). By classical theory of Kirillov model (see for example [3, § 3.5]), we have $\pi_\infty(\Delta)|_{M(\mathbb{Q}_p)} \cong W_E$. By (11) and using (12) (13), we see $\mathrm{Ker}(\delta^\vee)|_{M(\mathbb{Q}_p)} \cong W_E\otimes_E (x^{-1}\otimes x^{1-k})$. We define $F(\mathrm{Ker}(\delta^\vee))$ exactly in the same way as for $\mathrm{GL}_2(\mathbb{Q}_p)$-representations (noting in the definition of $F(-)$, we actually only use the $M(\mathbb{Q}_p)$-action). By [5, Lemma 3.3.5 (2)], $F(\mathrm{Ker}(\delta^\vee))$ is representable by $\mathcal{R}_E(x^{-1})/t$.

By (9), we have a commutative diagram

$$\begin{array}{ccc} \pi_c(\Delta,k)^\vee/u_+\pi_c(\Delta,k)^\vee & \longrightarrow & \pi^\vee/u_+\pi^\vee \\ {\scriptstyle \jmath_1}\big\downarrow & & \big\downarrow \\ \check{\Delta}_r/t & \longrightarrow & \check{\Delta}_r/t \oplus \mathcal{R}_E^r(x^{1-k})/t, \end{array} \tag{14}$$

such that each vertical map becomes surjective if we tensor the corresponding source by $\mathcal{R}_E^r$. As the bottom horizontal map is obviously injective, we deduce using (10) that $\pi_{\mathrm{alg}}(\Delta,k)^\vee[u_+] \subset \mathrm{Ker}\,\jmath_1$ and hence $\jmath_1$ factors through (a continuous $\mathcal{R}_E^+$-linear $(\psi,\Gamma)$-equivariant map)

$$\jmath_1' : \mathrm{Ker}(\delta^\vee)^\vee \longrightarrow \check{\Delta}_r/t$$

which is surjective after tensoring the source by $\mathcal{R}_E^r$. However, as $F(\mathrm{Ker}(\delta^\vee))$ is represented by $\mathcal{R}_E(x^{-1})/t$, we see $\jmath_1'$ factors through (enlarging $r$ if needed) $\mathcal{R}_E^r(x^{-1})/t \to \check{\Delta}_r/t$, which can not be surjective, a contradiction.

(2) We show $M(\neq 0)$ is stabilized by $\mathrm{GL}_2(\mathbb{Q}_p)$ hence isomorphic to $\pi_{\mathrm{alg}}(\Delta,k)^\vee$, which will lead to a contradiction (and will conclude the proof of the theorem) as the extension $\pi$ is non-split. We begin with the following claim.

**Claim:** For $v\in\pi^\vee$, the followings are equivalent:

(1) $v \in M$,

(2) $t^k v = 0$,

(3) $t^n v = 0$ for $n$ sufficiently large.

We prove the claim. Since $M \hookrightarrow \pi_{\mathrm{alg}}(\Delta, k)^\vee$ is $\mathcal{R}_E^+$-equivariant and $\pi_{\mathrm{alg}}(\Delta, k)^\vee$ is annihilated by $t^k$, we see (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (3) is trivial. Suppose $t^n v = 0$ for some $n$, then $\mathrm{pr}_1 \circ \jmath(t^n v) = t^n \mathrm{pr}_1 \circ \jmath(v) = 0$. Since $\check{\Delta}_r$ has no $t$-torsion, we see $\mathrm{pr}_1 \circ \jmath(v) = 0$, i.e. $v \in M$.

For $v \in \pi^\vee$, $b \in B(\mathbb{Q}_p)$, we have $t^n(bv) = (u_+)^n \cdot (bv) = b(\mathrm{Ad}_{b^{-1}}(u_+)^n \cdot v)$. If $t^n v = 0$, then $\mathrm{Ad}_{b^{-1}}(u_+)^n \cdot v = 0$ thus $t^n(bv) = 0$. By the claim, we see $M \subset \pi_{\mathrm{alg}}(\Delta, k)^\vee$ is stabilized by $B(\mathbb{Q}_p)$.

Next we show $M$ is stabilized by $u_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}_2$. Since $M$ is a $B(\mathbb{Q}_p)$-submodule of $\pi_{\mathrm{alg}}(\Delta, k)^\vee$, we see for any $v \in M$, the $\mathfrak{b}$-module generated by $v$ is finite dimensional and is spanned by eigenvectors of $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{gl}_2$. Using the relation $[(u_+)^{n+1}, u_-] = n(u_+)^n(h+n)$ in $\mathrm{U}(\mathfrak{gl}_2)$, we deduce for $v \in M$, $t^n(u_- \cdot v) = (u_+)^n u_- \cdot v$=0 for $n$ sufficient large and hence $u_- \cdot v \in M$ by the claim. Consequently, $M$ is a $\mathrm{U}(\mathfrak{gl}_2)$-submodule of $\pi^\vee$ and the injection $M \hookrightarrow \pi_{\mathrm{alg}}(\Delta, k)^\vee$ is $\mathrm{U}(\mathfrak{gl}_2)$-equivariant. We deduce then any vector $v$ in $M$ is annihilated by $(u_-)^k$.

Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$. For $v \in M$, we have $t^k w v = w(\mathrm{Ad}_w(u_+)^k \cdot v) = w(u_-^k \cdot v) = 0$. Thus $M$ is stabilized by $w$, hence is stabilized by $\mathrm{GL}_2(\mathbb{Q}_p)$ (recalling $M$ is $B(\mathbb{Q}_p)$-invariant). Since $\pi_{\mathrm{alg}}(\Delta, k)$ is irreducible, we deduce $M \cong \pi_{\mathrm{alg}}(\Delta, k)^\vee$. As previously discussed, this finishes the proof. □

*Proof of Corollary 2.6.* The "if" part is a trivial consequence of Theorem 2.5. Assume now

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_c(\Delta, k), \pi_\infty \otimes_E W) \neq 0.$$

Then $\pi_\infty \otimes_E W$ has the same central character and infinitesimal character as $\pi_c(\Delta, k)$. By [9, Prop. 3.1.1], one deduces $W \cong \mathrm{Sym}^{k-1} E^2$.

Similarly as in Corollary 2.4 (using Corollary 3.3, [5, Thm. 3.3.1 & Thm. 4.1.5]), we have a morphism

$$\mathcal{E}' : \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\pi_c(\Delta, k), \pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2\right) \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma)}(\mathcal{R}_E(x^{1-k})/t^k, \check{\Delta}).$$

By the same argument as in the proof of Theorem 2.5 (with $\pi_{\mathrm{alg}}(\Delta, k)$ replaced by $\pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2$), the morphism is injective. Suppose $\pi_\infty$ is not isomorphic to $\pi_\infty(\Delta)$ and there exists a non-split $[\pi] \in \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\left(\pi_c(\Delta, k), \pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2\right)$. Let $[\check{D}] := \mathcal{E}'([\pi])$, and let $[\pi(D)] := \mathcal{E}^{-1}([\check{D}])$. The pull-back of $\pi_c(\Delta, k)$ of $\pi(D) \oplus \pi \twoheadrightarrow \pi_c(\Delta, k)^{\oplus 2}$ via the diagonal map gives a non-split extension $\tilde{\pi}$ of $\pi_c(\Delta, k)$ by $(\pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2) \oplus \pi_{\mathrm{alg}}(\Delta, k)$ satisfying $\tilde{\pi}/\pi_{\mathrm{alg}}(\Delta, k) \cong \pi$ and $\tilde{\pi}/(\pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2) \cong \pi(D)$. By [5, Thm. 4.1.5], $F(\tilde{\pi})$ is representable by an extension of $(\mathcal{R}_E(x^{1-k})/t^k)^{\oplus 2}$ by $\check{\Delta}$ such that the pull-back of either of the two factors $\mathcal{R}_E(x^{1-k})/t^k$ is isomorphic to $\check{D}$. We deduce then $F(\tilde{\pi})$ is representable by $\check{D} \oplus \mathcal{R}_E(x^{1-k})/t^k$. We have thus a continuous $\mathcal{R}_E^+$-linear $(\psi, \Gamma)$-equivariant morphism when $r$ is sufficiently large:

$$\jmath : \tilde{\pi}^\vee \longrightarrow \check{D}_r \oplus \mathcal{R}_E^r(x^{1-k})/t^k$$

such that the morphism becomes surjective if we tensor the source by $\mathcal{R}_E^r$ (by similar arguments as for the surjectivity of (8)). Let $M$ be the kernel of $\mathrm{pr}_1 \circ \jmath$. Since $F(\pi(D))(\check{D}) \cong \mathrm{End}_{(\varphi,\Gamma)}(\check{D}) \cong E$, the restriction of $\mathrm{pr}_1 \circ \jmath$ on $\pi(D)^\vee$ is equal, up to non-zero scalars, to the morphism $\pi(D)^\vee \to \check{D}$ in [9, Prop. 2.20] hence is injective. Using a similar exact sequence as in (10) with $\pi$ replaced by $\pi(D)$, we can deduce $\pi(D)[u_+]|_{M(\mathbb{Q}_p)} \cong (W_E \otimes_E (x^{-1} \otimes x^{k-1}))^{\oplus 2}$ (see also [9, Remark 3.3.2]). Now by the same arguments as in the proof of Theorem 2.5 $\big($with $\pi_c(\Delta, k)$ replaced by $\pi(D)$ and $\pi_{\mathrm{alg}}(\Delta, k)$ replaced by $\pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2\big)$, one can prove $M$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant, and is isomorphic to $(\pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2)^\vee$. Hence $\tilde{\pi} \cong \pi(D) \oplus \pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2$ and then $\pi \cong \tilde{\pi}/\pi_{\mathrm{alg}}(\Delta, k) \cong \pi_c(\Delta, k) \oplus \pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2$ $\big($noting $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\pi_{\mathrm{alg}}(\Delta, k), \pi_\infty \otimes_E \mathrm{Sym}^{k-1} E^2\big) = 0$ by assumption$\big)$, a contradiction. □


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