Topics in Geometry: Almgren-Pitts Min-max Theory

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This course is divided into two parts: harmonic map theory and Almgren-Pitts min-max theory. The second part covers the basic setups and regularity theory, and extends the discussion to some recent developments in the field, including the notion of volume spectrum and their Weyl Law, the proof of Yau's Conjecture on the existence of infinitely many closed minimal hypersurfaces, and the Multiplicity One Conjecture. This part involves Geometric Measure Theory.

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1 Preliminaries from Geometric Measure Theory

1.1 Theory of varifolds

Definition. Let $A \subseteq \mathbb{R}^{n+k}$. We define the n-dimensional Hausdorff measure

$$\mathcal{H}^n(A) = \lim_{\delta \to 0} \mathcal{H}^n_\delta(A),$$

where for each $\delta > 0$, $\mathcal{H}^n_\delta$ is defined by taking $\mathcal{H}^n_\delta(\emptyset) = 0$ and

$$\mathcal{H}^n_\delta(A) = \omega_n \inf \sum_{j=1}^{\infty} \left(\frac{\operatorname{diam} C_j}{2} \right)^n,$$

where $\omega_n = \operatorname{Vol}(B^n(0))$ and the infimum is taken over all $\bigcup_j C_j$ such that $\operatorname{diam} C_j < \delta$ and $A \subseteq \bigcup_j C_j$.

Definition. Let μ be an outer measure on $\mathbb{R}^{n+k}$ and let $x \in \mathbb{R}^{n+k}$. We define the n-dimensional upper and lower densities $\Theta^{*n}(\mu, x), \Theta^n_*(\mu, x)$ by

$$\Theta^{*n}(\mu, x) = \limsup_{\rho \to 0} \frac{\mu(B_\rho(x))}{\mathcal{H}^n(B_\rho(x))}$$
$$\Theta^n_*(\mu, x) = \liminf_{\rho \to 0} \frac{\mu(B_\rho(x))}{\mathcal{H}^n(B_\rho(x))}$$

If $\Theta^{*n}(\mu, x) = \Theta^n_*(\mu, x)$, then the common value will be denoted $\Theta^n(\mu, x)$.

Let (M^n, g) be a n-dimensional smooth Riemannian manifold embedded in some $\mathbb{R}^N$. We denote by $G_k(M)$ the Grassmannian bundle of un-oriented k-planes over M. That is,

$$G_k(M) := \{(x, P) : x \in M, P \subset T_x M \text{ is a } k\text{-dimensional subspace}\}.$$

When $U \subset M$, we have $G_k(U) = G_k(M)|_U$.

Definition. A k-varifold V on U is a Radon measure on $G_k(U)$.

Denote $\mathcal{V}_k(U) := \{\text{all } k\text{-varifolds}\}$. Given $V \in \mathcal{V}_k(U)$, there exists a Radon measure μ_V on U defined by $\mu_V(A) := V(\pi^{-1}(A))$, where $A \subset U$. We call μ_V the weight of V and $||V||(U) := \mu_V(U)$ (or $\mathbf{M}(V)$) the mass of V. The following lemma is a compactess result for k-varifolds.

Lemma 1.1. The set $A \subset \mathcal{V}_k(U)$ such that $\mathbf{M}(V) \leq C < \infty$ is satisfied for all $V \in A$ is weakly compact. When U is closed, $\mathbf{M} : A \to \mathbb{R}_{\geq 0}$ is continuous w.r.t. the weak topology.

Definition. A $\mathcal{H}^k$-measurable set $M \subset U^n \hookrightarrow \mathbb{R}^n$ is countably k-rectifiable if $M \subset \bigcup_{j=0}^{\infty} M_j$ such that $\mathcal{H}^k(M_0) = 0$ and $M_j \subset F_j(A_j)$ for all $j \geq 1$, where $F_j : A_j \subset \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz for each j.

Definition. Let $M \subset U^n \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set and let $\theta > 0$ be a locally $\mathcal{H}^k$-integrable function on M. We say that $P^k \hookrightarrow \mathbb{R}^n$ is an approximate tangent plane of M at x w.r.t. θ if $\forall f \in C^\infty_c(\mathbb{R}^n)$,

$$\lim_{\lambda \to 0} \int_{\eta_{x,\lambda}(M)} f(y) \theta(x + \lambda y) d\mathcal{H}^k(y) = \theta(x) \int_P f(y) d\mathcal{H}^k(y),$$

where $\eta_{x,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ is the blow-up at x defined by $\eta_{x,\lambda}(y) = (y - x)/\lambda$.

The following theorem gives the important characterization of countably k-rectifiable sets in terms of existence of approximate tangent planes w.r.t. a multiplicity function.

Theorem 1.2 (First Rectifiable Theorem)**.** Let $M \subset U^n \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set. M is countably k-rectifiable iff there exists a locally $\mathcal{H}^k$-integrable function $\theta > 0$ on M and a unique approximate tangent plane T_xM for $\mathcal{H}^k$-a.e. $x \in M$.

Lemma 1.3. Given a countably k-rectifiable set $M \hookrightarrow U$ and a locally $\mathcal{H}^k$-integrable function $\theta > 0$ on M, we can define a k-varifold $V := V(M, \theta)$ such that

$$V(M,\theta)(A) = \int_{\{x \in M : (x, T_xM) \in A\}} \theta(x) d\mathcal{H}^k(x), \quad \forall A \subset G_k(U).$$

A natural question to ask is when a general varifold is rectifiable, i.e. when a measure is given by Lipschitz subsets. This will be answered later by the rectifiability theorem.

Now, we move on to discuss the first variation of varifolds. Let $U \subset \mathbb{R}^n$ and let $V \subset \mathcal{V}_k(U)$. Suppose that $\varphi : U \to U' \cong U$ is a diffeomorphism. We may view φ as a map $\varphi : G_k(U) \to G_k(U)$ defined by $\varphi(x, S) = (\varphi(x), d\varphi_x(S))$, where S is a k-plane. Note that $d\varphi_x|_S$ is a $n \times k$ matrix while $(d\varphi_x|_S)^*$ is a $k \times n$ matrix. Hence, $(d\varphi_x|_S)^* \circ d\varphi_x|_S$ is a $k \times k$ matrix and the Jacobian of $d\varphi_x|_S : S \to d\varphi_x(S)$ is

$$J\varphi(x, S) = \det[(d\varphi_x|_S)^* \circ d\varphi_x|_S]^{\frac{1}{2}}.$$

Then the pushforward of V is defined as

$$(\varphi_\# V)(A) := \int_{\varphi^{-1}(A)} J\varphi(x, S) dV(x, S), \quad \forall A \subset G_k(U).$$

Given $X \in \mathfrak{X}_c(U)$ a compactly supported smooth vector field in U, we have the local flow $\varphi^X : (-\epsilon, \epsilon) \times U \to U$ with $\frac{d}{dt}\varphi^X(t,p) = X(\varphi^X(t,p))$. Then the first variation of V can be explicitly computed as

$$\delta V(x) := \frac{d}{dt}\bigg|_{t=0} ||(\varphi_t^X)_\# V|| = \int_{G_k(U)} \operatorname{div}_S X dV(x, S).$$

Here, $\operatorname{div}_S X := \sum_{i=1}^k \langle \nabla_{\tau_i} X, \tau_i \rangle$, where $\{\tau_1, \ldots, \tau_k\}$ forms an orthonormal basis of S.

Remark. Recall that for minimal submanifolds $\Sigma^k \hookrightarrow U$, we have the first variation formula

$$\delta \Sigma(x) := \frac{d}{dt}\bigg|_{t=0} \operatorname{Vol}(\varphi_t^X(\Sigma)) = \int_\Sigma \operatorname{div}_{T_x\Sigma} X \, \mathrm{dVol}\,.$$

We present a proof of the first variation formula of varifolds. Let $(x, S) \in G_k(U)$ and suppose that $\{\tau_1, \ldots, \tau_k\}$ forms an orthonormal basis of S. Note that $\varphi_t^X(x) = x + tX(x) + o(t^2)$ and $(D_{\tau_j}\varphi_t^X)^l = \tau_j^l + tD_{\tau_j}X^l + o(t^2)$. A simple calculation leads to

$$\begin{aligned}
((d\varphi_t^X|_S)^* \circ d\varphi_t^X|_S)_{ij} &= (D_{\tau_i}\varphi_t^X)^l \cdot (D_{\tau_j}\varphi_t^X)^l \\
&= (\tau_i^l + tD_{\tau_i}X^l + o(t^2))(\tau_j^l + tD_{\tau_j}X^l + o(t^2)) \\
&= \delta_{ij} + t(\tau_i \cdot D_{\tau_j}X + \tau_j \cdot D_{\tau_i}X) + o(t^2)
\end{aligned}$$

and

$$\frac{d}{dt}\bigg|_{t=0} \det[(d\varphi_t^X|_S)^* \circ d\varphi_t^X|_S]^{\frac{1}{2}} = \frac{d}{dt}\bigg|_{t=0} [1 + \operatorname{Tr}(\tau_i \cdot D_{\tau_j}X + \tau_j \cdot D_{\tau_i}X)]^{\frac{1}{2}}$$

$$
\begin{aligned}
&= \frac{d}{dt}\bigg|_{t=0} [1 + 2t \operatorname{div}_S X + o(t)]^{\frac{1}{2}} \\
&= \operatorname{div}_S X,
\end{aligned}
$$

which complete the proof.

Definition. $V \in \mathcal{V}_k(U)$ is stationary in U if $\delta V(X) \equiv 0$, $\forall X \in \mathfrak{X}_c(U)$.

Remark. Stationary varifolds can be viewed as generalization of minimal surfaces.

Example 1.4. Triple junction surfaces are stationary.

Recall that given a minimal submanifold $\Sigma^k \hookrightarrow \mathbb{R}^n$ with $x_0 \in \Sigma$, we have for all $B_\sigma(x_0) \subset B_\rho(x_0) \subset \mathbb{R}^n$,

$$
\frac{\operatorname{Vol}(B_\rho(x_0) \cap \Sigma)}{\rho^k} - \frac{\operatorname{Vol}(B_\sigma(x_0) \cap \Sigma)}{\sigma^k} = \int_{(B_\rho(x_0) \setminus B_\sigma(x_0)) \cap \Sigma} \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}} \, \mathrm{dVol}\,.
$$

The following theorem provides a monotonicity formula for varifolds analogous to that for minimal submanifolds.

Theorem 1.5 (Monotonicity Formula)**.** Let $V \in \mathcal{V}_k(U)$ be stationary in U. For all $B_\sigma(0) \subset B_\rho(0) \subset U$, we have

$$
\frac{\mu_V(B_\rho(0))}{\rho^k} - \frac{\mu_V(B_\sigma(0))}{\sigma^k} = \int_{G_k(B_\rho(0) \setminus B_\sigma(0))} \frac{|D_S^\perp r|^2}{r^k} dV(x,S).
$$

Here, $D_S^\perp = P_{S^\perp}(\nabla r)$, where $S^\perp$ is the orthogonal complement of k-plane $S \subset \mathbb{R}^n$.

Corollary 1.6. Let $V \in \mathcal{V}_k(U)$ be stationary in U. Then, $\Theta(||V||, x)$ exists everywhere in U and is bounded.

Definition. $V \in \mathcal{V}_k(U)$ is said to have locally bounded first variation in U if for each $W \subset\subset U$, there is a constant $C > 0$ such that $|\delta V(X)| \le C \sup_U |X|$ for all compactly supported continuous vector fields X in U.

Theorem 1.7 (General Monotonicity Formula)**.** Suppose that $V \in \mathcal{V}_k(U)$ has locally bounded first variation in U. Let $x \in U$ such that there is $0 < \rho_0 < \operatorname{dist}(x, \partial U)$ and $\Lambda \ge 0$ with

$$
||\delta V||(B_\rho(x)) \le \Lambda \mu_V(B_\rho(x)), \quad 0 < \rho < \rho_0.
$$

Then for all $0 < \sigma \le \rho < \rho_0$,

$$
\Theta^k(||V||, x) \le e^{\Lambda\sigma} \frac{\mu_V(B_\sigma(x))}{\omega_k \sigma^k} \le e^{\Lambda\rho} \frac{\mu_V(B_\sigma(x))}{\omega_k \rho^k} - \frac{1}{\omega_k} \int_{G_k(B_\rho(x) \setminus B_\sigma(x))} \frac{|D_S^\perp r|^2}{r^k} dV(y,S).
$$

Definition. Given $V, W \in \mathcal{V}_k(U)$, the varifold distance between V and W is defined as

$$
\mathbf{F}(V,W) := \sup \left\{ \int_{G_k(U)} f(x,S) dV(x,S) - \int_{G_k(U)} f(x,S) dW(x,S) \right\},
$$

where the supremum is taken over all $f \in \operatorname{Lip}(G_k(W))$ with $||f||_\infty \le 1$ and $\operatorname{Lip}(f) \le 1$.

Definition. Let $V \in \mathcal{V}_k(U)$ and let $x \in U$. We denote by $\mathrm{VarTan}(V, x)$ the varifold tangent at x, which is the set of all weak limits

$$\mathrm{VarTan}(V, x) := \{C \in \mathcal{V}_k(\mathbb{R}^n) : C = \lim_{\lambda_i \to 0} (\eta_{x,\lambda_i})_\# V\}.$$

Let $V_i = (\eta_{x,\lambda_i})_\# V$ and suppose that x is any point of U such that $\lim_{\rho\to 0} \rho^{1-k} ||\delta V||(B_\rho(x)) = 0$. By the lower semicontinuity of the first variation, we have

$$||\delta C||(B_\rho(x)) = \liminf_{i\to\infty} ||\delta V_i||(B_\rho(x)) = \liminf_{i\to\infty} \lambda_i^{1-k} ||\delta V||(B_\rho(x)) = 0.$$

This shows that C is stationary in $\mathbb{R}^{k+l}$. One may further deduce from definition of C that

$$\frac{||C||(B_\rho(0))}{\omega_k \rho^k} = \lim_{\lambda_i \to 0} \frac{||V_i||(B_\rho(0))}{\omega_k \rho^k} = \lim_{\lambda_i \to 0} \frac{||V||(B_\rho(x)}{\omega_k \lambda_i^k \rho^k} = \Theta^k(||V||, x).$$

Since $\delta C = 0$, the monotonicity formula implies that

$$\int_{G_k(B_\rho(0))} \frac{|D_S^\perp r|^2}{|x|^k} dC(x, S) = 0, \quad \forall \rho > 0.$$

Then $P_{S^\perp}(x) = 0$ for all $(x, S) \in \mathrm{spt}\, C$. By choosing an appropriate vector field X and substituting into the ODE obtained by the first variation, one may conclude that

$$\lambda^k ||C||(\eta_{0,\lambda}(A)) = ||C||(A), \quad \forall A \subset \mathbb{R}^{k+l}, \lambda > 0.$$

Theorem 1.8 (Rectifiability Theorem)**.** Let $V \in \mathcal{V}_k(U)$ be stationary in U. If $\Theta^k(||V||, x) > 0$ for $||V||$-a.e. $x \in U$, then V is a k-rectifiable varifold. Indeed, $V = V(M, \theta)$, where $M = \{x \in U : \Theta^k(||V||, x) > 0\}$ is a countably k-rectifiable set and $\theta > 0$ is a locally $\mathcal{H}^k$-integrable function on M.

Corollary 1.9. Assume $\Theta^k(||V||, x) \geq C_0 > 0$ for $||V||$-a.e. $x \in U$. Then C is a k-rectifiable varifold. Moreover, $(\eta_{0,\lambda})_\# C = C$.

Theorem 1.10 (Constancy Theorem)**.** Let $V \in \mathcal{V}_k(U)$ be stationary in U and let $M^k \hookrightarrow U$ be a connected, C^∞-embedded submanifold. If $\mathrm{spt}\, ||V|| \subset M^k$, then $V = c \cdot V(M)$.

Theorem 1.11 (Compactness Theorem)**.** Let $\{V_j\}$ be a sequence of k-varifolds each stationary in U. Suppose that $\Theta^k(||V_j||, x) \geq 1$ for $||V_j||$-a.e. $x \in U$ and $\sup_j\{||V_j||(K)\} \leq C(K), \forall K \subset\subset U$. Then a subsequence of $\{V_j\}$ converges weakly (in the sense of Radon measures) to some k-rectifiable varifold $V \in \mathcal{V}_k(U)$. Moreover, we have $\Theta^k(||V||, x) \geq 1$ for $||V||$-a.e. $x \in U$ and the lower semicontinuity $||\delta V||(W) \leq \liminf_{j\to\infty} ||\delta V_j||(W)$ for each $W \subset\subset U$.

Remark. An important additional result (also due to Allard [1]) is the Integral Compactness Theorem, which asserts that if all V_j above are integer multiplicity, then V is also integer multiplicity. We refer to [21] for a detailed proof.

Definition. Given $V, W \in \mathcal{V}_k(U)$, the varifold distance between V and W is defined as

$$\mathbf{F}(V, W) := \sup_{f \in \mathrm{Lip}(G_k(U)), ||f||_\infty \leq 1, \mathrm{Lip}\, f \leq 1} \left\{ \int_{G_k(U)} f(x, S) dV(x, S) - \int_{G_k(U)} f(x, S) dW(x, S) \right\}.$$

Remark. On $\{V \in \mathcal{V}_k(M) : ||V|| \leq L < \infty\}$, the weak topology coincides with the $\mathbf{F}$-distance topology.

Theorem 1.12 (Maximum Principle)**.** Let $V \in \mathcal{V}_k(U)$ where $U \subset \mathbb{R}^n$ is open. Suppose that $\operatorname{spt}||V|| \subset B_s(0) \setminus B_t(0)$ with $0 < t < s$ and $B_s(0) \subset U$. Then $\operatorname{spt}||V|| \cap \partial B_s(0) = \emptyset$.

Corollary 1.13. Suppose that $\operatorname{spt}||V|| \cap \partial B_s(0) \neq \emptyset$. Then $\operatorname{spt}||V|| \cap (U \setminus \overline{B_s(0)}) \neq \emptyset$.

Theorem 1.14 (Sard Theorem)**.** Let $V \in \mathcal{V}_k(U)$ be stationary in U and k-rectifiable. Let $p \in U$ and let $B_\rho(p) \subset\subset U$. Consider

$$T = \{y \in \operatorname{spt}||V|| \cap B_\rho(p) : T_y V \pitchfork T_y \partial B_{d(x,y)}(x)\}.$$

Then T is a dense subset of $\operatorname{spt}||V|| \cap B_\rho(p)$.

1.2 Sets of locally finite perimeters

Definition. $\Omega \subset \mathbb{R}^{n+1}$ has locally finite perimeter if the characteristic function χ_Ω is of bounded variation in U, that is, $\forall X \in \mathfrak{X}_c^1(U)$, $\operatorname{supp}|X| \subset W \subset\subset U$,

$$\int_\Omega \operatorname{div} X d\mathcal{H}^{n+1} \leq C(W) \sup|X|.$$

If we view the integral as a functional on $X : U \to \mathbb{R}^{n+1}$ which is bounded on compact subsets, then by the Riesz representation theorem, there is a Radon measure $\mu_{\partial\Omega} = |D\chi_\Omega|$ in U and a $\mu_{\partial\Omega}$-measurable vector field $v = (v_1, \ldots, v_{n+1})$ with $|v| = 1$ for $\mu_{\partial\Omega}$-a.e. x in U such that

$$\int_\Omega \operatorname{div} X d\mathcal{H}^{n+1} = \int_U X \cdot v d\mu_{\partial\Omega}, \quad \forall X \in \mathfrak{X}_c^1(U).$$

If ∂C is C^∞-embedded, by the divergence theorem, we have

$$\int_\Omega \operatorname{div} X d\mathcal{H}^{n+1} = \int_{\partial\Omega} X \cdot v_{\partial\Omega} d\mathcal{H}^n.$$

This implies that $\mu_{\partial\Omega} = \mathcal{H}^n \llcorner \partial\Omega$ and $v_{\partial\Omega}$ is the inward unit normal to $\partial\Omega$. The bounded variation condition of χ_Ω in U reduces to $\mathcal{H}^n(\partial\Omega \cap W) \leq C(W)$. In general, we interpret $\mu_{\partial\Omega}$ as a "generalized boundary measure" and v as a "generalized inward unit normal."

