Hawkes Processes as Branching Processes

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Marked Hawkes Process

A **Hawkes** process is a **Point Process** such that its intensity *jumps* with the process itself. Let N denotes the Hawkes process, its intensity reads

$$\lambda_t = \lambda_0 + \alpha \int_0^{t^-} e^{-\beta\,(t-s)}\, dN_s.$$

We can replace N by the related random measure ν.

In this situation we can add a mark to ν. We will suppose the random measure ν defined on $\mathbb{R}_+^2$.

The new point process $N^{(P)}$ has the following intensity

$$\lambda_t = \lambda_0 + \int_0^{t^-} \int_{\mathbb{R}_+} \zeta\, e^{-\beta\,(t-s)}\, \nu(d\zeta, ds).$$

The constant α is replaced by the mark ζ that is a sequence of random variables.

Some remarks

- In financial point of view, the presence of marks is natural, it can explain the different degree of impact. For instance on order books (large vs small orders), credit risk (recovery), etc. In these case the mark are observable.
- The mark ζ can then be unobservable or partially observable and a filtering procedure is needed, see for instance Callegaro et al (2016).
- a marked Hawkes process is a **Point Process**. In particular $\Delta N_\tau^{(P)} = 1$. This is the reason why we prefer to call this process a marked Hawkes process instead a compound Hawkes process, See for instance Bernis et al. (2017).
- We can define a compound Hawkes process in the following way

$$N_t^{(C)} := \int_0^t \zeta\, \nu(d\zeta, ds) = \sum_i \zeta_i\, \mathbb{I}_{\tau_i \leq t}$$

whereas

$$N_t^{(P)} = \int_0^t \nu(d\zeta, ds) = \sum_i \mathbb{I}_{\tau_i \leq t}$$

- Note that $N^{(P)}$ and $N^{(C)}$ are càdlàg whereas λ is làdcàg.

Comparaison with Hawkes processes

A large part of the results about Hawkes process can be easily extended

.

- Extension of the main results in probability change, see Bernis et al. (2017)
- Maximum Likelihood estimator, almost the same result if the marks are observable.
- Linearity of the model $\rightarrow$ a large part of the formulae are still valid with $\alpha := \mathbb{E}[\zeta]$.

A little history to start

Galton and Watson 1875

The decay of the families of men who occupied conspicuous positions in past times has been a subject of frequent remark, and has given rise to various conjectures. It is not only the families of men of genius or those of the aristocracy who tend to perish, but it is those of all with whom history deals, in any way, even of such men as the burgesses of towns, concerning whom Mr. Doubleday has inquired and written. The instances are very numerous in which surnames that were once common have since become scarce or have wholly disappeared. The tendency is universal, and, in explanation of it, the conclusion has been hastily drawn that a rise in physical comfort and intellectual capacity is necessarily accompanied by diminution in “fertility”.

Question : are the distinguished families to die more likely out than ordinary ones ?

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Question : are the distinguished families to die more likely out than ordinary ones ?

Galton answer : study the fertility data and link it to the probability of extinction of a family. Let $\{p_i\}_{i=0,1,\dots}$ be the probability that a man has i sons. What is the probability that the male line is extinct after k generations ?

The probability generation function is $f(s) = \sum_{i=0}^{\infty} p_i\, s^i$.

Galton Watson model : a discrete-discrete framework

Looking at a generation k, we denote by Z_k the number of descendants.

Z_n is then a Markov chain.

It is reasonable to assume that the chain is time-invariant. **Questions :** why ? why not ?

each family evolves independently to the other ones.

The transition probability is given by $P_{i,j} := \mathbb{P}\left[Z_{n+1} = j | Z_n = i\right]$.

The probability generation function of Z_n is $f_n(s) = \sum_{i=0}^{\infty} \mathbb{P}[Z_{n-1} = i] \; [f(s)]^i$ and by iteration $f_n(s) = f_{n-1}(f(s))$.Then f_n is the n-iterate function of f.