Definition. Let $\Omega \subset \mathbb{R}^{n+1}$ be a set of locally finite perimeter. Define the reduced boundary $\partial^*\Omega$ in U by

$$\partial^*\Omega = \left\{ x \in U : \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} v(x) d\mu_{\partial\Omega}}{\mu_{\partial\Omega}(B_\rho(x))} \text{ exists and has length } 1 \right\}.$$

By the density theorem, we have $\mu_{\partial\Omega}(U \setminus \partial^*\Omega) = 0$. Hence, $\mu_{\partial\Omega} = \mu_{\partial\Omega} \llcorner \partial^*\Omega$. The following theorem gives a characterization of $\partial^*\Omega$ as a countably n-rectifiable set.

Theorem 1.15. Suppose that $\Omega \subset \mathbb{R}^{n+1}$ has locally finite perimeter in U. Then $\partial^*\Omega$ is countably n-rectifiable and $\mu_{\partial\Omega} = \mathcal{H}^n \llcorner \partial^*\Omega$. At each $x \in \partial^*\Omega$, the approximate tangent plane T_x exists, has multiplicity 1, and is given by

$$T_x = \{y \in \mathbb{R}^{n+1} : y \cdot v(x) = 0\}.$$

Definition. Denote the set of all sets with locally finite perimeter in U by $\mathcal{C}(U)$. A set $\Omega \in \mathcal{C}(U)$ is called a Caccioppoli set in U.

Theorem 1.16 (Compactness Theorem)**.** Given $\{\Omega_j\} \subset \mathcal{C}(U)$ with

$$\sup_j \mu_{\partial\Omega_j}(W) \leq C(W) < \infty, \quad \forall W \subset\subset U,$$

then a subsequence $\{\Omega_{j'}\}$ converges weakly to a limit $\Omega_\infty \in \mathcal{C}(U)$ in the sense that

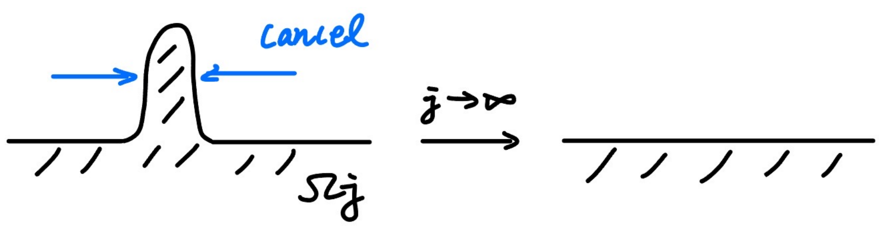
- $\chi_{\Omega_{j'}} \to \chi_{\Omega_\infty}$ in $L^1_{\text{loc}}(U, \mathcal{H}^{n+1})$;
- $v_{\partial\Omega_j} d\mu_{\partial\Omega_j} \to v_{\partial\Omega_\infty} d\mu_{\partial\Omega_\infty}$.

Moreover, we have the lower semicontinuity $\mu_{\partial\Omega_\infty}(W) \leq \liminf_{j\to\infty} \mu_{\partial\Omega_j}(W)$ for each $W \subset\subset U$.

Remark. The first condition is equivalent to $\mathcal{H}^{n+1}(\Omega_{j'} \triangle \Omega_\infty) = \text{Vol}(\Omega_{j'} \triangle \Omega_\infty) \to 0$, where $\triangle$ is the symmetric difference of two sets. The second condition is equivalent to $[\partial\Omega_j]$ converges as currents to $[\partial\Omega_\infty]$, where $[\partial\Omega]$ is in the dual space of $\Lambda^n(U)$ and given ω a n-form and $\{e_1, \ldots, e_n\}$ an orthonormal basis of $T_x\partial\Omega$,

$$[\partial\Omega](\omega) = \int_{\partial\Omega} \langle e_1^* \wedge \cdots \wedge e_n^*, \omega \rangle d\mathcal{H}^n.$$

Example 1.17. Due to the cancellations, the sequence Ω_j on left converges as Caccioppoli sets/currents to Ω on right. However, $\partial\Omega_j$ does not converge to $\partial\Omega$ in the sense of measure.



Definition. Given any $\Omega \in \mathcal{C}(U)$, the mass norm of Ω and $\partial\Omega$ are defined to be

$$\mathbf{M}(\Omega) = \int_{\Omega\cap U} d\mathcal{H}^{n+1} = \text{Vol}(\Omega \cap U), \quad \mathbf{M}(\partial\Omega) = \int_{\partial\Omega\cap U} d\mathcal{H}^n = \mu_{\partial\Omega}(U).$$

For each $W \subset\subset U$, define

$$\mathbf{M}_W(\Omega) = \int_{\Omega\cap W} d\mathcal{H}^{n+1} = \text{Vol}(\Omega \cap W), \quad \mathbf{M}_W(\partial\Omega) = \int_{\partial\Omega\cap W} d\mathcal{H}^n = \mu_{\partial\Omega}(W).$$

Moreover, given any pair $\Omega_1, \Omega_2 \in \mathcal{C}(U)$, we set

$$\mathbf{M}(\partial\Omega_1, \partial\Omega_2) = \mathbf{M}(\partial\Omega_1 - \partial\Omega_2).$$

Remark. If $\mathbf{M}(\partial\Omega_1, \partial\Omega_2) << 1$, then $\text{Vol}(\partial\Omega_1 \triangle \partial\Omega_2)$ is very small.

Definition. Given any pair $\Omega_1, \Omega_2 \in \mathcal{C}(U)$, the flat metric is defined as

$$\mathcal{F}(\Omega_1, \Omega_2) := \mathcal{F}(\partial\Omega_1, \partial\Omega_2) := \inf\{\mathbf{M}(T) + \mathbf{M}(S) : \partial\Omega_1 - \partial\Omega_2 = T + \partial S\},$$

where the infimum is taken over all integer rectifiable n-current T and integer rectifiable $(n+1)$-current S such that S is a filling of $\partial\Omega_1 - \partial\Omega_2 - T$.

By considering $S = 0$ in the definition of flat metric, we immediately obtain the following corollary, which suggests that flat metric is weaker than mass norm.

Corollary 1.18. $\mathcal{F}(\partial\Omega_1, \partial\Omega_2) \leq \mathbf{M}(\partial\Omega_1 - \partial\Omega_2)$.

Proposition 1.19. There exists $\delta << 1$ such that if $\Omega_1, \Omega_2 \in \mathcal{C}(U)$ with $\Omega_1, \Omega_2 \subset U$ and $\mathcal{F}(\Omega_1, \Omega_2) < \delta$, and moreover if $\mathbf{M}(\Omega_1 - \Omega_2) < \mathrm{Vol}(U)/2$, then

$$\mathcal{F}(\Omega_1, \Omega_2) = \min\{\mathbf{M}(\Omega_1 \triangle \Omega_2), \mathbf{M}(U \setminus (\Omega_1 \triangle \Omega_2)).\}$$

Definition. Given any $\Omega \in \mathcal{C}(U)$ with $\Omega \subset U$, the flat norm is defined as

$$\mathcal{F}(\Omega) := \mathcal{F}(\Omega, \emptyset) = \min\{\mathrm{Vol}(\Omega), \mathrm{Vol}(U \setminus \Omega)\}.$$

The following proposition indicates that under certain condition, convergence as Caccioppoli sets is equivalent to convergence in flat metric.

Proposition 1.20. On the set $\{\partial\Omega : \Omega \in \mathcal{C}(U), \mathbf{M}(\partial U) \leq L < \infty\}$, we have

$$\Omega_j \to \Omega_\infty \Longleftrightarrow \mathcal{F}(\Omega_j, \Omega_\infty) \to 0.$$

Recall that if $\Omega \in \mathcal{C}(U)$, then the reduced boundary $\partial^*\Omega$ is a countably n-rectifiable set. Moreover, at each $x \in \partial^*\Omega$, the approximate tangent space T_x exists and has multiplicity 1. Then it is natural to define a n-rectifiable varifold corresponding to the pair $(\partial^*\Omega, \theta \equiv 1)$.

Definition. Given $\Omega \in \mathcal{C}(U)$, we denote by $|\partial\Omega|$ the n-rectifiable varifold induced by the countably n-rectifiable set $\partial^*\Omega$.

Definition. Given any pair $\Omega_1, \Omega_2 \in \mathcal{C}(U)$, the $\mathbf{F}$-metric is defined as

$$\mathbf{F}(\Omega_1, \Omega_2) = \mathcal{F}(\Omega_1, \Omega_2) + \mathbf{F}(|\partial\Omega_1|, |\partial\Omega_2|),$$

where $\mathbf{F}$ denotes the varifold distance.

Remark. By definition, $\Omega_j \to \Omega_\infty$ under the $\mathbf{F}$-metric $\Longleftrightarrow$ $\Omega_j \to \Omega_\infty$ weakly and $|\partial\Omega_j| \to |\partial\Omega_\infty|$.

Among all Caccioppoli sets in U, we pay special attention to those that are locally mass minimizing. Such sets possess good regularity results, which have been established by De Giorgi, Federer-Fleming, Almgren, and Simons through a series of works.

Definition. Say that $\Omega \in \mathcal{C}(U)$ is locally mass minimizing if $\forall p \in U$, $\exists \delta > 0$ such that $\forall \Omega' \in \mathcal{C}(U)$ with $\Omega' \triangle \Omega \subset\subset B_r(p)$, then $\mathbf{M}(\partial\Omega) \leq \mathbf{M}(\partial\Omega')$.

Theorem 1.21 (De Giorgi, Federer-Fleming, Almgren, Simons, see [21])**.** Suppose that $\Omega \in \mathcal{C}(U)$ is locally mass minimizing in U. Then

- For $3 \leq (n+1) \leq 7$, $\partial\Omega$ is a C^∞-embedded minimal hypersurface;
- For $n+1 = 8$, $\partial\Omega$ is a C^∞-embedded minimal hypersurface away from discrete singular points;
- For $n+1 > 8$, $\partial\Omega$ is a C^∞-embedded minimal hypersurface away from a singular set $\mathrm{Sing}(\Sigma)$ of Hausdorff codimension 7.

1.3 Mod-2 flat chains

Let (M^{n+1}, g) be a $(n+1)$-dimensional closed Riemannian manifold. Assume that (M, g) is isometrically embedded in some Euclidean space $\mathbb{R}^N$. The spaces we will work with are:

- the space $\mathbf{P}_k(\mathbb{R}^N; G)$ of k-dimensional polyhedral chains in $\mathbb{R}^N$ with coefficients in G;
- the space $\mathbf{I}_k(M; \mathbb{Z}_2)$ of k-dimensional flat chains in $\mathbb{R}^N$ with coefficients in $\mathbb{Z}_2$ and support contained in M;
- the space $\mathcal{Z}_k(M; \mathbb{Z}_2)$ of flat chains $T \in \mathbf{I}_k(M; \mathbb{Z}_2)$ such that $\partial T = 0$.

For every $P \in \mathbf{P}_k(\mathbb{R}^N; G)$, we may write $P = \sum_{i=1}^l a_i[P_i]$, where $a_i \in G$ and $\{P_1, \ldots, P_l\}$ are disjoint polyhedrons. Define the mass norm

$$\mathbf{M}(P) := \sum_{i=1}^l a_i \mathcal{H}^k(P_i)$$

and the flat norm

$$\mathcal{F}(P) := \inf\{\mathbf{M}(R) + \mathbf{M}(Q) : P = R + \partial Q, R \in \mathbf{P}_k(\mathbb{R}^N; G), Q \in \mathbf{P}_{k+1}(\mathbb{R}^N; G)\}.$$

Since $\mathcal{F}$ defines a metric in $\mathbf{P}_k(\mathbb{R}^N; G)$, we may let $\mathcal{C}_k(\mathbb{R}^N; G)$ be the $\mathcal{F}$-completion of $\mathbf{P}_k(\mathbb{R}^N; G)$. Under this definition, $\mathcal{C}_k(\mathbb{R}^N; G)$ consists of flat k-chains over G.

When coefficients are taken to be $\mathbb{Z}_2$, we say that a flat k-chain T is rectifiable if T is the limit of C^1 flat k-chains in the $\mathbf{M}$-topology. Moreover, we have the following rectifiability result.

Theorem 1.22. Every flat k-chain T with coefficients in $\mathbb{Z}_2$ of finite mass is rectifiable.

Since every $T \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ has finite mass and finite boundary mass, we deduce that

$$\mathbf{I}_{n+1}(M; \mathbb{Z}_2) = \mathcal{C}(M).$$

The following lemma is a direct corollary of the constancy theorem.

Lemma 1.23. $\partial : \mathbf{I}_{n+1}(M; \mathbb{Z}_2) = \mathcal{C}(M) \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a double covering space.

Lemma 1.24. ∂ satisfies lifting properties, that is, given a map $\sigma : I^k = [0,1]^k \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ and $U_0 \in \mathcal{C}(M)$ such that $\partial U_0 = \sigma(0)$, there exists a unique map $U : I^k \to \mathcal{C}(M)$ such that $U(0) = U_0$ and $\partial U(t) = \sigma(t)$.

With the lemmas above, we are ready to prove the Almgren's isomorphism theorem for codimension 1 case.

Theorem 1.25 (Almgren's Isomorphism Theorem, [2, 15])**.** $\mathcal{Z}_n(M; \mathbb{Z}_2)$ is weakly homotopic to $\mathbb{R}P^\infty$.

Proof. Let $f : M \to \mathbb{R}$ be a Morse function with $f(M) = [0,1]$. Define a map $\Phi : \mathbb{R}P^\infty \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ by

$$\Phi([a_0, a_1, \ldots, a_k, 0, 0, \ldots]) = \partial\{x : a_0 + a_1 f(x) + \cdots + a_k f(x)^k < 0\}.$$

This map is well-defined as both sides are scaling invariant. We claim that Φ is a weak homotopy equivalence, i.e.

$$\Phi_* : \pi_k(\mathbb{R}P^\infty, *) \to \pi_k(\mathcal{Z}_n(M; \mathbb{Z}_2), 0)$$

are isomorphisms for all k.

To start with, we show that $\mathbf{I}_{n+1}(M;\mathbb{Z}_2) = \mathcal{C}(M)$ is contractible. Define the map $H : [0,1] \times \mathcal{C}(M) \to \mathcal{C}(M)$ by

$$H(t,\Omega) = \Omega \llcorner \{x : f(x) < t\}.$$

Note that $t \in [0,1] \mapsto \Omega \llcorner \{x : f(x) < t\}$ is continuous in the flat norm. One can further check that H is continuous and hence a homotopy between a constant map and the identity map. This shows that $\mathcal{C}(M)$ is contractible.

Since $\mathbb{R}P^\infty$ is an Eilenberg–MacLane space with $\pi_1(\mathbb{R}P^\infty, *) \cong \mathbb{Z}_2$ and $\pi_k(\mathbb{R}P^\infty, *) = 0$ for all $k \geq 2$, it is sufficient to show that $\pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) \cong \mathbb{Z}_2$ and $\pi_k(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) = 0$ for all $k \geq 2$. First, consider the case $k \geq 2$. Let $\Psi : I^k \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ be a map with $\Psi(\partial I^k) = 0$. By the lifting properties, there exists a unique map $U : I^k \to \mathcal{C}(M)$ with $U(0) = \emptyset$ and $\partial U(t) = \Psi(t)$. Moreover, we have $U(\partial I^k) = \emptyset$. Since $\mathcal{C}(M)$ is contractible, U is homotopic to a constant map relative to ∂I^k. This implies that Ψ is homotopic to a constant map relative to ∂I^k downstairs, i.e. $\pi_k(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) = 0$.

Now, consider the case $k = 1$. Let $\sigma : [0,1] \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ be a map with $\sigma(0) = \sigma(1) = 0$. By the lifting properties, there exists a unique map $U : [0,1] \to \mathcal{C}(M)$ with $U(0) = \emptyset$ and $\partial U(t) = \sigma(t)$. Since $\partial U(1) = \sigma(1) = 0$, we have $U(1) = \emptyset$ or $U(1) = M$ by the constancy theorem. Note that $U(1) = \emptyset \iff$ the lift of σ stays as a loop upstairs $\iff \sigma$ is homotopic to a constant map relative to $\{0,1\}$. If $U(1) = M$, then σ is not nullhomotopic downstairs and $\sigma : [0,1] \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ defined by $\sigma(t) = \partial\{x : f(x) < t\}$ generates $\pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) \cong \mathbb{Z}_2$. In this case, $\Phi|_{S^1} : S^1 \hookrightarrow \mathbb{R}P^\infty \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is given by

$$\Phi([\cos(\pi\theta), \sin(\pi\theta), 0, 0, \ldots]) = \partial\{\cos(\pi\theta) + \sin(\pi\theta) f < 0\} = \partial\{f < -\cot(\pi\theta)\},$$

which induces an isomorphism on fundamental groups. This completes the proof that Φ is a weak homotopy equivalence. $\square$

Corollary 1.26. The cohomology ring of $\mathcal{Z}_n(M;\mathbb{Z}_2)$ w.r.t. $\mathbb{Z}_2$ coefficients is

$$H^*(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{\lambda}],$$

where $\bar{\lambda}$ is the generator of $H^1(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2) \cong \mathbb{Z}_2$ (the fundamental cohomology class).

Remark. If $\sigma : S^1 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a loop, then

$$\bar{\lambda} \cdot [\sigma] \neq 0 \iff \sigma \text{ is homotopically nontrivial.}$$

2 Almgren-Pitts min-max theorem

2.1 Sweepout and width

Theorem 2.1 (Almgren-Pitts Min-max Theorem [3, 17, 18])**.** Let (M^{n+1}, g) be a closed Riemmanian manifold. Then there always exists a closed minimal hypersurface Σ^n such that outside a singular set $\mathrm{Sing}(\Sigma)$ of Hausdorff codimension 7, it is C^∞-embedded. In particular, if $3 \leq (n+1) \leq 7$, Σ is C^∞.

Definition. A sweepout (s.w.) is a map $\varphi : [0,1] \to (\mathcal{C}(M), \mathbf{F})$ such that

- φ is continuous w.r.t. the $\mathbf{F}$-metric;
- $\varphi(0) = \emptyset$ and $\varphi(1) = M$.

Example 2.2. Let $f : M \to \mathbb{R}$ be a Morse function. Then $\varphi : [0,1] \to \mathcal{C}(M)$ defined by $t \mapsto f^{-1}([0,t])$ is a sweepout. Note that φ is continuous w.r.t. $\mathcal{F}$ because $t \mapsto \mathrm{Vol}(f^{-1}([0,t]))$ is continuous, and φ is continuous w.r.t. $\mathbf{F}$ because $t \mapsto d\mathcal{H}^n|_{f^{-1}(t)}$ is continuous.

Lemma 2.3. Given any sweepout $\varphi : [0,1] \to \mathcal{C}(M), \mathbf{F})$, it is homotopic under the $\mathbf{F}$-metric to the sweepout by a Morse function.

Proof. Recall that in the proof of Theorem 1.25 we have shown that $(\mathcal{C}(M), \mathcal{F})$ is contractible. Since a $\mathcal{F}$-homotopy can be interpolated to a $\mathbf{F}$-homotopy, we conclude that $(\mathcal{C}(M), \mathbf{F})$ is contractible. □

Definition. The width of (M^{n+1}, g) is defined as

$$W := \inf_{\varphi \text{ is a s.w.}} \max_{x\in[0,1]} \mathbf{M}(\partial\varphi(x)).$$

Using the following lower bound for the isoperimetric profiles for small volumes, we show that the width is always positive.

Lemma 2.4. There exists constants $C_0 > 0$ and $V_0 > 0$ depending only on M such that

$$\mathrm{Area}(\partial\Omega) \geq C_0 \,\mathrm{Vol}(\Omega)^{\frac{n}{n+1}}, \quad \text{whenever } \Omega \in \mathcal{C}(M) \text{ and } \mathrm{Vol}(\Omega) \leq V_0.$$

Corollary 2.5. We have $W > 0$.

Proof. We shall present a heuristic proof here. Let $\varphi : [0,1] \to (\mathcal{C}(M), \mathbf{F})$ be a sweepout. Then the map $x \mapsto \mathrm{Vol}(\varphi(x))$ is continuous and there exists $x_0 \in (0,1)$ such that $\mathrm{Vol}(\varphi(x_0)) = V_0$. By the isoperimetric profiles for small volumes, we have

$$\max_{x\in[0,1]} \mathbf{M}(\partial\varphi(x)) \geq \mathbf{M}(\partial\varphi(x_0)) \geq C_0 \,\mathrm{Vol}(\varphi(x_0))^{\frac{n}{n+1}} = C_0 V_0^{\frac{n}{n+1}}.$$

Since φ is arbitrary, we conclude that $W > 0$. □

Definition. A minimizing sequence of sweepouts is a sequence $\{\varphi_j : [0,1] \to \mathcal{C}(M)\}$ such that $\max_{x\in[0,1]} \mathbf{M}(\partial\varphi_j(x)) \to W$ as $j \to \infty$.

Definition. The critical set of $\{\varphi_j\}$ is given by

$$C(\{\varphi_j\}) := \{V \in \mathcal{V}_n(M) : V = \lim_{i\to\infty} |\partial\varphi_{j_i}(x_i)| \text{ with } \mathbf{M}(\partial\varphi_{j_i}(x_i)) \to W\}.$$

Theorem 2.6. For every minimizing sequence $\{\varphi_j^*\}$, there exists another pull-tight minimizing sequence $\{\varphi_j\}$ that is homotopic to φ_j^* in $(\mathcal{C}(M), \mathbf{F})$ such that

- every $V \in C(\{\varphi_j\})$ is stationary;
- there exists $V_\infty \in C(\{\varphi_j\})$ such that $V_\infty = \sum_{i=1}^{l} m_i[\Sigma_i]$, where $\{\Sigma_1, \ldots, \Sigma_l\}$ are disjoint closed, C^∞-embedded minimal hypersurfaces away from a singular set of Hausdorff codimension 7.

2.2 Tightening process

In this section, we construct the tightening map adapted to the area functional (i.e. pseudo-gradient flow of $\mathbf{M}$ over $\mathcal{V}_n(M)$) and prove that after applying the tightening map to a minimizing sequence, every element in the critical set is stationary.

Given the width defined above and $A_\infty = \{V \in \mathcal{V}_n(M) : ||V||(M) \leq W + 1, V \text{ is stationary}\}$, the existence of a pseudo-gradient flow of $\mathbf{M}$ over $\mathcal{V}_n(M)$ is guaranteed by the following proposition.

Proposition 2.7. There exists a continuous map

$$H : [0,1] \times (\mathcal{C}(M), \mathbf{F}) \cap \{\Omega : \mathbf{M}(\partial\Omega) \leq W+1\} \to (\mathcal{C}(M), \mathbf{F}) \cap \{\Omega : \mathbf{M}(\partial\Omega) \leq W+1\}.$$

such that

1. $H(0,\Omega) = \Omega$;
2. $H(t,\Omega) = \Omega$, if $|\partial\Omega| \in A_\infty$;
3. if $|\partial\Omega| \notin A_\infty$, then

$$\mathbf{M}(\partial H(1,\Omega)) - \mathbf{M}(\partial\Omega) \leq -L(\mathbf{F}(|\partial\Omega|, A_\infty)) < 0,$$

where $L : [0,\infty) \to [0,\infty)$ is a continuous map with $L(0) = 0$ and $L(t) > 0$ if $t > 0$.

Applying the tightening map H in Proposition 2.7 to a minimizing sequence of sweepouts yields that

Proposition 2.8 (Tightening)**.** Given any minimizing sequence of sweepouts $\{\varphi_j^*\}$ on (M^{n+1}, g). Let $\varphi_j(x) = H(1, \varphi_j^*(x)), \forall x \in [0,1]$. Then $\{\varphi_j\}$ is also a minimizing sequence of sweepouts. Moreover, $C(\{\varphi_j\}) \subset C(\{\varphi_j^*\})$ and every $V \in C(\{\varphi_j\})$ is stationary.

Proof. By property 3 and the definition of the width, we have

$$W \leq \max_{x\in[0,1]} \mathbf{M}(\partial\varphi_j(x)) \leq \max_{x\in[0,1]} \mathbf{M}(\partial\varphi_j^*(x)) \to W,$$

which implies that $\{\varphi_j\}$ is also a minimizing sequence of sweepouts.

Given any $V \in C(\{\varphi_j\})$, we know that $V = \lim_{i\to\infty} |\partial\varphi_{j_i}(x_i)|$, where $\mathbf{M}(\partial\varphi_{j_i}(x_i)) \to W$. If we denote $V^* = \lim_{i\to\infty} |\partial\varphi_{j_i}^*(x_i)|$, then

$$||V^*||(M) = \lim_{i\to\infty} |\partial\varphi_{j_i}^*(x_i)|(M) \leq W$$

and by property 3,

$$0 = \lim_{i\to\infty} \mathbf{M}(\partial\varphi_{j_i}(x_i)) - \lim_{i\to\infty} \mathbf{M}(\partial\varphi_{j_i}^*(x_i)) \leq -L(\lim_{i\to\infty} \mathbf{F}(|\partial\varphi_{j_i}^*(x_i)|, A_\infty)).$$

If follows that $\mathbf{F}(V^*, A_\infty) = 0$ and V^* is stationary. Since we have

$$V = \lim_{i\to\infty} |\partial H(1, \varphi_{j_i}^*(x_i))| = |\partial H(1, \lim_{i\to\infty} \varphi_{j_i}^*(x_i))| = |\partial \lim_{i\to\infty} \varphi_{j_i}^*(x_i))| = \lim_{i\to\infty} |\partial\varphi_{j_i}^*(x_i))| = V^*,$$

we conclude that $C(\{\varphi_j\}) \subset C(\{\varphi_j^*\})$. □

Now, we prove Proposition 2.7 by providing an explicit construction of the tightening map H, which involves three major steps.

Proof of Proposition 2.7. Step I: Annular decomposition. Fix $L > 0$ ($L = W+1$) and let

$$A^L = \{V \in \mathcal{V}_n(M) : ||V||(M) \leq L\};$$
$$A_\infty = \{V \in A^L : V \text{ is stationary}\};$$
$$A_j = \{V \in A^L : \frac{1}{2^j} \leq \mathbf{F}(V, A_\infty) \leq \frac{1}{2^{j-1}}\}, \quad j \in \mathbb{N}.$$

One can check that A_∞ and A_j for all $j \in \mathbb{N}$ are compact in A^L under the $\mathbf{F}$-metric.

Lemma 2.9. For each $j \in \mathbb{N}$, there exists $C_j > 0$ such that for each $V \in A_j$, there exists $X_V \in \mathfrak{X}^1(M)$ such that

$$||X_V||_{C^1(M)} \leq 1 \text{ and } \delta V(X_V) \leq -C_j < 0.$$

Proof. Suppose that such C_j does not exist. Then there exists $V_k \in A_j$ such that

$$\sup_{X \in \mathfrak{X}^1(M), ||X||_{C^1(M)} \leq 1} |\delta V_k(X)| \to 0 \quad \text{as } k \to \infty.$$

By compactness of A_j, the subsequence limit $V = \lim_{k\to\infty} V_k$ satisfies $V \in A_j$. However, we have

$$\delta V(X) = \lim_{k\to\infty} \delta V_k(X) = 0, \quad \forall X \in \mathfrak{X}^1(M), ||X||_{C^1(M)} \leq 1,$$

which contradicts with $V \in A_j$. □

Step II: A map from A^L to $\mathfrak{X}^1(M)$. Given $V \in A_j$, let X_V be given in Lemma 2.9. Since the map $(x, S) \mapsto \operatorname{div}_S X_V(x)$ is $C^0(G_n(M))$, we deduce that the map

$$W \mapsto \delta W(X_V) = \int_{G_n(M)} \operatorname{div}_S X_V(x) dW(x, S)$$

is continuous in $\mathbf{F}$. Therefore, $\forall V \in A_j$, $\exists 0 < r_V < 1/2^{j+1}$ such that $\forall W \in U_{r_V}(V) = \{W \in V_n(M) : \mathbf{F}(W, V) < r_V\}$, we have

$$\delta W(X_V) \leq \frac{1}{2}\delta V(X_V) \leq -\frac{1}{2}C_j < 0.$$

Now $\{U_{r_V/2} : V \in A_j\}$ forms an open cover of A_j. By compactness of A_j, there exists finitely many

$$\{U_{r_{j,i}} : V_{j,i} \in A_j, 1 \leq i \leq q_j\}$$

with $r_{j,i} = r_{V_{j,i}}$ such that

- $\bigcup_i U_{r_{j,i}/2}(V_{j,i}) \supset A_j$
- $U_{r_{j,i}}(V_{j,i})$ are disjoint from A_k if $|k - j| \geq 2$.

In the following, we denote $U_{r_{j,i}}(V_{j,i}), U_{r_{j,i}/2}(V_{j,i})$, and $X_{V_{j,i}}$ by $U_{j,i}$, $\widetilde{U}_{j,i}$, and $X_{j,i}$ respectively. By writing $\psi_{j,i}(V) = \mathbf{F}(V, A^L \setminus \widetilde{U}_{j,i})$ and letting

$$\varphi_{j,i}(V) = \frac{\psi_{j,i}(V)}{\sum\{\psi_{p,q}(V) : p \in \mathbb{N}, 1 \leq q \leq q_p\}},$$

we see that $\{\varphi_{j,i} : j \in \mathbb{N}, 1 \leq i \leq q_j\}$ forms a partition of unity subordinate to the covering $\{\widetilde{U}_{j,i}\}$.

The map $X : A^L \to \mathfrak{X}^1(M)$ is defined by

$$X(V) = \mathbf{F}(V, A_\infty) \sum_{j \in \mathbb{N}, 1 \leq i \leq q_j} \varphi_{j,i}(V) X_{j,i}.$$

Lemma 2.10. We have that

1. the map $X : A^L \to \mathfrak{X}^1(M)$ defined above is continuous w.r.t. the C^1-topology on $\mathfrak{X}(M)$;
2. for every $V \in A_j$, let $\rho(V)$ be the smallest radius of the ball $\widetilde{U}_{k,i}$ which contains V. Then we have
$$\delta W(X(V)) \leq -\frac{1}{2^{j-1}} \min\{C_{j-1}, C_j, C_{j+1}\}, \quad \forall W \in U_{\rho(V)}(V).$$

Proof. We only give a proof of 2 here. By construction, these $U_{k,i}$ that contains V must satisfy $|k-j| \le 1$. Hence, we have

$$\delta W(X_{k,i}) \le -\frac{1}{2}\min\{C_{j-1}, C_j, C_{j+1}\}, \quad \forall W \in U_{k,i}.$$

Assume that $W \in U_{\rho(V)}(V)$. Since $V \in \widetilde{U}_{k,i}$ and $\rho(V) = \min\{r_{k,i}/2\}$, we know that $W \in U_{k,i}$. As $\mathbf{F}(V, A_\infty) \le 1/2^{j-1}$, it follows that

$$\begin{aligned}\delta W(X(V)) &= \delta W(\mathbf{F}(V, A_\infty)\sum \varphi_{k,i}(V) X_{k,i}) \\ &\le -\frac{1}{2^{j-1}}\min\{C_{j-1}, C_j, C_{j+1}\}.\end{aligned}$$

□

Step III: A map from A^L to the space of isotopies. Given $V \in A^L$, let $\Phi^V : [0,\infty) \times M \to M$ be the flow of diffeomorphisms generated by $X(V)$, i.e.

$$\begin{cases}\Phi^V(0,p) = p \\ \frac{d}{dt}\Phi^V(t,p) = X(V)(\Phi^V(t,p))\end{cases}.$$

Lemma 2.11. The map $V \in A^L \mapsto \Phi^V(\cdot,\cdot) \in C^1([0,T] \times M, M)$ is continuous in the C^1-topology in $C^1([0,T] \times M, M)$.

Lemma 2.12. Let $\Phi(x,\cdot) \in C^0([0,1], \mathrm{Diff}^1(M))$. Then for every fixed $V \in \mathcal{V}_n(M)$, the map $x \mapsto (\Phi_x)_* V$ is continuous from $[0,1]$ to $\mathcal{V}_n(M)$.

Proof. Recall that

$$(\Phi_* V)(f) = \int_{G_n(M)} f(\Phi(p), d\Phi_p(S))|J\Phi(p,S)| dV(p,S).$$

The continuity of the map $x \mapsto (\Phi_x)_* V$ follows from the continuity of $x \mapsto |J\Phi_x(p,S)|$ in $C^0(G_n(M))$ and the continuity of $x \mapsto f(\Phi_x(p), (d\Phi_x)_p(S))$ in $C^0(G_n(M))$. □

Corollary 2.13. Let $\varphi : [0,1] \to (\mathcal{C}(M), \mathbf{F})$ be a sweepout. Write $\{\Omega_x = \varphi(x)\}$. Then for $t : [0,1] \to [0, T_0]$ a continuous function,

$$\{\widetilde{\Omega}_x = \Phi^{|\partial\Omega_x|}(t(x))(\Omega_x)\}$$

is also a sweepout.

Next, we can find two continuous functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ and

$$\delta W(X(V)) \le g(\mathbf{F}(V, A_\infty)), \quad \text{if } \mathbf{F}(W,V) \le \rho(\mathbf{F}(V, A_\infty))$$

In particular, if $\Omega \in \mathcal{C}(M)$ and $\mathbf{F}(|\partial\Omega|, V) \le \rho(\mathbf{F}(V, A_\infty))$, then

$$\delta|\partial\Omega|(X(V)) \le g(\mathbf{F}(V, A_\infty)).$$

Next, we construct a continuous time function $T : [0,\infty) \to [0,\infty)$ such that

- $\lim_{t\to 0} T(t) = 0$, and $T(t) > 0$ if $t > 0$;
- $\forall V \in A^L$, denote $\gamma = \mathbf{F}(V, A_\infty)$. Then $V_t = (\Phi^V(t))_* V \in U_{\rho(\gamma)}(V), \forall 0 \le t \le T(\gamma)$.

Note that $\forall V \in A_j$, $\rho = \rho(\gamma)$, $\exists T_V > 0$ such that $V_t \in U_\rho(V), \forall 0 \leq t \leq T_V$. By compactness of A_j and the continuity of the map $(t, V) \mapsto V_t$, we may choose T_V such that $T_V \geq T_j > 0$ for all $V \in A_j$, where T_j depends only on j. Using C^∞-cutoff functions, one can interpolate between T_j and obtain $T(t)$ above.

Now, define

$$\Psi^V(t, \cdot) = \Phi^V(T(\gamma)t, \cdot), \quad t \in [0,1].$$

Let $L : \mathbb{R}^+ \to \mathbb{R}^+$ with $L(\gamma) = T(\gamma)g(\gamma)$. Then $L(0) = 0$, and $L(\gamma) > 0$ if $\gamma > 0$. The map $H : [0,1] \times A^L \to A^L$ is defined as

$$H : (t, V) \mapsto V_t = \Omega_t = (\Psi^V(t))_* V \subset U_{\rho(\gamma)}(V) \subset U_{\rho(\gamma)}(|\partial\Omega|)$$

satisfying

1. $H(0, V) = V$;
2. If $V \in A_\infty$, then $X(V) = 0$ and hence $H(t, V) = V$;
3. If $V \notin A_\infty$, then $\gamma = \mathbf{F}(V, A_\infty) > 0$ and

$$||V_1(M)|| - ||V(M)|| = \int_0^{T(\gamma)} (\delta V_t)(X(V))dt.$$

Since $V_t \subset U_{\rho(\gamma)}(V)$, we have $\delta V_t(X(V)) \leq -g(\gamma)$, which implies that

$$||V_1(M)|| - ||V(M)|| \leq -T(\gamma)g(\gamma) = -L(\gamma) < 0.$$

Similarly, we may define $H : [0,1] \times (\mathcal{C}(M), \mathbf{F}) \cap \{\Omega : \mathbf{M}(\partial\Omega) \leq W+1\} \to (\mathcal{C}(M), \mathbf{F}) \cap \{\Omega : \mathbf{M}(\partial\Omega) \leq W+1\}$ by

$$H : (t, \Omega) \mapsto (\Psi^{|\partial\Omega|}(t))(\Omega)$$

satisfying

1. $H(0, \Omega) = \Omega$;
2. If $|\partial\Omega| \in A_\infty$, then $X(|\partial\Omega|) = 0$ and hence $H(t, \Omega) = \Omega$;
3. If $|\partial\Omega| \notin A_\infty$, then $\gamma = \mathbf{F}(|\partial\Omega|, A_\infty) > 0$ and

$$\mathbf{M}(\partial H(1, \Omega)) - \mathbf{M}(\partial\Omega) = \int_0^{T(\gamma)} (\delta|\partial\Omega_t|)(X(|\partial\Omega|))dt.$$

Since $\partial\Omega_t \subset U_{\rho(\gamma)}(\partial\Omega)$, we have $\delta|\partial\Omega_t|(X(|\partial\Omega|)) \leq -g(\gamma)$, which implies that

$$\mathbf{M}(\partial H(1, \Omega)) - \mathbf{M}(\partial\Omega) \leq -T(\gamma)g(\gamma) = -L(\gamma) < 0.$$

This completes the construction of the tightening map H. □

2.3 Almost minimizing

To begin with, we explain why one cannot expect the min-max solution to be locally mass minimizing. Then we introduce a notion of almost minimizing varifolds and present a proof of the existence of such a varifold from min-max construction. Finally, we formulate and solve a natural constrained minimization problem which will be used in the construction of replacements.