Easily defining $m := \mathbb{E}[Z_1]$ and $\sigma^2 := \mathbb{V}[Z_1]$ we obtain $\mathbb{E}[Z_n] = m^n$ and $\mathbb{V}[Z_n] = \sigma^2 \frac{m^n(m^n-1)}{m^2-m}$.

Theorem : if $m \leq 1$ the extinction is sure $\lim_{n\to\infty} f_n(0) = 1$. If $m > 1$, the probability of extinction is the only fixed point smaller than 1 of $f(s) = s$.

A different way

The previous explicit computation is straightforward but quite tedious.

Of course same arguments will apply in the continuous case.

However, a large part of the results can be obtained using a different point of view.

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Let ζ_i a sequence of random variable with values on $\mathbb{N}$. We interpret ζ_i as the number of sons of a father i.

We then assume that ζ_i are IID, that is the number of sons does not depends on the father, the family and on the generation.

What is the link between the population at generation n and $n+1$?

$$Z_{n+1} = \sum_{i=1}^{Z_n} \zeta_i.$$

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What is the limit for very large population and overlapping generations ?

$$dZ_t = \int_0^{Z_{t-}} \zeta \, \nu(d\zeta, dt).$$

This is the key stone of Dawson-Li representation of branching processes.

Branching property

Branching property :

A process X has the Branching Property if for any t and x, y in the state space of X, X_t^{x+y} is equal in law to the independent sum of X_t^x and X_t^y.

If a process X can be decomposed as $X = X^{(1)} + X^{(2)}$ where for $i = 1, 2$, $X^{(i)}$ satisfying the same SDE with $X_0 = X_0^{(1)} + X_0^{(2)}$, then the process is said a **branching process**.

Let denote by P_t^x the semigroup of X with initial condition $X_0 = x$, we have $P_t^{x+y} = P_t^x \star P_t^y$, where $\star$ denotes the convolution.

Convolution theorem

Let f and g be two functions, we denote by $\widetilde{f}$ and $\widetilde{g}$ their Laplace transform. We have the following result

$$\widetilde{(f \star g)} = \widetilde{f}\widetilde{g}.$$

Infinite divisibility : see Kyprianu

A random variable is characterized by its moment generating function (i.e. the Laplace transform)

$$\mathbb{E}^{x+y}\left[e^{-u\,X_t}\right]=\mathbb{E}^{x}\left[e^{-u\,X_t}\right]\;\mathbb{E}^{y}\left[e^{-u\,X_t}\right].$$

Iterating, it is clear that the process is infinitely divisible with respect to the initial condition

$$\mathbb{E}^{x}\left[e^{-u\,X_t}\right]=\left(\mathbb{E}^{\frac{x}{n}}\left[e^{-u\,X_t}\right]\right)^n.$$

Taking the logarithm we have

$$g(t,x,u):=-\log\mathbb{E}^{x}\left[e^{-u\,X_t}\right]=n\,g\left(t,\frac{x}{n},u\right).$$

By density of $\mathbb{Q}$ into $\mathbb{R}$, we have $g(t,x,u)=xg(t,1,u)=x\,b_t(u)$.

It is easy to see that $b_t(u)\geq 0$ and then $g(t,x,u)\leq g(t,y,u)$ for all $x<y$

$$e^{-x\,b_{t+s}(u)}=\mathbb{E}^{x}\left[e^{-u\,X_{t+s}}\right]=\mathbb{E}^{x}\left[\mathbb{E}\left[e^{-u\,X_{t+s}}\,|\,X_t\right]\right]=\mathbb{E}^{x}\left[e^{-X_t\,b_s(u)}\right]=e^{-x\,b_t(b_s(u))}.$$

That is the Semigroup law $b_{t+s}(u)=b_t(b_s(u))$. Then $b_t(u)=b_{t/n}(b_{t/n}(\ldots b_{t/n}(u)\ldots))$.