Ideally, assume that $\{\Sigma_x^n = \partial\varphi(x)\}_{x\in[0,1]}$ is an optimal sweepout in (M^{n+1}, g), i.e.

$$\max_{x\in[0,1]} \mathbf{M}(\Sigma_x) = W.$$

Suppose by contradiction that Σ_{x_0} is not locally mass minimizing whenever $\mathbf{M}(\Sigma_{x_0}) = W$. Then for every $x_0 \in [0,1]$, if $\mathbf{M}(\Sigma_{x_0}) = W$, there exists an open set $U_{x_0} \subset M$ such that Σ_{x_0} is not mass minimizing in U_{x_0}, i.e. there exists deformations $\Sigma_{x_0} \to \{\Sigma_{x_0,t}\}_{t\in[0,1]}$ such that

$$\Sigma_{x_0,t} \Delta \Sigma_{x_0} \subset\subset U_{x_0} \text{ and } \mathbf{M}(\Sigma_{x_0,1}) < \mathbf{M}(\Sigma_{x_0}) = W.$$

To derive a contradiction, we hope to deform nearby slices $\{\Sigma_x : |x - x_0| \ll 1\}$ parallelly to $\{\widetilde{\Sigma}_x : |x - x_0| \ll 1\}$ such that

$$\max_{x\in[0,1]} \mathbf{M}(\widetilde{\Sigma}_x) < W.$$

The issue is that to maintain $\{\widetilde{\Sigma}_x\}$ as a continuous family of x, we can only deform $\{\Sigma_x\}$ to time 1 for x very close to x_0. Hence, we have to deform $\{\Sigma_x : |x - x_0| \ll 1\}$ in another open set $U_{x_0'}$, and moreover we require that

$$\mathbf{M}(\Sigma_{x,t}) \leq \mathbf{M}(\Sigma_x) + \delta$$

for every $t \in [0,1]$ and $\delta \ll 1$. This sheds light on the following heuristic definition of almost minimizing.

Definition (Heuristic)**.** Whenever $\mathbf{M}(\Sigma_{x_0}) = W$, Σ_{x_0} is almost minimizing in the following sense: given any pair of disjoint open subsets $(U_1, U_2) \subset M$ with

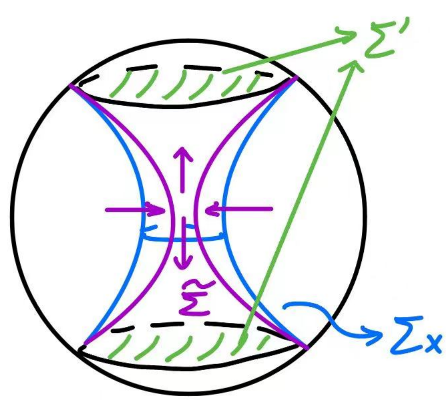
$$\text{dist}(U_1, U_2) > 2\min\{\text{diam}(U_1), \text{diam}(U_2)\},$$

there exists one of them, WLOG say U_1, such that Σ_{x_0} is (ϵ, δ)-almost minimizing in U_1, i.e. for any deformation $\Sigma_{x_0} \to \{\Sigma_{x_0,t}\}_{t\in[0,1]}$, if

1. $\Sigma_{x_0,t} \Delta \Sigma_{x_0} \subset\subset U_1$;
2. $\mathbf{M}(\Sigma_{x_0,t}) \leq \mathbf{M}(\Sigma_{x_0}) + \delta$ for every $t \in [0,1]$ and $\delta \ll 1$,

then we have $\mathbf{M}(\Sigma_{x_0}) \leq \mathbf{M}(\Sigma_{x_0,1}) + \epsilon$.

Example 2.14. In the ball B^3, consider the deformation of a catenoid Σ_x to top and bottom planes Σ'. We have $|\Sigma'| < |\Sigma_x| - \epsilon$. But to deform Σ_x to Σ', one has to pass through $\widetilde{\Sigma}$ with $|\widetilde{\Sigma}| > |\Sigma_x| + \delta$. Hence, this example does not violate the (ϵ, δ)-almost minimizing property.



Theorem 2.15. Assume all the above. Then there exists Σ_x with $\mathbf{M}(\Sigma_x)$ close to W such that Σ_x is almost minimizing in the above sense.

Heuristic Proof. Assume this is not true. Then for any such x, there exists a pair of disjoint open subsets $(U_{x,1}, U_{x,2}) \subset M$ with

$$\operatorname{dist}(U_{x,1}, U_{x,2}) > 2\min\{\operatorname{diam}(U_{x,1}), \operatorname{diam}(U_{x,2})\}$$

such that on each $U_{x,i}$ $(i = 1, 2)$, there exists a deformation $\Sigma_x \to \{\Sigma_{x,i,t}\}_{t\in[0,1]}$ with

1. $\Sigma_{x_i,t}\Delta\Sigma_x \subset\subset U_{x,i}$;
2. $\mathbf{M}(\Sigma_{x,i,t}) \le \mathbf{M}(\Sigma_x) + \delta$ for every $t \in [0, 1]$ and $\delta \ll 1$;
3. but $\mathbf{M}(\Sigma_{x,i,1}) \le \mathbf{M}(\Sigma_x) - \epsilon$.

To derive a contradiction, it is sufficient to patch the deformations (via a covering) to deform $\{\Sigma_x\}$ to $\{\Sigma''_x\}$ with $\max_{x\in[0,1]} \mathbf{M}(\Sigma''_x) < W$. Fix $x_0 \in (0, 1)$ with $\mathbf{M}(\Sigma_{x_0}) \ge W - \epsilon/4$. Choose $U_{x_0,1}$ with deformation $\{\Sigma_{x_0,t}\}_{t\in[0,1]}$. We can deform $\{\Sigma_x\}$ to $\{\Sigma_{x,t(x)}\}$ parallelly, where $t(x) = 1$ for x close to x_0 $(x \in B_{r_0/2}(x_0))$ and $t(x) = 0$ outside a small neighborhood of x_0 $(x \notin B_{r_0}(x_0))$. Then we have

$$\mathbf{M}\left(\Sigma_{x,t(x)}\right) \le \begin{cases} W - \frac{\epsilon}{2}, & \text{if } |x - x_0| \ll \frac{r_0}{2} \\ W + \frac{\epsilon}{4}, & \text{if } \frac{r_0}{2} \le |x - x_0| \le r_0 \end{cases}.$$

For convenience, write $\{\Sigma'_x = \Sigma_{x,t(x)}\}$. Pick $x_1 \in (0, 1)$ with $\mathbf{M}(\Sigma'_{x_1}) \ge W - \epsilon/4$, $r_0/2 \le |x_1 - x_0| \le r_0$. There exists a pair of disjoint open subsets $(U_{x_1,1}, U_{x_1,2}) \subset M$ with

$$\operatorname{dist}(U_{x_1,1}, U_{x_1,2}) > 2\min\{\operatorname{diam}(U_{x_1,1}), \operatorname{diam}(U_{x_1,2})\}$$

such that on each $U_{x_1,i}$ $(i = 1, 2)$, there exists a deformation of Σ'_{x_1}. By requirements, at least one of $\{U_{x_1,1}, U_{x_1,2}\}$ is disjoint from $U_{x,1}$, say $U_{x_1,2}$. As $\Sigma'_{x_1} = \Sigma_{x_1}$ outside $U_{x,1}$, we deform parallelly again to obtain Σ''_x such that

$$\mathbf{M}(\Sigma''_x) \le \mathbf{M}(\Sigma'_x) - \frac{\epsilon}{2} \le \mathbf{M}(\Sigma_x) + \frac{\epsilon}{4} - \frac{\epsilon}{2} \le W - \frac{\epsilon}{2}.$$

Hence, by a 2-step deformation process, we can deform $\{\Sigma_x\}$ to $\{\Sigma''_x\}$ with $\max_{x\in[0,1]} \mathbf{M}(\Sigma''_x) < W$, which gives the desired contradiction. □

Remark. The key part of the proof is "parallel deformations," which depends on the topology. If the slices are C^∞-embedded minimal hypersurfaces, then we deform in the C^∞-topology. If the distance between slices is measured by flat metric, then there are no deformations. In our case, we deform in the $\mathbf{M}$-topology.

Definition. Given $\epsilon, \delta > 0$ and an open set $U \subset M^{n+1}$, define

$$\begin{aligned} \mathcal{A}(U; \epsilon, \delta) := \{\Omega \in \mathcal{C}(M) \text{ such that if } \Omega = \Omega_0, \ldots, \Omega_m \in \mathcal{C}(M) \text{ satisfying} \\ 1.\operatorname{spt}(\Omega_i - \Omega) \subset\subset U; \\ 2.\mathcal{F}(\Omega_i, \Omega_{i+1}) \le \delta; \\ 3.\mathbf{M}(\partial\Omega_i) \le \mathbf{M}(\partial\Omega) + \delta, \\ \text{then } \mathbf{M}(\partial\Omega_m) \ge \mathbf{M}(\partial\Omega) - \epsilon\}. \end{aligned}$$

Definition. Say a varifold $V \in \mathcal{V}_n(M)$ is almost minimizing in U if there exists $\epsilon_i \to 0$, $\delta_i \to 0$, and $\Omega_i \in \mathcal{A}(U; \epsilon_i, \delta_i)$ such that $\mathbf{F}(|\partial\Omega_i|, V) \le \epsilon_i$ for every i.

Definition. A varifold $V \in \mathcal{V}_n(M)$ is almost minimizing in small annuli if $\forall p \in M$, $\exists r_{\text{am}}(p) > 0$ such that V is almost minimizing in $\mathcal{A}_{s,r}(p) = B_r(p) \setminus B_s(p)$ for all $0 < s < r < r_{\text{am}}(p)$.

Theorem 2.16 (Existence of almost minimizing varifold)**.** Let $\{\varphi_i : [0,1] \to \mathcal{C}(M)\}$ be a pull-tight minimizing sequence of sweepouts for (M^{n+1}, g). Then there exists $V \in C(\{\varphi_j\})$ such that

1. V is stationary;
2. V is almost minimizing in small annuli.

Proof. The fact that V is stationary follows from the pull-tight process. Suppose that V is not almost minimizing in small annuli. Then $\forall V \in C(\{\varphi_i\})$, $\exists p_V \in M$ such that $\forall \widetilde{r} > 0$, $\exists r, s > 0$ with $\widetilde{r} > r + 2s > r - 2s > 0$ and $\epsilon, \delta > 0$ such that $\forall \Omega \in \mathcal{C}(M)$, if $\mathbf{F}(|\partial\Omega|, V) < \epsilon$, then $\Omega \notin \mathcal{A}_{r-2s,r+2s}(p_V; \epsilon, \delta)$. Note that we can find

$$U_{V,1} = \mathcal{A}_{r_1-2s_1,r_1+2s_1}(p_V) \ \text{ and } \ U_{V,2} = \mathcal{A}_{r_2-2s_2,r_2+2s_2}(p_V)$$

such that

$$\operatorname{dist}(U_{V,1}, U_{V,2}) > 2\min\{\operatorname{diam}(U_{V,1}), \operatorname{diam}(U_{V,2})\}.$$

Since $V = \lim_{i\to\infty} |\partial\varphi_{j_i}(x_i)|$ with $\mathbf{M}(\partial\varphi_{j_i}(x_i)) \to W$, we know that for i large enough, $\partial\varphi_{j_i}(x_i) \notin \mathcal{A}_{r-2s,r+2s}(p_V; \epsilon, \delta)$. Since there are deformations on $U_{V,1}$ and $U_{V,2}$, one may follow the heuristic proof to patch them together and deform $\{\varphi_{j_i}\}$ to $\{\widetilde{\varphi}_{j_i}\}$ such that

$$\limsup_{i\to\infty} \max_{x\in[0,1]} \mathbf{M}(\partial\widetilde{\varphi}_{j_i}) < W,$$

which gives the desired contradiction. □

Now, we formulate and solve a natural constrained minimization problem which will be used in the construction of replacements.

Lemma 2.17. Given $\epsilon, \delta > 0$, an open set $U \subset M$, and $\Omega \in \mathcal{A}(U; \epsilon, \delta)$, we can do the following: for each $K \subset\subset U$, let

$$\begin{aligned} \mathcal{C}_\Omega := \{\Lambda \in \mathcal{C}(M) \text{ such that } \exists \Omega = \Omega_0, \ldots, \Omega_m = \Lambda \text{ satisfying} \\ 1.\, \operatorname{spt}(\Omega_i - \Omega) \subset K; \\ 2. \mathcal{F}(\Omega_i, \Omega_{i+1}) \le \delta; \\ 3. \mathbf{M}(\partial\Omega_i) \le \mathbf{M}(\partial\Omega) + \delta\}. \end{aligned}$$

Then there exists $\Omega^* \in \mathcal{C}(M)$ such that

(i) $\Omega^* \in \mathcal{C}_\Omega$ and $\mathbf{M}(\partial\Omega^*) = \inf\{\mathbf{M}(\partial\Lambda) : \Lambda \in \mathcal{C}_\Omega\}$;
(ii) Ω^* is locally mass minimizing in $\operatorname{int}(K)$;
(iii) $\Omega^* \in \mathcal{A}(U; \epsilon, \delta)$.

Proof. (i) Take a minimizing sequence $\{\Lambda_j\} \subset \mathcal{C}_\Omega$ with $\lim_{j\to\infty} \mathbf{M}(\partial\Lambda_j) = \inf\{\mathbf{M}(\partial\Lambda) : \Lambda \in \mathcal{C}_\Omega\}$. Since $\mathbf{M}(\partial\Lambda_j) \le \mathbf{M}(\partial\Omega) + \delta$ for all j, by compactness theorem we may assume that $\Lambda_j \to \Omega^*$. Moreover, we have the lower semicontinuity $\mathbf{M}(\partial\Omega^*) \le \inf\{\mathbf{M}(\partial\Lambda) : \Lambda \in \mathcal{C}_\Omega\}$. To check $\Omega^* \in \mathcal{C}_\Omega$, consider the sequence $\Omega = \Omega_0, \ldots, \Omega_m = \Lambda_j, \Omega_{m+1} = \Omega^*$. Observe that $\operatorname{spt}(\Lambda_j - \Omega) \subset K$ implies $\operatorname{spt}(\Omega^* - \Omega) \subset K$, which yields 1. For $j \gg 1$, we have $\mathcal{F}(\Lambda_j, \Omega^*) < \delta$, which yields 2. Since

$$\mathbf{M}(\partial\Omega^*) \le \inf\{\mathbf{M}(\partial\Lambda) : \Lambda \in \mathcal{C}_\Omega\} \le \mathbf{M}(\partial\Omega) + \delta,$$

we conclude that $\Omega^* \in \mathcal{C}_\Omega$ and its boundary is minimizing in the class.

(ii) It is sufficient to show that $\forall p \in \operatorname{spt}(\partial\Omega^*)$, $\exists r_p > 0$ such that $\mathbf{M}(\partial\Omega^*) \leq \mathbf{M}(\partial\Lambda)$, $\forall \Lambda \in \mathcal{C}(M)$ with $\Lambda\Delta\Omega^* \subset\subset B_{r_p}(p)$. Choose $r_p \ll 1$ such that

$$\mathbf{M}(\partial\Omega^* \llcorner B_{r_p}(p)) < \frac{\delta}{2}.$$

Suppose by contradiction that there exists $\Omega' \in \mathcal{C}(M)$ with $\Omega'\Delta\Omega^* \subset\subset B_r(p)$ but

$$\mathbf{M}(\partial\Omega' \llcorner B_r(p)) < \mathbf{M}(\partial\Omega^* \llcorner B_r(p)) < \delta/2.$$

Consider the sequence $\Omega = \Omega_0, \ldots, \Omega_m = \Omega^*, \Omega_{m+1} = \Omega'$, which clearly satisfies 1, 2, and 3. Hence, we have $\Omega' \in \mathcal{C}_\Omega$ with $\mathbf{M}(\partial\Omega') < \mathbf{M}(\partial\Omega^*)$, but this contradicts with (i).

(iii) Suppose by contradiction that there exists $\Omega^* = \Omega_0^*, \ldots, \Omega_m^* \in \mathcal{C}(M)$ such that

1. $\operatorname{spt}(\Omega_i^* - \Omega^*) \subset\subset U$;
2. $\mathcal{F}(\Omega_i^*, \Omega_{i+1}^*) < \delta$;
3. $\mathbf{M}(\partial\Omega_i^*) \leq \mathbf{M}(\Omega^*) + \delta$

but $\mathbf{M}(\partial\Omega_m^*) < \mathbf{M}(\Omega^*) - \epsilon$. Consider the sequence $\Omega = \Omega_0, \ldots, \Omega_m = \Omega^* = \Omega_0^*, \ldots, \Omega_m^*$, which clearly satisfies 1, 2, and 3. Hence, we have $\Omega_m^* \in \mathcal{C}_\Omega$ with

$$\mathbf{M}(\partial\Omega_m^*) \geq \mathbf{M}(\partial\Omega) - \epsilon \geq \mathbf{M}(\partial\Omega^*) - \epsilon,$$

which gives the desired contradiction. □

2.4 Replacements

Proposition 2.18 (Existence and properties of replacements)**.** Let $V \in \mathcal{V}_n(M)$ be almost minimizing in an open set $U \subset M$ and let $K \subset\subset U$ be a compact subset of U. Then there exists $V^* \in \mathcal{V}_n(M)$ called a replacement of V in K such that

1. $V \llcorner G_n(M \setminus K) = V^* \llcorner G_n(M \setminus K)$;
2. $||V||(M) = ||V^*||(M)$;
3. V^* is also almost minimizing in U;
4. $V^* = \lim_{i\to\infty} |\partial\Omega_i^*|$ for some $\Omega_i^* \in \mathcal{A}(U; \epsilon_i, \delta_i)$ with $\epsilon_i, \delta_i \to 0$ and Ω_i^* is locally mass minimizing in $\operatorname{int} K$ for all i.
5. if V is stationary in M, so is V^*.

Proof. By definition of almost minimizing, we may write $V = \lim_{i\to\infty} |\partial\Omega_i|$ for some $\Omega_i \in \mathcal{A}(U; \epsilon_i, \delta_i)$. By Lemma 2.17, for each i there is $\Omega_i^* \in \mathcal{C}_{\Omega_i}$ minimizing in the class such that Ω_i^* is locally mass minimizing in $\operatorname{int} K$ and $\Omega_i^* \in \mathcal{A}(U; \epsilon_i, \delta_i)$. Up to a subsequence, we have $V^* = \lim_{i\to\infty} |\partial\Omega_i^*|$.

Property 1 follows from the fact that $\operatorname{spt}(\Omega_i - \Omega_i^*) \subset K$. To see property 2, we observe that

$$\mathbf{M}(\partial\Omega_i) - \epsilon_i \leq \mathbf{M}(\partial\Omega_i^*) \leq \mathbf{M}(\partial\Omega_i)$$

for each i. Let $i \to \infty$ give property 2. Since $V^* = \lim_{i\to\infty} |\partial\Omega_i^*|$ for some $\Omega_i^* \in \mathcal{A}(U; \epsilon_i, \delta_i)$ with $\epsilon_i, \delta_i \to 0$, we know that V^* is also almost minimizing in U, which proves property 3. By the following lemma, V^* is stationary in U. Since $V^* = V$ in $G_n(M \setminus K)$, V^* is stationary in $M \setminus K$. Let $\varphi \in C_c^\infty(U)$ be a cutoff function with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in a small neighborhood of K. Then for all $X \in \mathfrak{X}_c(M)$, we may write $X = \varphi X + (1 - \varphi X)$ with $\operatorname{spt}(\varphi X) \subset U$ and $\operatorname{spt}((1-\varphi)X) \subset M \setminus K$. It follows that

$$\delta V^*(X) = \delta V^*(\varphi X) + \delta V^*((1-\varphi)X) = 0$$

for all $X \in \mathfrak{X}_c(M)$, i.e. V^* is stationary in M. □

Lemma 2.19. Under the hypotheses above, if V is stationary in U, so is V^*.

Proof. Suppose by contradiction that V^* is not stationary in U. Then there exists $\epsilon_0 > 0$ and $X \in \mathfrak{X}_c(U)$ such that

$$|\delta V(X)| = \left| \int_{G_n(M)} \operatorname{div}_S X(p) dV(p,S) \right| \geq \epsilon_0 \int_M |X| d\mu_V > 0.$$

By changing the sign of X if necessary, we may assume that

$$\int_{G_n(M)} \operatorname{div}_S X(p) dV(p,S) \leq -\epsilon_0 \int_M |X| d\mu_V < 0.$$

By continuity, there exists a constant $\epsilon_1(\epsilon_0, V, X) > 0$ such that for all $\Omega \in \mathcal{C}(M)$ with $\mathbf{F}(|\partial\Omega|, V) < 2\epsilon_1$, we have

$$\delta|\partial\Omega|(X) = \int_{\partial\Omega} \operatorname{div}_{\partial\Omega} X d\mu_{\partial\Omega} \leq \frac{\epsilon_0}{2} \int_M |X| d\mu_V < 0.$$

If $\mathbf{F}(|\partial\Omega|, V) < \epsilon_1$, by deforming Ω along the flow $\{\Phi^X(t) : 0 \leq t \leq \tau\}$ for a uniform $\tau > 0$, we obtain $\{\Omega_t\}$ such that

- the map $t \mapsto \Omega_t$ is continuous in $\mathcal{F}$-topology;
- $\mathbf{M}(\partial\Omega_t) \leq \mathbf{M}(\partial\Omega) - \epsilon_2$ for some constant $\epsilon_2(\epsilon_0, \epsilon_1, V, X) > 0$.

In summary, if we choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $\delta > 0$, then given $\Omega \in \mathcal{C}(M)$ with $\mathbf{F}(|\partial\Omega|, V) < \epsilon$, we have $\Omega \notin \mathcal{A}(U; \epsilon, \delta)$, which gives the desired contradiction. □

Proposition 2.20 (Regularity of replacements)**.** Let $V \in \mathcal{V}_n(M)$ be almost minimizing in an open set $U \subset M$ and let $K \subset\subset U$ be a compact subset of U. Then $V^* \llcorner \operatorname{int} K$ is an integer multiple of a C^∞-embedded minimal hypersurface away from a singular set $\operatorname{Sing}(\Sigma)$ of Hausdorff codimension 7.

Proof. By Proposition 2.18, we have $V^* = \lim_{i\to\infty} |\partial\Omega_i^*|$ for some $\Omega_i^* \in \mathcal{A}(U; \epsilon_i, \delta_i)$ with $\epsilon_i, \delta_i \to 0$ and Ω_i^* is locally mass minimizing in $\operatorname{int} K$ for all i. It follows from Theorem 1.21 that for each i, $\partial\Omega_i^* = \Sigma_i$ is a C^∞-embedded minimal hypersurface in $\operatorname{int} K$ away from a singular set of Hausdorff codimension 7. In particular, each Σ_i is stable. Then the Schoen-Simon-Yau curvature estimates (see Theorem 3.1) guarantees that Σ_i converges to $V^* \llcorner \operatorname{int}(K)$ in the C^∞-topology with a possibly increased multiplicity. □

Theorem 2.21 (Lipschitz regularity of the min-max varifold)**.** Let (M^{n+1}, g) be a Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Assume that $V \in \mathcal{V}_n(M)$ is stationary and almost minimizing in small annuli. Then

1. V is n-rectifiable;
2. $\operatorname{VarTan}(V, p)$ consists of integer multiple of planes in $\mathbb{R}^{n+1}$ and hence V is integer n-rectifiable.

Proof. We claim that $\Theta^n(V, p) > 0, \forall p \in \operatorname{spt} ||V||$. Then applying Theorem 1.8 gives 1. Pick $r_i \to 0$ and let V_i^* be replacements in $\mathcal{A}_{r_i, 2r_i}(p)$. It follows from Proposition 2.20 that $V_i^* \llcorner \mathcal{A}_{r_i,2r_i}(p) = \Sigma_i$ is a C^∞-embedded minimal hypersurface for each i. By the maximum principle, $\Sigma_i \cap \partial B_{3r_i/2}(p) \neq \emptyset$, so we may pick $y_i \in \Sigma_i \cap \partial B_{3r_i/2}(p)$. Then the monotonicity formula implies that

$$||V_i^*||(B_{r_i/2}(y_i)) = |\Sigma_i \llcorner B_{r_i/2}(y_i)| \geq \omega_n \left(\frac{r_i}{2}\right)^n.$$

Based on the estimate above, we conclude that

$$\Theta^n(V,p) \geq \liminf_{r_i \to 0} \frac{||V||(B_{2r_i}(p))}{\omega_n (2r_i)^n} \geq \frac{1}{4^n}.$$

To prove 2, we first apply the lemma below and obtain stable minimal hypersurfaces $\Sigma_i = V_i^* \llcorner \mathcal{A}_{r_i,2r_i}(p)$. Consider the rescalings $\tau_i = \tau_{p,r_i}$ defined by $\tau_i(x) = (x-p)/r_i$. We know that as $i \to \infty$,

- $\tau_i(M) \to T_pM = \mathbb{R}^{n+1}$ smoothly;
- $(\tau_i)_\# V \rightharpoonup W \in \mathrm{VarTan}(V,p)$ up to a subsequence;
- $(\tau_i)_\# V_i^* \rightharpoonup W^* \in \mathcal{V}_n(\mathbb{R}^{n+1})$ up to a subsequence.

By the properties of replacements, we deduce that

- $W = W^*$ in $G_n(\mathbb{R}^{n+1} \setminus \mathcal{A}_{1,2}(0))$;
- $||W||(B_R(0)) = ||W^*||(B_R(0)), \forall R > 2$;
- $W^* \llcorner \mathcal{A}_{1,2}(0)$ is the limit of stable minimal hypersurfaces $\Sigma_i^* = \tau_i(\Sigma_i) \subset \tau_i(\mathcal{A}_{r_i,2r_i}(p)) = \mathcal{A}_{1,2}(0))$.

Moreover, the monotonicity formula implies the uniform area bound

$$|\Sigma_i^*| = \frac{1}{r_i^n}|\Sigma_i| \leq \frac{1}{r_i^n}||V_i^*||(B_{2r_i}(p)) = \frac{1}{r_i^n}||V||(B_{2r_i}(p)) \leq C$$

for some constant $C > 0$. By the Schoen-Simon-Yau and Schoen-Simon curvature estimates (see Theorem 3.1), a subsequence of Σ_i^* converges graphically and smoothly to a C^∞-embedded minimal hypersurface Σ_∞. Since $\Theta^n(V,p) \geq C > 0$, by Corollary 1.9 we know that W is n-rectifiable in $\mathbb{R}^{n+1}$ and is a stationary cone, i.e. $\tau_r(W) = W$. Since V_i^* are almost minimizing in small annuli, the same density lower bound holds and W^* is also n-rectifiable. Moreover, we have

$$\Theta^n(W^*,0) = \frac{||W^*||(B_R(0))}{\omega_n R^n} = \frac{||W||(B_R(0))}{\omega_n R^n} = \Theta^n(W,0), \quad \forall R > 2$$

and W^* is also a stationary cone, i.e. $\tau_r(W^*) = W^*$. As $W = W^*$ outside $\mathcal{A}_{1,2}(0)$, we deduce that $W \equiv W^*$ in $\mathbb{R}^{n+1}$. By Simons' Theorem [22], which says that any smooth minimizing hypercone in $\mathbb{R}^{n+1}$ with $3 \leq (n+1) \leq 7$ is flat, we conclude that W is an integer multiple of planes. □

Lemma 2.22. Let $V \in \mathcal{V}_n(M)$ be almost minimizing in U. Then V is stable in U in the following sense: for all $X \in \mathfrak{X}_c(U)$ and the associated flow $\Phi^X(t)$, we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} ||(\Phi^X(t))_* V||(M) \geq 0.$$

Proof. Recall that

$$||(\Phi_t)_* V||(M) = \int_{G_n(M)} |J\Phi_t|(p,S) dV(p,S),$$

where $|J\Phi_t|(p,S) = \sqrt{\det(((d\Phi_t)_p|_S)^*(d\Phi_t)_p|_S)}$. Hence, the map $t \mapsto ||(\Phi_t)_* V||(M)$ is a smooth function. Now, we proceed as in the proof of Lemma 2.19 and conclude that V is stable in U. □

2.5 Smooth regularity

Theorem 2.23 (Smooth regularity)**.** Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Assume that $V \in \mathcal{V}_n(M)$ is stationary and almost minimizing in small annuli. Then $V = \sqcup_{i=1}^{l} m_i \Sigma_i$, where m_i are all positive integers and $\{\Sigma_1, \ldots, \Sigma_l\}$ is a disjoint collection of closed, C^∞-embedded minimal hypersurfaces.

Before proving the theorem, we shall recall the maximum principle for minimal surfaces. For each $x \in \partial B_s(p)$, if $x = \lim_{i\to\infty} x_i$ with $x_i \in \operatorname{spt}||V|| \cap B_s(p)$, then x is a limit point of $\operatorname{spt}||V|| \setminus \overline{B_s(p)}$. Suppose $0 < s < r < r_{\mathrm{am}}(p)$ and let $T = \{x \in \operatorname{spt}||V|| \cap B_s(p) : T_x V \pitchfork T_x \partial B_s(p)\}$. By Theorem 1.14, the set T is dense in $\operatorname{spt}||V|| \cap B_s(p)$ and hence $x \in \overline{T}$. On the other hand, we have

$$\overline{T} \cap \partial B_s(p) = \overline{\operatorname{spt}||V^*|| \cap B_s(p)} \cap \partial B_s(p) \subset \overline{\operatorname{spt}||V^*|| \setminus \overline{B_s(p)}} \cap \partial B_s(p).$$

Since $\operatorname{spt}||V^*|| = \Sigma$ on $\mathcal{A}_{s,r}(p)$, we conclude that $x \in \overline{\Sigma}$.

As a result, if Σ can be extended to a C^∞-embedded minimal hypersurface when $s \to 0$, then we can prove that $\operatorname{spt}||V|| \cap \mathcal{A}_{0,s}(p)$ is C^∞-embedded. However, when we decrease s and move inward, Σ might also change. To show that Σ is invariant when $s \to 0$, we use a 2-step replacement argument. By applying the argument infinitely many times, we obtain the smooth regularity in the punctured ball.

Proof of Theorem 2.23. Step I: Constructing successive replacements V^* and V^{**} on two overlapping concentric annuli. Pick $p \in \operatorname{spt}||V||$ and suppose $0 < s < r < r_{\mathrm{am}}(p)$. Let V^* be the replacement of V in $\mathcal{A}_{s,r}(p)$. Then $V^* = \Sigma_1$ is a C^∞-embdedded minimal hypersurface in $\mathcal{A}_{s,r}(p)$. Pick $0 < t_1 < s < t_2 < r$ such that $\partial B_{t_2}(p) \pitchfork \Sigma_1$. Let V^{**} be the replacement of V^* in $\mathcal{A}_{t_1,t_2}(p)$. Then $V^{**} = \Sigma_2$ is a C^∞-embdedded minimal hypersurface in $\mathcal{A}_{t_1,t_2}(p)$. Note that $V^{**} = V^*$ outside $\mathcal{A}_{t_1,t_2}(p)$.

Step II: Gluing the replacements smoothly as immersed hypersurfaces on the overlap. Our goal is to show that $\Sigma_1 = \Sigma_2$ in $\mathcal{A}_{s,t_2}(p)$. Recall that to glue solutions u_1, u_2 of the weak formulation of the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

along a common boundary Γ, we only need

$$\begin{cases} u_1 = u_2 & \text{on } \Gamma \\ \nabla u_1 = \nabla u_2 & \text{on } \Gamma \end{cases}.$$

In our case, it is sufficient to check that (i) Σ_2 glues to Σ_1 in C^0, i.e. $\overline{\Sigma}_2 \cap \partial B_{t_2}(p) = \Sigma_1 \cap \partial B_{t_2}(p)$; (ii) Σ_2 glues to Σ_1 in C^1.

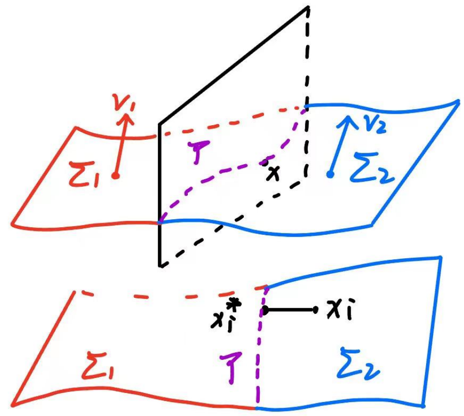
Consider (i) first. By the maximum principle, we have

$$\begin{aligned} \overline{\Sigma}_2 \cap B_{t_2}(p) &= \overline{\operatorname{spt}||V^{**}|| \cap B_{t_2(p)}} \cap \partial B_{t_2}(p) \\ &\subseteq (\operatorname{spt}||V^{**}|| \setminus B_{t_2}(p)) \cap \partial B_{t_2}(p) \\ &= (\Sigma_1 \setminus B_{t_2}(p)) \cap \partial B_{t_2}(p) \\ &= \Sigma_1 \cap \partial B_{t_2}(p) \end{aligned}$$

Conversely, fix $x \in \Sigma_1 \cap B_{t_2}(p)$. We know that $\forall C \in \operatorname{VarTan}(V^*, x)$, $C = T_x\Sigma_1$ with $T_x\Sigma_1 \pitchfork T_x B_{t_2}(p)$. Based on the Lipschitz regularity of the min-max varifold and the fact that

$$V^{**} = \begin{cases} \Sigma_1 & \text{outside } \overline{B_{t_2}(p)} \\ \Sigma_2 & \text{inside } B_{t_2}(p) \end{cases},$$

we know that $\forall C' \in \mathrm{VarTan}(V^{**}, x)$, $C' = T_x\Sigma_1$ with $T_x\Sigma_1 \pitchfork T_xB_{t_2}(p)$. By the maximum principle, we obtain $x \in \overline{\operatorname{spt}||V^{**}||} \cap \partial B_{t_2}(p)$ and $\Sigma_1 \cap B_{t_2}(p) \subseteq \overline{\Sigma}_2 \cap B_{t_2}(p)$.



For (ii), let x_i^* be the projection of x_i onto $\Gamma = \overline{\Sigma_2} \cap \partial B_{t_2}(p)$ inside Σ_2. Let $r_i = \mathrm{dist}_M(x_i, x_i^*) = \mathrm{dist}(x_i, \Gamma)$. Write $\tau_{x_i^*, r_i} : y \mapsto (y - x_i^*)/r_i$ and consider the blow-up limit $W = \lim_{i\to\infty}(\tau_{x_i^*, r_i})_\# V^{**}$ up to a subsequence. We claim that

- W is stationary;
- $\Theta^n(W, \cdot) > 0$;
- W is n-rectifiable;
- $W = \lim_{i\to\infty} \tau_{x_i^*, r_i}(\Sigma_1)$ in an half space and hence is equal to $T_x\Sigma_1$ in an half space.

To begin with, we check that

$$\frac{||W||(B_r(0))}{\omega_n r^n} = \Theta^n(W, 0) = l = \Theta^n(V^{**}, x), \quad \forall r > 0.$$

By the monotonicity formula, we have

$$\begin{aligned} \frac{||(\tau_{x_i^*, r_i})_\# V^{**}||(B_r(0))}{\omega_n r^n} &= \frac{||V^{**}||(B_{r_i r}(x_i^*))}{\omega_n (r_i r)^n} \\ &\leq e^{C\epsilon} \frac{||V^{**}||(B_\epsilon(x_i^*))}{\omega_n \epsilon^n} \\ &\leq e^{C\epsilon} \frac{||V^{**}||(B_{\epsilon + d(x, x_i^*)}(x))}{\omega_n \epsilon^n} \\ &\leq e^{C\epsilon} \left(1 + \frac{d(x, x_i^*)}{\epsilon}\right)^n \frac{||V^{**}||(B_{\epsilon + d(x, x_i^*)}(x))}{\omega_n (\epsilon + d(x, x_i^*))^n} \\ &\leq e^{C\epsilon} \left(1 + \frac{d(x, x_i^*)}{\epsilon}\right)^n (l + \delta). \end{aligned}$$

Given $\delta > 0$, there exists $\epsilon(\delta, x) > 0$ such that the last inequality holds true. As $\delta \to 0$ and $i \to \infty$, we obtain the desired inequality. The reverse inequality also follows from the monotonicity formula:

$$\frac{||(\tau_{x_i^*, r_i})_\# V^{**}||(B_r(0))}{\omega_n r^n} = \frac{||V^{**}||(B_{r_i r}(x_i^*))}{\omega_n (r_i r)^n} \geq l.$$

Hence, W is n-rectifiable and is a stationary cone. Since $W = l \cdot T_x\Sigma_1$ in an half space, we know from the half space theorem for minimal surfaces that $W = l \cdot T_x\Sigma_1$.

We proceed to show that $\forall x_i \in \Sigma_2$ with $x_i \to x \in \Sigma_1 \cap \partial B_{t_2}(p)$, we have $\lim_{i\to\infty} v^{\Sigma_2}(x_i) \to v^{\Sigma_1}(x)$, where $v^{\Sigma_i}(\cdot)$ is the unit normal vector field. Note that $\Sigma_2 \cap B_{r_i}(x_i)$ is a stable, C^∞-embedded minimal hypersurface in M. By the Schoen-Simon-Yau and Schoen-Simon curvature estimates (see Theorem 3.1), a subsequence of the blow-ups $\tau_{x_i^*,r_i}(\Sigma_2 \cap B_{r_i}(x_i))$ converges smoothly to a C^∞-embedded minimal hypersurface Σ_∞. Since $\Sigma_\infty = l \cdot T_x\Sigma_1$ in an half space, we know from the half space theorem for minimal surfaces that $\Sigma_\infty = l \cdot T_x\Sigma_1$. It follows that $v^{\Sigma_2}(x_i) \to v^{\Sigma_1}(x)$.

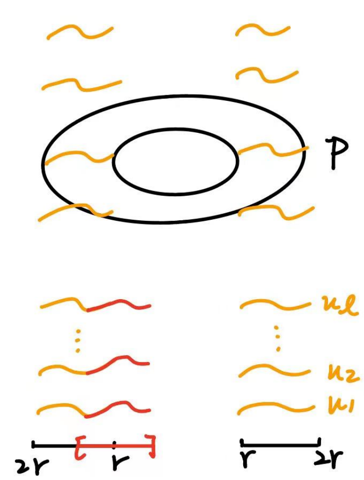
Step III: Extending the replacements down to the point p to get a C^∞-embedded minimal hypersurface Σ in the punctured ball. As $t_1 \to 0$, $\Sigma_1 \cup \Sigma_2$ extends by unique continuation to a C^∞-embedded minimal hypersurface Σ^* in $\mathcal{A}_{0,r}(p)$. For every replacement V^{**} in $\mathcal{A}_{t_1,t_2}(p)$, if $x \in \operatorname{spt}||V|| \cap \partial B_{t_1}(p)$ with $T_xV \pitchfork T_x\partial B_{t_1}(p)$, then we may apply the maximum principle and obtain $x \in \Sigma^*$. By Theorem 1.14, we have $\operatorname{spt}||V|| \cap \mathcal{A}_{0,s}(p) \subset \Sigma^*$. It follows from the constancy theorem that $\operatorname{spt}||V|| \cap \mathcal{A}_{0,s}(p)$ is an integer multiple of $\Sigma^* \cap \mathcal{A}_{0,s}(p)$. Moreover, one can check that $\Sigma^* \cap \mathcal{A}_{0,s}(p)$ is stable.