Lamperti representation

The function b is differentiable with respect to t and satisfies

$$b_t(u) + \int_0^t \Psi(b_s(u))\, ds = u,$$

where

$$\Psi(q) = \beta q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-q\,y} - 1 + q\,y)\,\pi(dy),$$

with π being a Lévy measure such that $\int_0^\infty (y \wedge y^2)\,\pi(dy) < \infty$.

idea see Le Gall

Apply Lévy-Khintchine formula for infinitely divisible process.

Moreover, a 1 to 1 relationship exists between Ψ and the semigroup.

Some examples

Extinction : $\Psi(q) = \beta\, q$. Then $X_t = X_0\, e^{-\beta\, t}$.

CIR model $\Psi(q) = \frac{1}{2}\, \sigma^2\, q^2$. Then

$$b_t(u) = \frac{u}{1 + \frac{\sigma^2}{2}\, u\, t}.$$

Pure jumps $\Psi(q) = \int_0^\infty (e^{-q\,y} - 1 + q\,y)\, \frac{dy}{y^{1+\alpha}} = u^\alpha$.

Exercise : Consider $dX_t = \sigma\, \sqrt{X_t}\, dW_t$ and show that $\exp\left\{ \frac{u\, X_t}{1 + \frac{\sigma^2}{2}\, u\, (T-t)} \right\}$ is a martingale.

Generator

We have the following result, see Kawazu and Watanabe (1971).

generator

Markov process X with state space $\mathbb{R}_+$ with Branching mechanism :

$$\Psi(q) = \beta\, q + \frac{1}{2}\sigma^2\, q^2 + \int_0^\infty (e^{-q\,u} - 1 + q\,u)\,\pi(du),$$

with $\sigma \geq 0$, $\beta \in \mathbb{R}$ and π being a Lévy measure such that $\int_0^\infty (u \wedge u^2)\,\pi(du) < \infty$.
The CBI process X has as generator the operator $\mathcal{L}$ acting on $C_0^2(\mathbb{R}_+)$ as

$$\mathcal{L}f(x) = \frac{\sigma^2}{2}\,x f''(x) - \beta\,x f'(x) + x\int_0^\infty \big(f(x+u) - f(x) - u f'(x)\big)\,\pi(du).$$

Dawson Li representation

Integral representation

The previous generator admits the following semigroup (via implicitly Hille-Yosida theorem).

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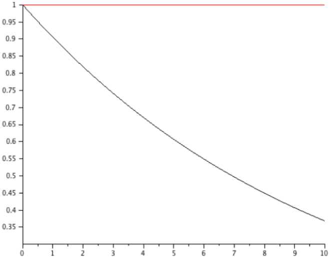
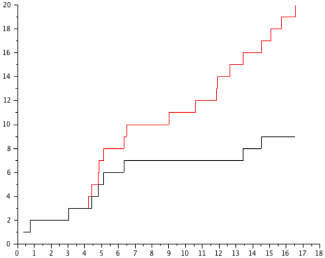
$$\begin{aligned} X_t &= -\int_0^t \int_0^{X_s} \beta \, du \, ds + \sigma \int_0^t \int_0^{X_s} W(ds, dv) \\ &\quad + \int_0^t \int_0^{X_{s-}} \int_{\mathbb{R}+} u \, \widetilde{N}(ds, dv, du), \end{aligned}$$

- $W(ds, dv)$: white noise on $\mathbb{R}^2_+$ with intensity $ds \, dv$,
- $\widetilde{N}(ds, dv, d\zeta)$: compensated Poisson random measure on $\mathbb{R}^3_+$ with intensity $ds \, dv \, \pi(du)$,

Similarity with Thinning procedure





Generalisation with exogenous term

Endogenous vs exogenous term

We have seen in Hawkes processes that jumps arise both for exogenous and endogenous raisons.

The equivalent phenomenon in population dynamic is the **immigration**. The immigration is independent of the size of the population and it is completely exogenous.

In Hawkes framework, immigration is a pure rate λ_0.

The class of Continuous State Branching Processes (CSBP) can be extended to Continuous state Branching processes with Immigration (CBI).

The main difference with Hawkes is that a pure external jumps source can be added. That is immigration is not forced to be a **rate** but can be a **non-decreasing Lévy processes**.

Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process X with state space $\mathbb{R}_+$ verifying

$$\mathbb{E}^x\left[e^{-p X_t}\right] = \exp\left[-x\, v(t,p) - \int_0^t \Phi\big(v(s,p)\big) ds\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

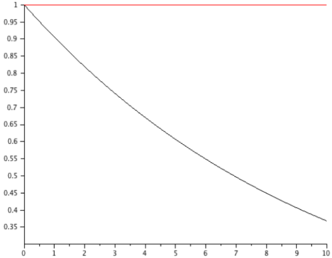
$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

, and Ψ and Φ are functions on $\mathbb{R}_+$ given by

$$\Psi(q) = \beta\, q + \frac{1}{2}\,\sigma^2\, q^2 + \int_0^\infty (e^{-q\,u} - 1 + q\,u)\,\pi(du),$$
$$\Phi(q) = \gamma\, q + \int_0^\infty (1 - e^{-q\,u})\,\nu(du),$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and π, ν being two Lévy measures such that $\int_0^\infty (u \wedge u^2)\,\pi(du) < \infty$ and $\int_0^\infty (1 \wedge u)\,\nu(du) < \infty$.

Thinning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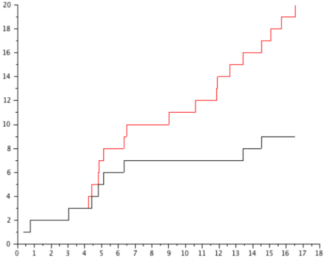
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Link to Hawkes process

- When $\sigma = 0$ and $\pi(du) = \delta_1(du)$, then X is given by

$$X_t = X_0 - \int_0^t \big(\beta + \pi(\mathbb{R})\big)\, X_s \, ds + \int_0^t \int_0^{X_{s-}} N(ds, dv) \tag{1}$$

  which is the intensity of Hawkes process $\int_0^t \int_0^{X_{s-}} N(ds, dv)$, N being the Poisson random measure with intensity $ds\, dv$.

- Consider a sequence $\left\{X_t^{(n)}, t \geq 0\right\}$ defined by (1) with parameters $(a/n, nb, \sigma_Z)$. Then

$$X_{nt}^{(n)}/n \xrightarrow{\mathcal{L}} Y_t \quad \text{in } D(\mathbb{R}_+),$$

  where $D(\mathbb{R}_+)$ is the Skorokhod space of càdlàg processes and

$$Y_t = \int_0^t a\,(b - Y_s)ds + \sigma_Z \int_0^t \int_0^{Y_s} W(ds, du).$$

  See Jiao et al. (2016).

- Jaisson and Rosenbaum (2015) : nearly unstable Hawkes process converges, after suitable scaling, to a CIR process.

α-CIR

We will extend the usual CIR

CIR model in finance

- mean reverting models : energy, commodities, co-integrated assets.
- Interest rates : Cox - Ingersoll - Ross model
- Volatility : Heston model

In order to include a positive α-stable branching driver.

The α-CIR model setup : Integral representation (Dawson-Li)

Integral form by using the random fields

$$
\begin{aligned}
X_t = X_0 + \int_0^t a\,(b - X_s)\,ds + \sigma \int_0^t \int_0^{X_s} W(ds, du) \\
+ \sigma_Z \int_0^t \int_0^{X_{s-}} \int_{\mathbb{R}+} \zeta\, \widetilde{N}(ds, du, d\zeta),
\end{aligned}
\tag{2}
$$

- $W(ds, du)$: white noise on $\mathbb{R}_+^2$ with intensity $dsdu$,
- $\widetilde{N}(ds, du, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $ds\,du\,\mu(d\zeta)$,
- $\mu(d\zeta)$: a Lévy measure satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\,\mu(d\zeta) < \infty$.

We choose the Lévy measure to be

$$
\mu(d\zeta) = -\frac{1_{\{\zeta>0\}}\,d\zeta}{\cos(\pi\alpha/2)\,\Gamma(-\alpha)\,\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \tag{3}
$$

For existence and uniqueness of the solution see Dawson and Li (2012), Theorem 3.1 and Li and Ma (2015) Theorem 2.1.