Step IV: Showing that the singularity of $\Sigma = \Sigma^* \cap \mathcal{A}_{0,s}(p)$ at p is removable. That is, we need to verify the following proposition.

Proposition 2.24 (Removable singularity)**.** Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \le (n+1) \le 7$. Assume that $\Sigma \subset \mathcal{A}_{0,s}(p)$ is a 2-sided, stable, C^∞-embedded minimal hypersurface with $\operatorname{Area}(\Sigma) \le C$. Then Σ extends to be a C^∞-embedded minimal hypersurface in $B_s(p)$.

Proof. To begin with, we show that $\forall r_i \to 0$, the blow-ups $\tau_{r_i}(\Sigma \cap \mathcal{A}_{r_i,2r_i}(p))$ converges weakly to an integer multiple of P, where $P \subset T_pM^{n+1}$ is a n-plane. As in the proof of Theorem 2.21, the monotonicity formula gives a uniform area bound, which together with the 2-sided stability of $\tau_{r_i}(\Sigma\cap\mathcal{A}_{r_i,2r_i}(p))$ implies that a subsequence of $\tau_{r_i}(\Sigma\cap\mathcal{A}_{r_i,2r_i}(p))$ converges graphically and smoothly to a 2-sided, stable, C^∞-embedded minimal hypersurface Σ_∞ of multiplicity m. Since $m \cdot \Sigma_\infty$ is a smooth minimizing hypercone in $\mathbb{R}^{n+1}$, we know from Simons' Theorem [22] that Σ_∞ is a plane.

A major concern is that the tangent cone we obtained depends on the choice of blow-up sequences. That is, given $r_i \to 0$ and $r_i' \to 0$, we might have $\tau_{r_i}(\Sigma\cap\mathcal{A}_{r_i,2r_i}(p)) \to m\cdot P$ and $\tau_{r_i'}(\Sigma\cap\mathcal{A}_{r_i',2r_i'}(p)) \to m \cdot P'$ with $P \neq P'$. But based on the previous arguments, we have the following corollary.



Corollary 2.25. There exists $r_0 << 1$ such that $\forall r < r_0$, $\exists P_r^n$ such that

$$\Sigma \cap \mathcal{A}_{r,2r}(p) = \sqcup_{i=1}^{l} m_i \cdot \operatorname{Graph} u_i,$$

where Graph u_i denotes the graph of u_i over P_r^n with $u_1 < \cdots < u_l$ and $\sum_{i=1}^l m_i = m$.

Note that $\Sigma \cap \mathcal{A}_{r,2r}(p) = \sqcup_{i=1}^l m_i \cdot \Sigma_i(r), \forall 0 < r < r_0$. Using the corollary, we may extend each $\Sigma_i(r)$ to a connected Σ_i in $\mathcal{A}_{0,r_0/2}(p)$. We claim that

$$\frac{\text{Area}(\Sigma_i \cap B_r(0))}{\omega_n r^n} \to 1 \quad \text{as } r \to 0.$$

Once we show the claim, $\tau_{p,r_j}(\Sigma_i)$ converges weakly to a plane of multiplicity 1. Since Σ_i converges to a plane of multiplicity at least 1, we have

$$\frac{\text{Area}(\Sigma_i \cap B_r(0))}{\omega_n r^n} \geq 1.$$

Conversely, since

$$\sum_{i=1}^l m_i = m = \lim_{r\to 0} \frac{\text{Area}(\Sigma \cap B_r(0))}{\omega_n r^n} = \lim_{r\to 0} \sum_{i=1}^l m_i \frac{\text{Area}(\Sigma_i \cap B_r(0))}{\omega_n r^n},$$

we have

$$\frac{\text{Area}(\Sigma_i \cap B_r(0))}{\omega_n r^n} \leq 1,$$

which completes the proof of the claim.

We proceed to show that Σ_i extends across $\{p\}$ to a C^∞-embedded minimal hypersurface. A key ingredient is the following theorem, which forms a pillar of the theory of minimal surfaces. A proof can be found in [21] or [9].

Theorem 2.26 (Allard Regularity Theorem [1])**.** Let $V \in \mathcal{V}_n(B_r^{n+1}(p))$ be stationary in $B_r(p)$ with $\Theta^n(||V||, x) \geq 1, \forall$ a.e. $x \in \text{spt}\,||V||$. Moreover, assume that there exists $\epsilon > 0$ such that

$$\frac{||V||(B_r(p))}{\omega_n r^n} \leq 1 + \epsilon.$$

Then $V \llcorner B_{r/2}(p)$ is a graph of $C^{1,\alpha}$ functions over some plane P.

By Allard Regularity Theorem, Σ_i extends to be a C^∞-embedded minimal hypersurface in $B_s(p)$. By the maximum principle, we have $\Sigma_1 = \cdots = \Sigma_l$. This completes the whole proof. □

□

3 Weyl Law for the volume spectrum and Yau's Conjecture

3.1 Convergence of minimal hypersurfaces

Definition. Let (M^{n+1}, g) be a closed Riemannian manifold and let $U \subset M$ be an open set. A sequence $\{\Sigma_i\}$ of C^∞-embedded minimal hypersurfaces in U with $\partial\Sigma_i \cap U = \emptyset$ is said to converge to a C^∞-embedded Σ_∞ in U if

- $\forall p \in \Sigma_\infty$, $\exists$ a neighborhood $B \subset U$ of p such that $\Sigma_i \cap B$ is a multi-sheeted graphs of $u_{i_1}, \ldots, u_{i_{l_i}}$ over $\Sigma_\infty \cap B$ and $u_{i_j} \to 0$ smoothly as $j \to \infty$.

Prior to stating Sharp's Compactness Theorem, we shall recall the definitions of second variation, stability, and Morse index of minimal hypersurface as well as review the Schoen-Simon-Yau and Schoen-Simon curvature estimates.

For an embedded minimal hypersurface $\Sigma \hookrightarrow U$, the second variation is defined for all $X \in \mathcal{X}_c(U)$ and the associated flows Φ_t^X of X:

$$\delta^2\Sigma(X,X) = \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Area}(\Phi_t^X(\Sigma)).$$

This is a quadratic form on $\Gamma(TM|_\Sigma)$.

Definition. Say that Σ is stable if $\delta^2\Sigma(X,X) \geq 0, \forall X \in \Gamma_c(TM|_\Sigma)$.

Definition. The Morse index of Σ is the maximum number a set of linearly independent vector fields in $\Gamma_c(TM|_\Sigma)$ along which $\delta^2\Sigma$ is negatively definite.

Remark. In the case Σ is 2-side (there exists a normal vector field ν), $X = \varphi\nu$ with $\varphi \in C_c^\infty(\Sigma)$, the second variation formula becomes:

$$\delta^2\Sigma(\varphi\nu,\varphi\nu) = \int_\Sigma \left[|\nabla_\Sigma\varphi|^2 - (|A^\Sigma|^2 + \mathrm{Ric}^M(\nu,\nu))\varphi^2\right] d\mu_\Sigma = -\int_\Sigma \varphi L_\Sigma \varphi d\mu_\Sigma,$$

where $L_\Sigma\varphi := -\Delta_\Sigma\varphi - (|A^\Sigma|^2 + \mathrm{Ric}^M(\nu,\nu)))\varphi$ is the stability operator.

By the classical spectral theory of linear elliptic operators, there exists a discrete spectrum $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ with $L_\Sigma\varphi_i = \lambda_i\varphi_i$ for each i. Note that λ_1 is simple and φ_1 cannot change sign.

Remark. Σ is stable iff $\lambda_1 \geq 0$. The Morse index is given by

$$\mathrm{index}(\Sigma) = \#\{\lambda_i : \lambda_i < 0\}$$

while the nullity is given by

$$\mathrm{nul}(\Sigma) = \#\{\lambda_i : \lambda_i = 0\}.$$

Theorem 3.1. (Schoen-Simon-Yau [19], Schoen-Simon [18]) Assume $3 \leq (n+1) \leq 7$. Let Σ^n be a C^∞-embedded minimal hypersurface in an open set $U \subset M^{n+1}$ with $\partial\Sigma \cap U = \emptyset$. If

- Σ is 2-sided and stable,
- $\mathcal{H}^n(\Sigma) \leq C$,

then

$$\sup_{x \in \Sigma\cap U} |A^\Sigma|^2(x)\,\mathrm{dist}^2(x,\partial U) \leq C_1,$$

where $C_1 = C_1(C, M)$ is a constant.

Corollary 3.2. With all conditions above, let $\{\Sigma_i\}$ be a sequence of embedded minimal hypersurfaces satisfying

- Σ_i is 2-sided and stable for each i,
- $\mathcal{H}^n(\Sigma_i) \leq C$ for some uniform constant C,

then a subsequence of $\{\Sigma_i\}$ converges smoothly to a 2-sided stable embedded limit Σ_∞ possibly with integer multiplicity.

Now, we are ready to state the compactness theorem for minimal hypersurfaces with bounded index. We no longer require Σ_i to be stable but instead impose a uniform bound on Morse index.

Theorem 3.3 (Sharp [20])**.** Let Σ_i be closed, embedded minimal hypersurfaces in (M^{n+1}, g) with $3 \leq (n+1) \leq 7$. Assume that

- $\operatorname{index}(\Sigma_i) \leq k$ for some uniform constant k,
- $\mathcal{H}^n(\Sigma_i) \leq C$ for some uniform constant C.

Then a subsequence of $\{\Sigma_i\}$ converges smoothly to a closed, embedded limit Σ_∞ possibly with integer multiplicity in the following sense: there exists a set of at most k points $\{P_1, \dots, P_l\}$, $l \leq k$ such that $\forall U \subset\subset M \setminus \{P_1, \dots, P_l\}$, $\Sigma_i \to m\Sigma_\infty$ smoothly.

- Assume Σ_∞ is 2-sided:
  1. When $m = 1$, $\Sigma_i \to \Sigma_\infty$ smoothly globally and there exists a nontrivial $\varphi \in C^\infty(\Sigma_\infty)$ such that $L_{\Sigma_\infty}\varphi = 0$ (a Jacobi field),
  2. When $m > 1$, there exists $\varphi > 0, \varphi \in C^\infty(\Sigma_\infty)$ such that $L_{\Sigma_\infty}\varphi = 0$. This implies $\lambda_1(L_{\Sigma_\infty}) = 0$ and Σ_∞ is weakly stable.
- Assume Σ_∞ is 1-sided:
  1. If all Σ_i are 2-sided, then $m > 1$. The 2-sided connected double cover $\widetilde{\Sigma}_\infty \to \Sigma_\infty$ is weakly stable.
  2. If $m = 1$, then Σ_i are all 1-sided for $i \gg 1$ and $\Sigma_i \to \Sigma_\infty$ smoothly globally and $\widetilde{\Sigma}_\infty \to \Sigma_\infty$ admits a Jacobi field.

The following lemma gives the lower semicontinuity of Morse index under the smooth convergence, which will be used in the proof of Theorem 3.3.

Lemma 3.4. If $\Sigma_i \to m\Sigma_\infty$ smoothly in U with $\partial\Sigma_i \cap U = \emptyset$, then

$$\operatorname{index}(\Sigma_\infty) \leq \liminf_{i\to\infty} \operatorname{index}(\Sigma_i).$$

Proof of Theorem 3.3. We divide the proof into four steps: (i) convergence away from $\{P_1, \dots, P_l\}$; (ii) removable singularity of Σ_∞ across $\{P_{\infty,1}, \dots, P_{\infty,l}\}$; (iii) construction of the Jacobi field φ; (iv) removable singularity of φ when $m > 1$.

For (i), we have the first fact: $\forall \{U_1, \dots, U_{k+1}\}$ disjoint open sets in M, $\exists\ U_j$ such that Σ is stable in U_j, where $\operatorname{index}(\Sigma) \leq k$. To check this, we proceed by contradiction. If Σ is unstable in all U_j, then $\exists X_j \in \mathfrak{X}_c(U_j)$ such that $\delta^2\Sigma(X_j, X_j) < 0$. Since X_j's have pairwise disjoint support, $\{X_1, \dots, X_{k+1}\}$ is a linearly independent set, which contradicts with $\operatorname{index}(\Sigma) \leq k$. The second fact is a direct consequence of the first fact, which says: $\forall r > 0$, $\exists$ at most k points $\{P_1^\Sigma, \dots, P_l^\Sigma\}, l \leq k$ such that Σ is stable in any $B_r(p) \subset M \setminus \bigcup_{j=1}^l B_r(P_j^\Sigma)$.

Now, $\forall r > 0$, $\forall \Sigma_i$, $\exists \{B_r(P_{i,j})\}_{j=1}^{l_i}$ such that Σ_i is stable in any ball $B_r(p) \subset M \setminus \bigcup_{j=1}^{l_i} B_r(P_{i,j}^\Sigma)$. For each $j = 1, \dots, l$, let $\{P_{i,j}\} \to \{P_{\infty,j}\}$. Then Σ_i is stable in any ball $B_r(p) \subset M \setminus \bigcup_{j=1}^l B_{2r}(P_{\infty,j})$ for $i \gg 1$. By Theorem 3.1, a subsequence of $\{\Sigma_i\}$ converges smoothly to Σ_∞ in $M \setminus \bigcup_j B_{2r}(P_{\infty,j})$. Let $r \to 0$, and a further subsequence of $\{\Sigma_i\}$ converges smoothly to Σ_∞ in $M \setminus \{P_{\infty,1}, \dots, P_{\infty,l}\}$. By Lemma 3.4, we conclude that there are at most k such points.

For (ii), we claim that $\forall P_{\infty,j}$, $\exists r \ll 1$ such that $\Sigma_\infty|_{\mathcal{A}_{0,r}(P_{\infty,j})}$ is stable. To check this, we proceed by contradiction. If $\Sigma_\infty|_{\mathcal{A}_{0,r}(P_{\infty,j})}$ is not stable, then we may use cutoff functions to construct an infinitely number of linearly independent $X_\alpha \in \mathfrak{X}_c(M)$ supported in $\mathcal{A}_{0,r}(P_{\infty,j})$ with $\delta^2\Sigma_\infty(X_\alpha, X_\alpha) < 0$. This contradicts with $\operatorname{index}(\Sigma_i) \leq k$ for $i \gg 1$.

When $m = 1$, $\Sigma_i \to \Sigma_\infty$ smoothly in $M \setminus \{P_{\infty,1}, \ldots, P_{\infty,l}\}$. By Allard Compactness Theorem [1], $\Sigma \to \Sigma_\infty$ as varifolds. Moreover, we have

$$\frac{\mathcal{H}^n(\Sigma_i \cap B_r(p))}{\omega_n r^n} \to \frac{\mathcal{H}^n(\Sigma_\infty \cap B_r(p))}{\omega_n r^n} \to 1 \quad \text{as } r \to 0.$$

By Allard Regularity Theorem (see Theorem 2.26), we conclude that the convergence must be smooth and graphical everywhere over M.

For (iii), assume that Σ is a graph over Σ_∞. Then there is a function $u \in C^\infty(\Sigma_\infty)$ such that

$$\Sigma = \{\exp_x(u(x))\nu_{\Sigma_\infty}(x) : x \in \Sigma_\infty\},$$

where ν_{Σ_∞} is the unit normal of Σ_∞ pointing toward Σ. Let $\nu(x,t) = \frac{d}{dt}\exp_x(t\nu_{\Sigma_\infty})$ and let $Z(x,t) = \eta(x)\nu_{\Sigma_\infty}$, where $\eta \in C_c^\infty(\Sigma_\infty)$ is a test function. Since Σ and Σ_∞ are minimal hypersurfaces, we have

$$\begin{cases} \operatorname{div}_\Sigma Z = H_\Sigma = 0 \\ \operatorname{div}_{\Sigma_\infty} Z = H_{\Sigma_\infty} = 0 \end{cases}.$$

Denote the path of smooth hypersurfaces by $\Sigma_t := \{\exp_x(tu(x)\nu_\Sigma(x)) : x \in \Sigma_\infty\}$, $t \in [0,1]$. From the fundamental theorem of Calculus we deduce

$$0 = \operatorname{div}_\Sigma Z - \operatorname{div}_{\Sigma_\infty} Z = \int_0^1 \frac{d}{dt}(\operatorname{div}_{\Sigma_t} Z)dt.$$

Write $X = \partial t = u(x)\nu(x,t)$. A further computation leads to

$$\begin{aligned}
\frac{d}{dt}(\operatorname{div}_{\Sigma_t} Z) &= \frac{d}{dt} g_t^{ij}\langle \nabla_{\partial i} Z, \partial_j\rangle \\
&= -\, g_t^{ik} g_t^{jl} \frac{d}{dt}(g_t)_{kl}\langle \nabla_{\partial i} Z, \partial_j\rangle + g_t^{ij}\langle \nabla_{\partial t}\nabla_{\partial i} Z, \partial_j\rangle + g_t^{ij}\langle \nabla_{\partial i} Z, \nabla_{\partial t}\partial_j\rangle \\
&= -\, 2\langle \nabla_{e_i} X, e_j\rangle\langle \nabla_{e_i} Z, e_j\rangle + g_t^{ij}\langle \nabla_{\partial i}\nabla_X Z - R(X, \partial_i)Z, \partial_j\rangle \\
&\quad + g_t^{ij}\langle \langle \nabla_{e_i} Z, e_k\rangle e_k + \langle \nabla_{e_i} Z, \nu_{\Sigma_t}\rangle \nu_{\Sigma_t}, \langle \nabla_{e_j} X, e_l\rangle e_l + \langle \nabla_{e_j} X, \nu_{\Sigma_t}\rangle \nu_{\Sigma_t}\rangle \\
&= -\, 2\langle \nabla_{e_i} X, e_j\rangle\langle \nabla_{e_i} Z, e_j\rangle + g_t^{ij}\langle \nabla_{\partial i}\nabla_X Z, \partial_j\rangle - \operatorname{Ric}^M(X, Z) \\
&\quad + \langle \nabla_{e_i} X, e_j\rangle\langle \nabla_{e_i} Z, e_j\rangle + \langle \nabla_{e_i} X, \nu_{\Sigma_t}\rangle\langle \nabla_{e_i} Z, \nu_{\Sigma_t}\rangle \\
&= g_t^{ij}\langle \nabla_{\partial i}\nabla_X Z, \partial_j\rangle - \operatorname{Ric}^M(X, Z) - \langle \nabla_{e_i} X, e_j\rangle\langle \nabla_{e_i} Z, e_j\rangle + \langle \nabla_{e_i} X, \nu_{\Sigma_t}\rangle\langle \nabla_{e_i} Z, \nu_{\Sigma_t}\rangle.
\end{aligned}$$

Note that

$$\nabla_X Z = \nabla_{u(x)\nu_t}(\eta(x)\nu_t) = u(x)\eta(x)\nabla_{\nu_t}\nu_t = 0.$$

Since we have

$$\lim_{t\to 0}\frac{d}{dt}(\operatorname{div}_{\Sigma_t} Z) = -\operatorname{Ric}^M(\nu_{\Sigma_\infty}, \nu_{\Sigma_\infty})u\cdot\eta - |A^{\Sigma_\infty}|^2 u\cdot\eta + \nabla_{\Sigma_\infty} u\cdot\nabla_{\Sigma_\infty}\eta,$$

the function u solves the following equation for all $\eta \in C_c^\infty(\Sigma_\infty)$:

$$0 = \int_{\Sigma_\infty} \nabla_{\Sigma_\infty} u\cdot\nabla_{\Sigma_\infty}\eta - (\operatorname{Ric}^M(\nu_{\Sigma_\infty}, \nu_{\Sigma_\infty}) + |A^{\Sigma_\infty}|^2)u\cdot\eta + o(\cdots).$$

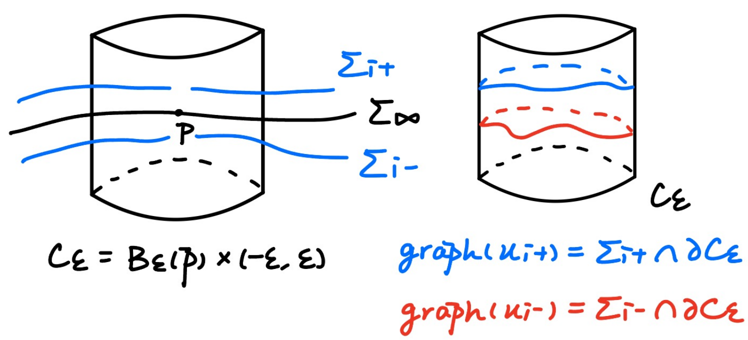
When $m = 1$, we know from Allard Regularity Theorem (see Theorem 2.26) that $\Sigma_i \to \Sigma_\infty$ smoothly. Let $\widetilde{u}_i$ be the height function of Σ_i over Σ_∞. The standard elliptic estimates give a smooth control over the L^2-renormalized function $u_i := \widetilde{u}_i/||\widetilde{u}_i||_{L^2(\Sigma_\infty)}$ and convergence of u_i to a nontrivial solution φ to the Jacobi equation:

$$-\Delta_M\varphi - (|A^{\Sigma_\infty}|^2 + \mathrm{Ric}^M(\nu_{\Sigma_\infty}, \nu_{\Sigma_\infty})) = 0.$$

When $m > 1$, let $\widetilde{u}_i$ be the height function of Σ_{i+} over Σ_{i-}, where Σ_{i+} and Σ_{i-} denote the outmost sheets. For $\Omega \subset\subset \Sigma_\infty$ and some fixed point $y \in \Omega$, we have a Harnack estimate for the renormalized function $u_i := \widetilde{u}_i/\widetilde{u}_i(y)$, which gives an L^∞ estimate. By standard elliptic estimates, we conclude that u_i converges locally and smoothly to a nontrivial solution φ to the Jacobi equation. Moreover, the maximum principle gives $\varphi > 0$ outside the singular set.

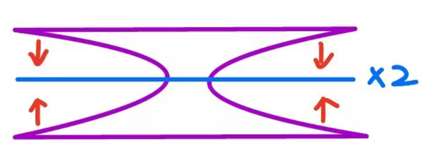
Finally, we prove (iv) when $m > 1$. Let $p \in \Sigma_\infty$ be a singularity and let $B_\epsilon(p)$ be a ball of radius ϵ in Σ_∞ around p. For $\epsilon > 0$, consider the cylindrical neighborhoods $C_\epsilon = B_\epsilon(p) \times (-\epsilon, \epsilon)$ around p. We have the following facts:

1. $\Sigma_i \cap C_\epsilon \to \Sigma_\infty \cap C_\epsilon$ in Hausdorff distance,
2. if $\Sigma_{i\pm} \cap (\partial B_\epsilon(p) \times (-\epsilon, \epsilon)) = \mathrm{Graph}_{\Sigma_\infty}(u_{i\pm})$, then $u_{i+} > u_{i-}$ and $u_{i\pm} \to 0$ smoothly in a neighborhood of $\Sigma_\infty \cap \partial B_\epsilon(p)$,
3. fix $u_{i\pm} : \partial B_\epsilon(p) \to (-\epsilon, \epsilon)$. By the Inverse Function Theorem, $\forall |t| \leq \delta(\epsilon)$, $\exists$ a foliation of minimal hypersurfaces $\Sigma_{i\pm,t}$ in C_ϵ with $\partial\Sigma_{i\pm,t} = \mathrm{Graph}_{\Sigma_\infty}(u_{i\pm} + t)$.



By standard elliptic estimates, $\Sigma_{i\pm,0} \to \Sigma_\infty$ smoothly. Let $\widetilde{u}_i$ be the height function of $\Sigma_{i+,0}$ over $\Sigma_{i-,0}$. Fact 1 and the maximum principle imply that $\Sigma_i \cap C_\epsilon$ should lie within $\Sigma_{i-,0}$ and $\Sigma_{i+,0}$. Hence for some fixed point y in a smaller domain, the normalized function $u_i(x) := \widetilde{u}_i/\widetilde{u}_i(p)$ is uniformly bounded. From this we deduce that φ admits a global bound, yielding full regularity over all of M. By the maximum principle, φ must remain strictly positive on M, which completes the whole proof. □

Remark. When $m > 1$, we cannot expect smooth and graphical convergence over all of M. As an example, consider a sequence of catenoids converging to a plane of multiplicity 2. The Allard Regularity Theorem fails to apply and the convergence is not smooth across the center.



3.2 Volume spectrum and Weyl Law

Given $m \in \mathbb{N}$, I^m denotes the m-dimensional cube $I^m = [0,1]^m$. For each $j \in \mathbb{N}$, $I(1,j)$ denotes the cube complex on I^1 whose 1-cells and 0-cells are, respectively,

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}] \ldots, [1-3^{-j}, 1] \text{ and } [0], [3^{-j}], \ldots, [1-3^{-j}], [1].$$

We denote by $I(m,j)$ the cell complex on I^m:

$$I(m,j) = I(1,j) \otimes \cdots \otimes I(1,j) \quad (m \text{ times}).$$

Then $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ is a q-cell of $I(m,j)$ if and only if α_i is a cell of $I(1,j)$ for each i and $\sum_{i=1}^m \dim(\alpha_i) = q$.

Definition. $X \subset I(m,j)$ is a cube complex if it is a union of cells in $I(m,j)$.

Definition ([14])**.** Let X be a cube complex and let $p \in \mathbb{N}$. A continuous map $\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ is a p-sweepout if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2).$$

This is equivalent to say that there exists $\lambda \in H^1(X; \mathbb{Z}_2)$ such that

- for any loop $\gamma : S^1 \to X$, we have $\lambda(\gamma) \neq 0$ iff $\Phi \circ \gamma : S^1 \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ is homotopically nontrivial.
- $\lambda^p = \lambda \smile \cdots \smile \lambda \neq 0$ in $H^p(X; \mathbb{Z}_2)$.

Remark. 1. If $\dim X < p$, then there are no p-sweepouts.
2. If Φ is a p-sweepout, then Φ is a $(p-1)$-sweepout.

Given A a symmetric $N \times N$ matrix with $Av_n = \lambda_n v_n$ $(\lambda_1 \leq \lambda_2 \leq \cdots)$, the Rayleigh formula gives a min-max characterization of eigenvalues of A:

$$\lambda_k = \inf_{\mathbb{R}^k \hookrightarrow \mathbb{R}^N} \max_{v \in \mathbb{R}^k \setminus \{0\}} \frac{\langle Av, v\rangle}{\langle v, v\rangle} = \inf_{\mathbb{R}P^{k-1} \hookrightarrow \mathbb{R}P^{N-1}} \max_{[v] \in \mathbb{R}P^{k-1}} Q([v]),$$

where $Q(v) = \langle Av, v\rangle / \langle v, v\rangle$ is scaling invariant.

Let (M^{n+1}, g) be a compact Riemannian manifold isometrically embedded in $\mathbb{R}^N$. Given the Laplacian $\Delta : W^{1,2}(M) \to W^{1,2}(M)$, we also have a min-max characterization of eigenvalues of Δ:

$$\lambda_k = \inf_{\mathbb{R}^k \hookrightarrow W^{1,2}(M)} \max_{u \in \mathbb{R}^k \setminus \{0\}} \frac{\int_M |\nabla u|^2}{\int_M u^2} = \inf_{\mathbb{R}P^{k-1} \hookrightarrow \mathbb{R}P^{\infty}} \max_{[u] \in \mathbb{R}P^{k-1}} Q([u]),$$

where $Q(u) = (\int_M |\nabla u|^2)/(\int_M u^2)$ is scaling invariant. In 1911, Weyl proved an asymptotic formula for the sequence of eigenvalues $\{\lambda_p\}_{p \in \mathbb{N}}$ that impacted mathematics profoundly. The celebrated Weyl Law [27] states that

$$\lim_{p \to \infty} \lambda_p \cdot p^{-\frac{2}{n+1}} = a(n) \operatorname{vol}(M)^{-\frac{2}{n+1}},$$

where $a(n) = 4\pi^2 \operatorname{vol}(B)^{-2/(n+1)}$ and B is the unit ball in $\mathbb{R}^{n+1}$.

In this section, we present a summary of Liokumovich-Marques-Neves's proof on the Weyl Law for the volume spectrum [11] that was conjectured by Gromov [6]. We shall start by introducing the p-width.

Definition. Let $p \in \mathbb{N}$. The p-width (p-th volume spectrum) of (M, g) is the number

$$\omega_p(M, g) := \inf_{\Phi \in \mathcal{P}_p} \sup_{x \in \mathrm{dmn}(\Phi)} \mathbf{M}(\Phi(x)),$$

where

$$\mathcal{P}_p = \{\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2) : \Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2)\}$$

and $\mathrm{dmn}(\Phi)$ denotes the domain of Φ.

Remark. Since every p-sweepout is a $(p-1)$-sweepout, we see that $\{\omega_p(M, g)\}_{p \in \mathbb{N}}$ is a monotone increasing sequence.

Now, the Weyl Law for the volume spectrum is formulated as below.

Theorem 3.5 (Weyl Law, Liokumovich-Marques-Neves 16 [11])**.** There exists a constant $a(n) > 0$ such that, for every compact Riemannian manifold (M^{n+1}, g) with (possibly empty) boundary, we have

$$\lim_{p \to \infty} \omega_p(M) p^{-\frac{1}{n+1}} = a(n) \operatorname{Vol}(M)^{\frac{n}{n+1}}.$$

The Weyl Law for the volume spectrum is first proven for Lipschitz domains and then modified to prove for compact Riemannian manifolds. One of the main tools in the proofs is the Lusternik-Schnirelmann inequality, which is stated below.

Lemma 3.6 (Lusternik-Schnirelmann Inequality [6, 7])**.** Let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz domain with $\operatorname{Vol}(\Omega) = 1$. Let $\{\Omega_i^*\}_{i=1}^N$ be disjoint Lipschitz subsets of Ω. For every $p \in \mathbb{N}$, we have

$$\omega_p(\Omega) \geq \sum_{i=1}^N \omega_{p_i}(\Omega_i^*),$$

where $p_i = \lfloor p \operatorname{Vol}(\Omega_i^*) \rfloor$.

Proof. Fix $\epsilon > 0$ and pick $\Phi \in \mathcal{P}_p(\Omega)$. Consider

$$U_i := \{x \in X : \operatorname{Area}(\Phi(x) \cap \Omega_i^*) < \omega_{p_i}(\Omega_i^*) - \frac{\epsilon}{N}\}.$$

The map $\Phi : U_i \to \mathcal{Z}_n(\Omega_i^*; \mathbb{Z}_2)$ defined by restricting currents to Ω_i^*, i.e. $\Phi(x) = \Phi(x) \cap \Omega_i^*$ does not belong to $\mathcal{P}_{p_i}(\Omega_i^*)$. Once we show that $X \setminus \bigcup_{i=1}^N U_i \neq \emptyset$, we may pick $x_0 \in X \setminus \bigcup_{i=1}^N U_i$ and obtain

$$\operatorname{Area}(\Phi(x_0) \cap \Omega_i^*) \geq \omega_{p_i}(\Omega_i^*) - \frac{\epsilon}{N}$$

for every i. It follows that

$$\operatorname{Area}(\Phi(x_0)) \geq \sum_{i=1}^N \operatorname{Area}(\Phi(x_0) \cap \Omega_i^*) \geq \sum_{i=1}^N \omega_{p_i}(\Omega_i^*) - \epsilon.$$

Since ϵ and Φ are arbitrary, we derive the inequality.

To verify that $X \setminus \bigcup_{i=1}^N U_i \neq \emptyset$, we may assume $X = \bigcup_{i=1}^N U_i$ and proceed by contradiction. Let $\iota_i : U_i \hookrightarrow X$ denote the inclusion map. The LES of cohomology for the pair (X, U_i) with $\mathbb{Z}_2$ coefficients is given by

$$\cdots \longrightarrow H^{p_i}(X, U_i; \mathbb{Z}_2) \xrightarrow{j^*} H^{p_i}(X; \mathbb{Z}_2) \xrightarrow{\iota_i^*} H^{p_i}(U_i; \mathbb{Z}_2) \longrightarrow \cdots$$

As $\Phi \notin \mathcal{P}_{p_i}(\Omega_i^*)$, we have $\iota_i^*(\lambda^{p_i}) = 0$, where λ is the generator of $H^1(X;\mathbb{Z}_2)$. By exacteness, $\lambda^{p_i} = j^*\lambda_i$ for some $\lambda_i \in H^{p_i}(X, U_i; \mathbb{Z}_2)$. By considering the relative cup product

$$H^{p_1}(X, U_1; \mathbb{Z}_2) \times \cdots \times H^{p_N}(X, U_N; \mathbb{Z}_2) \xrightarrow{\smile} H^{\bar{p}}(X, \bigcup_{i=1}^N U_i; \mathbb{Z}_2) = 0$$

with $\bar{p} \le p$, we obtain $\lambda_1 \smile \cdots \smile \lambda_N = 0$. This contradicts with

$$j^*(\lambda_1 \smile \cdots \smile \lambda_N) = j^*\lambda_1 \smile \cdots \smile j^*\lambda_N = \lambda^{\bar{p}} \neq 0$$

since every p-sweepout is a $\bar{p}$-sweepout for $\bar{p} \le p$.

□

Corollary 3.7. Let $\widetilde{\omega}_p(\Omega) := \omega_p(\Omega) p^{-\frac{1}{n+1}}$ be the renormalized volume spectrum and let $\Omega_i := \mathrm{Vol}(\Omega_i^*)^{-1/(n+1)}\Omega_i^*$ be a domain similar to Ω_i^*. Then

$$\widetilde{\omega}_p(\Omega) \ge \sum_{i=1}^N \mathrm{Vol}(\Omega_i^*)\widetilde{\omega}_{p_i}(\Omega_i) - \frac{C(n,\Omega)}{pV},$$

where $C(n,\Omega)$ is a positive constant and $V = \min\{|\Omega_1^*|, \ldots, |\Omega_N^*|\}$.

Proof. A direct calculation using the Lusternik-Schnirelmann inequality leads to

$$\begin{aligned}
\widetilde{\omega}_p(\Omega) &= p^{-\frac{1}{n+1}}\omega_p(\Omega) \\
&\ge p^{-\frac{1}{n+1}} \sum_{i=1}^N \omega_{p_i}(\Omega_i^*) \\
&= p^{-\frac{1}{n+1}} \sum_{i=1}^N \mathrm{Vol}(\Omega_i^*)^{\frac{n}{n+1}} \omega_{p_i}(\Omega_i) \\
&= \sum_{i=1}^N \mathrm{Vol}(\Omega_i^*)^{\frac{n}{n+1}} \left(\frac{p_i}{p}\right)^{\frac{1}{n+1}} \widetilde{\omega}_{p_i}(\Omega_i) \\
&\ge \sum_{i=1}^N \mathrm{Vol}(\Omega_i^*)^{\frac{n}{n+1}} \left(\mathrm{Vol}(\Omega_i^*) - \frac{1}{p}\right)^{\frac{1}{n+1}} \widetilde{\omega}_{p_i}(\Omega_i) \\
&= \sum_{i=1}^N \mathrm{Vol}(\Omega_i^*) \left(1 - \frac{1}{p\,\mathrm{Vol}(\Omega_i^*)}\right)^{\frac{1}{n+1}} \widetilde{\omega}_{p_i}(\Omega_i) \\
&\ge \sum_{i=1}^N \mathrm{Vol}(\Omega_i^*)\widetilde{\omega}_{p_i}(\Omega_i) - \frac{C(n,\Omega)}{pV}
\end{aligned}$$

for some positive constant $C(n,\Omega) > 0$. □

Theorem 3.8 (Weyl Law for Cubes)**.** Let $C = [0,1]^{n+1}$. There exists a constant $a(n) > 0$ such that

$$\lim_{p\to\infty} \widetilde{\omega}_p(C) = a(n).$$

Proof. Our goal is to check that

$$\limsup_{p\to\infty} \widetilde{\omega}_p(C) = \liminf_{p\to\infty} \widetilde{\omega}_p(C).$$

Pick $\{p_l\}, \{q_j\} \subset \mathbb{N}$ such that

$$\limsup_{p\to\infty} \widetilde{\omega}_p(C) = \lim_{l\to\infty} \widetilde{\omega}_{p_l}(C) \quad \text{and} \quad \liminf_{p\to\infty} \widetilde{\omega}_p(C) = \lim_{j\to\infty} \widetilde{\omega}_{q_j}(C).$$

Fix p_l and consider $q_j \gg p_l$ with $N_j \sim q_j/p_l \in \mathbb{N}$. By dividing C into a disjoint collection of subcubes $\{C_i^*\}_{i=1}^{N_j}$ of the same volume and applying the Lusternik-Schnirelmann inequality, we obtain that

$$\widetilde{\omega}_{q_j}(C) \geq \sum_{i=1}^{N_j} \mathrm{Vol}(C_i^*)\widetilde{\omega}_{p_l}(C) - \frac{C(n)}{q_j \,\mathrm{Vol}(C_i^*)}.$$

Since $q_j \,\mathrm{Vol}(C_i^*) \sim q_j/N_j \sim p_l$ and $\lim_{j\to\infty} N_j \,\mathrm{Vol}(C_i^*) = 1$, we have

$$\liminf_{p\to\infty} \widetilde{\omega}_p(C) \geq \lim_{j\to\infty} N_j \,\mathrm{Vol}(C_i^*)\widetilde{\omega}_{p_l}(C) - \frac{C(n)}{p_l} = \widetilde{\omega}_{p_l}(C) - \frac{C(n)}{p_l}.$$

As $p_l \to \infty$, we obtain the desired equality. □

Theorem 3.9 (Weyl Law for Domains)**.** For every compact Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$ with $\mathrm{Vol}(\Omega) = 1$, we have

$$\lim_{p\to\infty} \widetilde{\omega}_p(\Omega) = a(n).$$

Proof. It is sufficient to check that

$$\liminf_{p\to\infty} \widetilde{\omega}_p(\Omega) \geq a(n) \quad \text{and} \quad \limsup_{p\to\infty} \widetilde{\omega}_p(\Omega) \leq a(n).$$

For the lower bound, we prove by chopping the domain into cubes and then applying the Lusternik Schnirelmann inequality. For every $\epsilon > 0$, there exists a collection of cubes $\{C_i^*\}_{i=1}^N$ with pairwise disjoint interiors contained in Ω such that $\sum_{j=1}^N \mathrm{Vol}(C_i^*) \geq 1 - \epsilon$. For every $p \gg 1$,

$$\widetilde{\omega}_p(\Omega) \geq \sum_{i=1}^{N} \mathrm{Vol}(C_i^*)\widetilde{\omega}_{\lfloor p\,\mathrm{Vol}(C_i^*)\rfloor}(C) - \frac{C(n,\Omega)}{p \min\{\mathrm{Vol}(C_i^*)\}}.$$

As $p \to \infty$, we have

$$\liminf_{p\to\infty} \widetilde{\omega}_p(\Omega) \geq \left(\sum_{i=1}^{N} \mathrm{Vol}(C_i^*)\right) a(n) = (1-\epsilon)a(n),$$

which gives the desired lower bound as $\epsilon \to 0$.