The α-CIR model setup

We consider the following *usual* SDE

$$r_t = r_0 + \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \sqrt{r_s}\,dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha}\,dZ_s \tag{4}$$

- $B = (B_t, t \geq 0)$ a Brownian motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive α-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with
$$\mathbb{E}\left[e^{-q\,Z_t}\right] = \exp\left\{-\frac{t\,q^\alpha}{\cos(\pi\,\alpha/2)}\right\}, \quad q \geq 0.$$
- B and Z are independent.

Z_t follows the α-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.
The existence of a unique strong solution for the SDE (4) follows from Fu and Li (Theorem 5.3, 2010).

A natural extension of the CIR model : aggregate

- When $\sigma_Z = 0$, we recover the CIR model.
- When $\alpha = 2$, it also reduces to a CIR model but with volatility parameter $(\sigma^2 + 2\,\sigma_Z^2)^{1/2}$.
- The difference of Z from a Brownian motion is controlled by the tail index α :

  $\diamond$ $\alpha = 2$: Z is a Brownian motion scaled by $\sqrt{2}$;

  $\diamond$ $\alpha < 2$: Z is a pure jump process with heavy tails. More as α close to 1, more likely Z_t takes values far from median ;

  $\diamond$ comparison with Poisson process : Z has an infinite number of (small) jumps over any time interval, allowing it to capture the extreme activity.

Similar properties with CIR model I

Boundary condition :

The point 0 is an inaccessible boundary if and only if $2\,a\,b \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

See Duhalde et al. (2014) Theorem 2. This result is important in order to have a good shape of the ergodic distribution.

Ergodic law :

The process is exponentially ergodic, the limit distribution denoted by r_∞ satisfies

$$\mathbb{E}\left[e^{-pr_\infty}\right] = \exp\left\{-\int_0^p \frac{abq}{aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha}dq\right\}.$$

See Li (2011) Theorem 3.20 and page 67, see also Li and Ma (2013) Theorem 2.6. Important result for calibration in some applications.

Similar properties with CIR model II

Branching property :

r can be decomposed as $r = r^{(1)} + r^{(2)}$ where for $i = 1, 2$, $r^{(i)}$ is an α-CIR$(a, b^{(i)}, \sigma, \sigma_Z, \alpha)$ process such that $r_0 = r_0^{(1)} + r_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$.

See Dawson and Li (2006).
This property is a direct consequence of

- linearity of integrals,
- homogeneity of measures.

Important in hierarchical applications (interest rates, defaults rates, liquidity risk) since the self-similarity can reduce the complexity of the model.

Equivalence of two representations

Then the root representation (4) and the integral representation (2) are equivalent in the following sense :

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.

See Theorem 9.32 in Li (2011).
The equivalence is useful since we have two ways to study the properties of the model.

Simulation of processes Z and X with different α

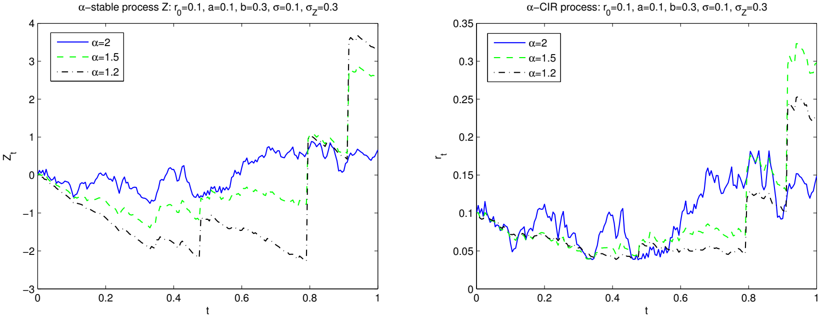

FIGURE: Three parameters of α : 2 (blue), 1.5 (green) and 1.2 (black).

Motivation

- Current sovereign bond markets in the Euro zone :
  ◇ persistency of low interest rates
  ◇ significant fluctuations at local extent.

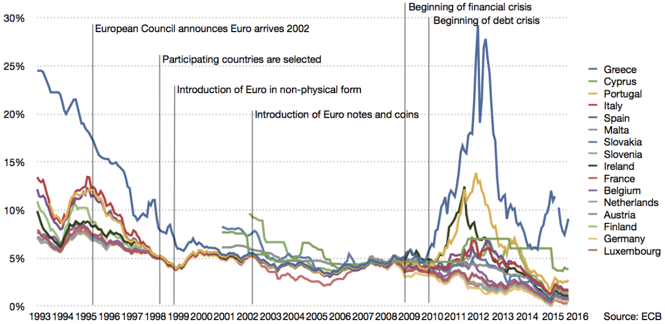

FIGURE: Long term interest rates of Euro area countries.

Motivation

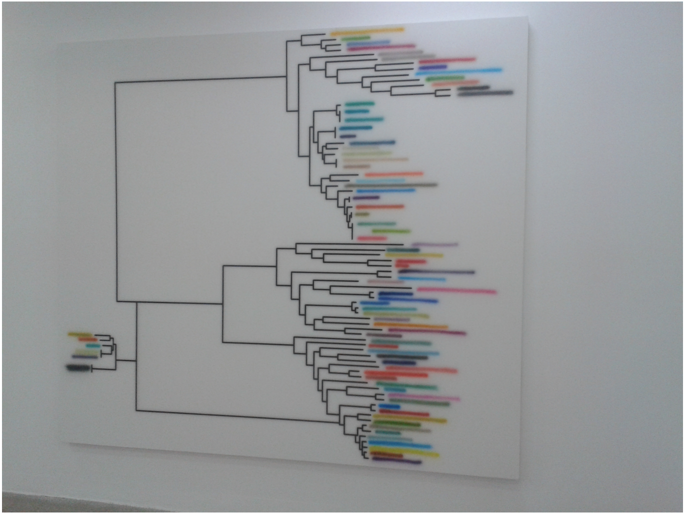

FIGURE: Avery Singer, Sailor

Modeling approaches

- Large fluctuations in financial data motivate the introduction of jumps in the interest rate dynamics : Eberlein & Raible (1999), Filipović, Tappe & Teichmann (2010)...
- Hawkes process to model the "self-exciting" feature : Aït-Sahalia & Jacod (2009), Errais, Giesecke & Goldberg (2010), Dassios & Zhao (2011), Rambaldi, Pennesi & Lillo (2014), and Jaisson & Rosenbaum (2015)...
- Difficulty : jump presence v.s. trend of low interest rate .

Plan of this work

- Objective : a new model of interest rate (α-CIR model) for these seemingly puzzling phenomena in a unified and parsimonious framework.
- Jump diffusion model as natural extension of the CIR model, using the α-stable branching processes.
- The bond price decreases with the parameter α, which allows to respond to the low interest rate behavior. Surprising results :
  - when α decreases, the tails are heavier. The “risks” become larger but the bond price increases....
  - comparison with CIR model with “Poisson” α-stable jumps.
  - different effect if jumps are in branching mechanism or immigration rate, very low interest rates in α-CIR model.

Locally equivalent CIR process with jumps

- Consider the α-CIR process with initial value r_0 and introduce

$$\begin{aligned}\lambda_t = r_0 + \int_0^t a\,(b-\lambda_s)\,ds + \sigma\int_0^t\int_0^{\lambda_s} W(ds,du)\\ + \sigma_Z\int_0^t\int_0^{r_0}\int_{\mathbb{R}+}\zeta\,\widetilde{N}(ds,du,d\zeta),\end{aligned} \tag{5}$$

  where the processes W and $\widetilde{N}$ are (almost) the same as in (4).
- the above CIR process with jumps can be written as

$$d\lambda_t = r_0 + a\,(b-\lambda_t)\,dt + \sigma\sqrt{\lambda_t}\,dB_t + \sigma_Z\,\sqrt[\alpha]{r_0}\,dZ_t,$$

- The implicit negative drifts lead to a linear decay for λ_t while an exponential decay for r_t : when σ_Z increases, the decreasing drift plays a more important role in α-CIR than in equivalent CIR with jumps.

Comparison between α-CIR and CIR with α-stable jumps (continued)

- Separating small and large jumps in CIR with jumps, we get

$$\lambda_t = r_0 + \int_0^t a\left(b - \frac{\sigma_Z\, r_0\, \Theta(\alpha, y)}{a} - \lambda_s\right) ds + \sigma \int_0^t \int_0^{\lambda_s} W(ds, du) + \dots$$
$$+ \sigma_Z \int_0^t \int_0^{r_0} \int_0^y \zeta\, \widetilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_0} \int_y^{\infty} \zeta\, N(ds, du, d\zeta),$$

where

$$\Theta(\alpha, y) = \frac{2}{\pi}\, \alpha\, \Gamma(\alpha - 1)\, \frac{\sin(\pi\alpha/2)}{y^{\alpha-1}}.$$

- In a similar way, the α-CIR process can be written as

$$r_t = r_0 + \int_0^t \widetilde{a}(\alpha, y)\left(\widetilde{b}(\alpha, y) - r_s\right) ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) + \dots$$
$$+ \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta\, \widetilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_y^{\infty} \zeta\, N(ds, du, d\zeta),$$

where

$$\widetilde{a}(\alpha, y) = a + \sigma_Z\, \Theta(\alpha, y), \quad \widetilde{b}(\alpha, y) = \frac{ab}{a + \sigma_Z\, \Theta(\alpha, y)}.$$

Link with the CBI processes

Let r be an α-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process. Then r is a CBI with

$$\text{branching mechanism} : \Psi(q) = a\,q + \frac{\sigma^2}{2}\,q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}\,q^\alpha \tag{6}$$

$$\text{immigration rate} : \Phi(q) = a\,b\,q. \tag{7}$$

Consequences :

- As $t \to +\infty$, r_t has a limite distribution r_∞, given by

$$\mathbb{E}[e^{-pr_\infty}] = \exp\left\{-\int_0^p \frac{\Phi(q)}{\Psi(q)}dq\right\}, \quad p \geq 0.$$

- Laplace transform

$$\mathbb{E}\left[e^{-\xi r_t - p\int_0^t r_s\,ds}\right] = \exp\left(-r_0\,v(t,\xi,p) - \int_0^t \Phi\big(v(s,\xi,p)\big)\,ds\right),$$

$$\text{with } \partial_t v(t,\xi,p) = -\Psi(v(t,\xi,p)) + p, \quad v(0,\xi,p) = \xi.$$

- Let $r^{(\alpha)}$ be α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, $\alpha \in (1, 2]$. Then $r^{(\alpha)} \xrightarrow{\mathcal{L}} r^{(2)}$ in $D(\mathbb{R}_+)$ as $\alpha \to 2$.

Application to bond pricing

For simplicity, we assume that the short rate r is given by an α-CIR$(a, b, \sigma, \sigma_Z, \mu, \alpha)$ model under $\mathbb{Q}$.

- Zero-coupon bond price :

$$B(t,T) = \exp\Big(- r_t\, v(T-t) - a\, b \int_0^{T-t} v(s)ds\Big),$$

  where $v(\cdot)$ is given by

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi(v(t)), \quad v(0) = 0,$$

  with $\Psi(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha$.

- We have

$$v(t) = f^{-1}(t) \text{ where } f(t) = \int_0^t \frac{dx}{1-\Psi(x)} \tag{8}$$

Proposition

The function $v(\cdot)$ is increasing with respect to $\alpha \in (1,2]$. In particular, the bond price $B(0,T)$ is decreasing with respect to α.

- α characterizes the tail fatness : when α decreases, it is more likely to take values far away from median and have large jumps.
- Generalized Blumenthal-Getoor index (e.g. Aït-Sahalia and Jacod, 2009) $\inf\{\beta > 0 : \sum_{0\leq s\leq T} \Delta r_s^{\beta} < \infty,\ a.s.\}$ equals $\inf\{\beta > 0 : \int_0^T r_s ds \int_0^1 u^{\beta}\mu(du) < \infty,\ a.s.\} = \alpha$.
- The above proposition shows that the α-CIR model is suitable to describe the phenomenon of low interest rate trend with jumps.

Bond behaviour of α-CIR model

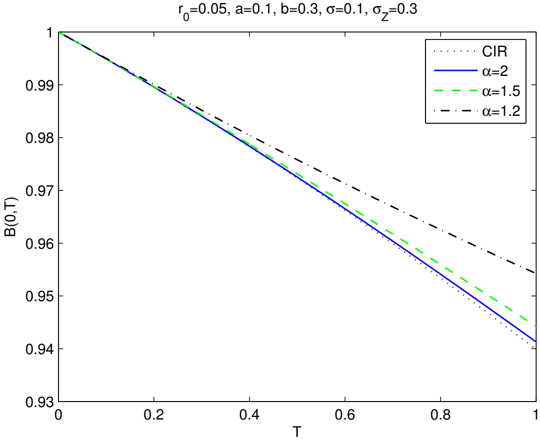

FIGURE: Bond price is decreasing w.r.t. α, curve CIR corresponds to $\sigma_Z = 0$.

Yield curve

$$Y(t,\theta) = -\frac{1}{\theta}\ln B(t,\theta) = r_t\frac{v(\theta)}{\theta} + \frac{a\,b}{\theta}\int_0^\theta v(s)\,ds.$$

According to Keller-Ressel and Steiner (08), define $x_0 > 0$ as the unique solution of $\Psi_\alpha(x_0) = 1$.

Long term yield : $Y(t,\theta) = a\,b\,x_0$.

Normal Yield curve : $Y(t,\theta)$ is increasing if $r_t < a\,b/\Psi'_\alpha(x_0)$.

Inverse Yield curve : $Y(t,\theta)$ is decreasing if $r_t > b$.

Humped Yield curve : $Y(t,\theta)$ has one maximum and no minimum if $a\,b/\Psi'_\alpha(x_0) < r_t < b$.

Related results for forward curve.

Jump behavior

- The jumps, especially the large jumps capture the significant changes in the interest rate and may imply the downgrade risk of credit quality.
- Fix $y > 0$ and define the first time that the jump of r is large than $\sigma_Z\, y$, i.e. $\tau_y = \inf\{t > 0 : \Delta r_t > \sigma_Z y\}$.
- Consider the truncated process $r^{(y)}$ as

$$r_t^{(y)} = r_0 + \int_0^t \widetilde{a}(\alpha, y)\left(\widetilde{b}(\alpha, y) - r_s\right) ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta\, \widetilde{N}(ds, du, d\zeta).$$

- It is also a CBI process which coincides with r up to τ_y, and with the branching mechanism given by

$$\Psi^{(y)} = \Psi + \sigma_Z^\alpha \int_y^\infty (1 - e^{-q\zeta})\, \mu(d\zeta).$$

Probability law of the first large jump

We have

$$\mathbb{P}(\tau_y > t) = \exp\left(-l(y,t)r_0 - ab\int_0^t l(y,s)ds \right),$$

where $l(y,t)$ is the unique solution of

$$\frac{dl}{dt}(y,t) = \sigma_Z^\alpha \int_y^\infty \mu(d\zeta) - \Psi^{(y)}\big(l(y,t)\big),$$

with initial condition $l(y,0) = 0$.

- Equivalent form :

$$\mathbb{P}(\tau_y > t) = \mathbb{E}\left[\exp\left\{ -\sigma_Z^\alpha \Big(\int_y^\infty \mu(d\zeta) \Big) \Big(\int_0^t r_s^{(y)} ds \Big) \right\} \right].$$

which is a bond price written on the auxiliary rate $r^{(y)}$ weighted by the measure μ restricted on (y, ∞).

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Thank you for your attention !

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