For the upper bound, we prove by rescaling domains to fill in the cube and then applying the Lusternik Schnirelmann inequality. For every $\epsilon > 0$, there are pairwise disjoint regions $\{\Omega_i^*\}_{i=1}^N$ contained in C such that $\sum_{i=1}^N \mathrm{Vol}(\Omega_i^*) \geq 1 - \epsilon$. Observe that

$$a(n) = \lim_{p\to\infty} \widetilde{\omega}_p(C) \geq \mathrm{Vol}(\Omega_1^*) \limsup_{p\to\infty} \widetilde{\omega}_{p_1}(\Omega) + \sum_{i=2}^{N} \mathrm{Vol}(\Omega_i^*) \liminf_{p\to\infty} \widetilde{\omega}_{p_i}(\Omega).$$

Since $\liminf_{p\to\infty} \widetilde{\omega}_p(\Omega) \geq a(n)$ and $1 - \sum_{i=2}^N \mathrm{Vol}(\Omega_i^*) \leq \mathrm{Vol}(\Omega_1^*) + \epsilon$, we deduce that

$$a(n)(\mathrm{Vol}(\Omega_1^*) + \epsilon) \geq a(n)\left(1 - \sum_{i=2}^{N} \mathrm{Vol}(\Omega_i^*)\right) \geq \mathrm{Vol}(\Omega_1^*) \limsup_{p\to\infty} \widetilde{\omega}_p(\Omega),$$

which gives the desired upper bound as $\epsilon \to 0$. □

Theorem 3.10 (Weyl Law for Compact Manifolds)**.** For every compact Riemannian manifold (M^{n+1}, g) with $\mathrm{Vol}(M) = 1$, we have

$$\lim_{p\to\infty} \widetilde{\omega}_p(M, g) = a(n).$$

Proof. It is sufficient to check that

$$\liminf_{p\to\infty} \widetilde{\omega}_p(M, g) \geq a(n) \quad \text{and} \quad \limsup_{p\to\infty} \widetilde{\omega}_p(M, g) \leq a(n).$$

For the lower bound, note that for every $\epsilon > 0$, there exists $\bar{r} > 0$ such that for all $r \leq \bar{r}$, we have $\mathcal{B}_r(p) \overset{\text{bilip}}{\simeq} B_r(0)$, where $\mathcal{B}_r(p)$ is a ball in $(M, g) \setminus \partial M$ around p and $B_r(0)$ is a ball in $(\mathbb{R}^{n+1}, g_0)$ around the origin. In particular, if $(1+\epsilon)^{-2} g \leq g_0 \leq (1+\epsilon)^2 g$, then

$$(1+\epsilon)^{-(n+1)} \mathrm{Vol}(\mathcal{B}_r(p)) \leq |B_r(0)| \leq (1+\epsilon)^{n+1} \mathrm{Vol}(\mathcal{B}_r(p))$$

and

$$\omega_p(\mathcal{B}_r(p)) \geq (1+\epsilon)^{-n} \omega_p(B_r(0)), \quad \forall p \in \mathbb{N}.$$

Choose a collection of pairwise disjoint geodesic balls $\mathcal{B}_i \subset M \setminus \partial M$ with $r_i \leq \bar{r}$ such that $\sum_{i=1}^N \mathrm{Vol}(\mathcal{B}_i) \geq 1/(1+\epsilon)$. Let B denote a ball in $\mathbb{R}^{n+1}$ of unit volume and let B_i denote an Euclidean ball with the same radius as $\mathcal{B}_i, i = 1, \dots, N$. By the Lusternik-Schnirelmann inequality, we obtain that

$$\begin{aligned}
\widetilde{\omega}_p(M) &= p^{-\frac{1}{n+1}} \omega_p(M) \\
&\geq p^{-\frac{1}{n+1}} \sum_{i=1}^N \omega_{\lfloor p \,\mathrm{Vol}(\mathcal{B}_i) \rfloor}(\mathcal{B}_i) \\
&\geq p^{-\frac{1}{n+1}} \sum_{i=1}^N (1+\epsilon)^{-n} |B_i|^{\frac{n}{n+1}} \omega_{p_i}(B) \\
&= (1+\epsilon)^{-n} \sum_{i=1}^N \left(\frac{p_i}{p|B_i|} \right)^{\frac{1}{n+1}} |B_i| \widetilde{\omega}_{p_i}(B) \\
&= (1+\epsilon)^{-n} \sum_{i=1}^N |B_i| \left(\frac{|\mathrm{Vol}(\mathcal{B}_i)|}{|B_i|} - \frac{1}{p|B_i|} \right)^{\frac{1}{n+1}} \widetilde{\omega}_{p_i}(B) \\
&\geq (1+\epsilon)^{-2n-1} \sum_{i=1}^N \mathrm{Vol}(\mathcal{B}_i) \left(\frac{|\mathrm{Vol}(\mathcal{B}_i)|}{|B_i|} - \frac{1}{p|B_i|} \right)^{\frac{1}{n+1}} \widetilde{\omega}_{p_i}(B).
\end{aligned}$$

As $p \to \infty$, we have

$$\begin{aligned}
\liminf_{p\to\infty} \widetilde{\omega}_p(M) &\geq (1+\epsilon)^{-2n-2} \sum_{i=1}^N \mathrm{Vol}(\mathcal{B}_i) a(n) \\
&\geq (1+\epsilon)^{-2n-3} a(n),
\end{aligned}$$

which gives the desired lower bound as $\epsilon \to 0$.

For the upper bound, the strategy is to first construct a connected region $\Omega \subset \mathbb{R}^{n+1}$ by decomposing M into almost Euclidean regions and adding tiny tubes to connect their bilipschitz images in $\mathbb{R}^{n+1}$. Then given a p-sweepout Φ of Ω, we cook up a p-sweepout Ψ of M whose elements have masses comparable with those of Φ. As $p \to \infty$, the increased mass is negligible compared to $p^{1/(n+1)}$, which gives the desired upper bound.

Let $\mathcal{C} = \{\mathcal{C}_i\}_{i=1}^N$ be a collection of domains such that (i) for all $i = 1, \dots, N$, $\mathcal{C}_i \overset{\text{bilip}}{\simeq} C_i \subset \mathbb{R}^{n+1}$ with bilipschitz constant $(1+\epsilon/2)$; (ii) $\mathcal{C}$ is a covering of M; (iii) $\mathcal{C}_i$'s have mutually disjoint interiors. By connecting the N disjoint regions $C_i \subset \mathbb{R}^{n+1}$ consecutively by tiny tubes, we obtain a conncted Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$ that satisfies

$$|\Omega| \leq (1+\epsilon)^{n+1} \operatorname{Vol}(M) = (1+\epsilon)^{n+1}.$$

Consider $\Phi \in \mathcal{P}_p(\Omega)$ with $X = \operatorname{dmn}(\Phi)$. By restricting the cycles to C_i, we obtain $\Phi_i \in \mathcal{P}_p(C_i)$ with domain X satisfying $\Phi_i^* \bar{\lambda} = \lambda = \Phi^* \bar{\lambda}$ and

$$\mathbf{M}(\Phi_i(x)) \leq (1+\epsilon)^n \mathbf{M}(\Phi(x) \llcorner C_i), \quad \forall x \in X.$$

We shall use the maps $\{\Phi_i\}_{i=1}^N$ to cook up a p-sweepout of M. Since $\Phi_i(x)$ has boundary in ∂C_i, one may choose $Z_i(x) \in \mathbf{I}_{n+1}(C_i; \mathbb{Z}_2)$ such that the cycle $\partial Z_i(x)$ coincides with $\Phi_i(x)$ on the interior of C_i. Note that the choice is not unique and $C_i + Z_i(x)$ is an alternative. Let $\widetilde{Z}_i$ denote the bilipschitz image of Z_i in $\mathcal{C}_i$. Given x, we argue that a choice of Z_1 induces choices of $Z_2, \dots, Z_n$ such that $(\partial \widetilde{Z}_1 + \cdots + \partial \widetilde{Z}_N)(x)$ is a relative cycle of M independent of the choices of Z_1. Then we show that the map Ψ defined by $\Psi(x) = (\partial \widetilde{Z}_1 + \cdots + \partial \widetilde{Z}_N)(x)$ is the desired p-sweepout of M.

For each $i = 1, \dots, N$, set

$$SX_i := \{(x, Z) : x \in X, \Phi_i(x) - \partial Z \in \mathbf{I}_n(\partial C_i; \mathbb{Z}_2)\} \subset X \times \mathbf{I}_{n+1}(C_i; \mathbb{Z}_2).$$

Let $\tau_i : SX_i \to X$ be the projection map and we claim that τ_i is a double covering space for all i ($\tau_i^{-1}(x) = \{(x, Z_x), (x, C_i + Z_x)\}$). The proof is analogous to the verification of $\mathcal{C}(M)$ as the double covering space of $\mathcal{Z}_n(M; \mathbb{Z}_2)$, which is a direct corollary of the constancy theorem. Under the bijective correspondence

$$\{\text{double covering spaces of } X\}/\cong \iff \operatorname{Hom}(\pi_1(X), \mathbb{Z}_2) \cong H^1(X; \mathbb{Z}_2),$$

one can check that the element $\sigma_i \in H^1(X; \mathbb{Z}_2)$ that classifies SX_i is identical to λ for all i. As a result, SX_1 is isomorphic to SX_i for all i and let $F_i : SX_1 \to SX_i$ be the corresponding isomorphism.

For each $i = 1, \dots, N$, by composing the projection map $SX_i \to \mathbf{I}_{n+1}(C_i; \mathbb{Z}_2)$ with the bilipschitz diffeomorphism from C_i to $\mathcal{C}_i$, we form the map $E_i : SX_i \to \mathbf{I}_{n+1}(\mathcal{C}_i; \mathbb{Z}_2)$. Define $\hat{\Psi} : SX_1 \to \mathcal{Z}_{n,\text{rel}}\,(M, \partial M; \mathbb{Z}_2)$ by

$$\hat{\Psi}(y) = \sum_{i=1}^N \partial(E_i \circ F_i(y)).$$

The map is continuous in the flat topology with $\hat{\Psi}(x, C_1 + Z) = \hat{\Psi}(x, Z)$. Hence, $\hat{\Psi}$ descends to a map $\Phi : X \to \mathcal{Z}_{n,\text{rel}}\,(M, \partial M; \mathbb{Z}_2)$ continuous in the flat topology. By lifting a homotopically nontrivial loop $\gamma : S^1 \to X$ upstairs and comparing $\Psi^* \bar{\lambda}(\gamma)$ with $\lambda(\gamma)$, we deduce that $\Psi^* \bar{\lambda} = \lambda$. As $\lambda^p \neq 0$, this shows that Ψ is a p-sweepout of M.

For all $x \in X$, we claim that

$$\mathbf{M}(\Psi(x)) \leq (1+\epsilon)^{2n} \mathbf{\Phi}(\mathbf{x}) + (1+\epsilon)^n \sum_{i=1}^N |\partial C_i|.$$

To see this, we choose $(x, Z) \in SX_1$. Since $\partial Z_i - \Phi_i(x) \in \mathbf{I}_n(\partial C_i; \mathbb{Z}_2)$, we have

$$\mathbf{M}(\partial Z_i) \leq \mathbf{M}(\Phi(x)) + |\partial C_i| \leq (1+\epsilon)^n \mathbf{M}(\Phi(x) \llcorner C_i) + |\partial C_i|.$$

It follows that

$$\mathbf{M}(\Psi(x)) \leq (1+\epsilon)^n \sum_{i=1}^{N} \mathbf{M}(\partial Z) \leq (1+\epsilon)^{2n} \mathbf{M}(\Phi(x)) + (1+\epsilon)^n \sum_{i=1}^{N} |\partial C_i|.$$

Given $\delta > 0$, pick $\Phi \in \mathcal{P}_p(\Omega)$ such that $\sup_{x \in X} \mathbf{M}(\Phi(x)) \leq \omega_p(\Omega) + \delta$. We have the following estimate

$$\begin{aligned}
\omega_p(M) \leq \sup_{x \in X} \mathbf{M}(\Psi(x)) &\leq (1+\epsilon)^{2n} \sup_{x \in X} \mathbf{M}(\Phi(x)) + (1+\epsilon)^n \sum_{i=1}^{N} |\partial C_i| \\
&\leq (1+\epsilon)^{2n} (\omega_p(\Omega) + \delta) + (1+\epsilon)^n \sum_{i=1}^{N} |\partial C_i| \\
&= (1+\epsilon)^{2n} \omega_p(\Omega) + (1+\epsilon)^n \sum_{i=1}^{N} |\partial C_i|, \quad \text{as } \delta \to 0.
\end{aligned}$$

Dividing the estimate above by $p^{1/(n+1)}$ and letting $p \to \infty$, we obtain that

$$\limsup_{p \to \infty} \widetilde{\omega}_p(M) \leq a(n)(1+\epsilon)^{2n} |\Omega|^{\frac{n}{n+1}} \leq a(n)(1+\epsilon)^{3n},$$

which gives the desired upper bound as $\epsilon \to 0$. □

3.3 Positive Ricci curvature case

In the early 80's, Yau formulated a conjecture [31, Problem 88] on the existence of infinitely many closed minimal surfaces in an arbitrary closed 3-manifold. This conjecture has been confirmed by combining works of Marques-Neves [14] and Song [25] as follows.

Theorem 3.11 (Marques-Neves [14], A. Song [25])**.** In any closed Riemannian manifold of dimension at least 3 and at most 7, there exist infinitely many distinct closed, C^∞-embedded minimal hypersurfaces.

In the following sections, we shall present the proofs of Yau's Conjecture in positive Ricci curvature case, generic metric case, and general case. To begin with, consider the positive Ricci curvature case.

Theorem 3.12 (Marques-Neves 13 [14])**.** Let (M^{n+1}, g) be a compact Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. If the Ricci curvature of g is positive, then M contains an infinite number of distinct closed, C^∞-embedded minimal hypersurfaces.

The following theorem is essential in the sense that it links the Almgren-Pitts min-max theory and the definition of the volume spectrum.

Theorem 3.13 (Min-max Theorem associated with p-width)**.** Let (M^{n+1}, g) be a compact Riemannian manifold of dimension $3 \leq (n+1) \leq 7$ and let $p \in \mathbb{N}$. There exists a disjoint collection of closed, C^∞-embedded minimal hypersurfaces $\{\Sigma^n_{p,k}\}$ such that

$$\omega_p(M, g) = \sum_{k=1}^{l_p} m_k^p \operatorname{Area}(\Sigma_{p,k}).$$

Proof. By definition, there exists $\phi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ such that

$$\max_{x \in X_i} \mathbf{M}(\Phi_i(x)) \to \omega_p(M, g).$$

Let $X_i^{(p)}$ denote the p-th skeleton of X_i. We have $\Phi_i \circ i : X_i^p \hookrightarrow X_i \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$. We claim that $\Phi_i|_{X_i^{(p)}}$ is a p-sweepout. By cellular homology, we have $H_p(X_i, X_i^{(p)}) = 0$. Then the universal coefficient theorem implies that

$$H^p(X_i, X_i^{(p)}; \mathbb{Z}_2) \cong \mathrm{Hom}(H_p(X_i, X_i^{(p)}), \mathbb{Z}_2) = 0.$$

The LES of cohomology for the pair $(X_i, X_i^{(p)})$ with $\mathbb{Z}_2$ coefficients is given by

$$\cdots \longrightarrow H^p(X_i, X_i^{(p)}; \mathbb{Z}_2) = 0 \xrightarrow{j^*} H^p(X_i; \mathbb{Z}_2) \xrightarrow{i^*} H^p(X_i^{(p)}; \mathbb{Z}_2) \longrightarrow \cdots$$

By exactness, i^* is injective. It follows that

$$\left(\Phi_i|_{X_i^{(p)}}\right)^* (\bar{\lambda}^p) = i^* \circ \left(\Phi_i|_{X_i^{(p)}}\right)^* (\bar{\lambda}^p) \neq 0.$$

Hence, $\Phi_i : X_i^{(p)} \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ is a p-sweepout.

For each Φ_i, consider

$$\Pi_i = \{\Psi : X_i^{(p)} \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2) : \Psi \text{ is homotopic to } \Phi_i\}$$

with the fixed parameter space $X_i^{(p)}$ and define the min-max value as

$$\mathbf{L}(\Pi_i) = \inf_{\Psi \in \Pi_i} \max_{x \in X_i^{(p)}} \mathbf{M}(\Psi(x)).$$

Since $\mathbf{L}(\Pi_i) > 0$ and $3 \leq (n+1) \leq 7$, the Almgren-Pitts Min-max Theorem guarantees that there exists a disjoint collection of closed C^∞-embedded hypersurfaces $\Sigma_{i,k}^n$ such that

$$\mathbf{L}(\Pi_i) = \sum_{k=1}^{l_i} m_{i,k} \,\mathrm{Area}(\Sigma_{i,k}).$$

Based on the fact that

$$\omega_p(M, g) \leq \mathbf{L}(\Pi_i) \leq \max_{x \in X_i} \mathbf{M}(\Phi_i(x)) \to \omega_p(M, g) \text{ as } i \to \infty,$$

we deduce that $L(\Pi_i) \to \omega_p$ as $i \to \infty$. By a result of Marques-Neves, we have the upper Morse index bound

$$\sum_{k=1}^{l_i} \mathrm{index}(\Sigma_{i,k}) \leq \dim X_i^{(p)} = p.$$

By Sharp's Compactness Theorem, we conclude that

$$\bigcup_{k=1}^{l_i} m_{i,k} \Sigma_{i,k} \to \bigcup_{k=1}^{l_p} m_k^p \Sigma_{p,k} \text{ as } i \to \infty$$

and

$$\omega_p(M, g) = \sum_{k=1}^{l_p} m_k^p \,\mathrm{Area}(\Sigma_{p,k}).$$

□

Theorem 3.14 (Gromov 88 [5], Guth 09 [7], Marques-Neves 13 [14])**.** There exists a positive constant $C = C(M)$ such that

$$\omega_p(M, g) \leq Cp^{\frac{1}{n+1}}.$$

Proof. It is sufficient to show this for $M = I^{n+1} = [0,1]^{n+1}/\sim$, where opposite faces of I^{n+1} are identified. We denote by $I(n+1,k)$ the cell complex on I^{n+1}. To start with, let $v \in \mathbb{R}^{n+1}$ and define a Morse function $f_v : I^{n+1} \to \mathbb{R}$ by $f_v(x) = \langle x, v\rangle$. Let $C(k)$ consist of all centers of $(n+1)$-cells in $I(n+1,k)$. We claim that for almost all $v \in S^n$, the level set $f_v^{-1}(t)$ contains at most one point in $C(k)$. To see this, one first observe that the set $\{x - y : x, y \in C(k)\}$ is finite. Then

$$B = \{v \in S^n : \langle v, x - y\rangle \neq 0, \forall x, y \in C(k)\}$$

is open with full measure in S^n, which proves the claim.

Now, our propose is to apply Guth's bend-and-cancel argument. Note that if a hyperplane P passes through the center, then we cannot radially project it to cells in the n-skeleton $I(n+1,k)_n$ and cancel the mass. Hence, we need to consider two separate cases: (i) $P \cap B_{\epsilon 3^{-k}}(C(k))$; (ii) $P \setminus B_{\epsilon 3^{-k}}(C(k))$.

For case (i), the claim above implies that

$$f(B_{\epsilon 3^{-k}}(x)) \cap f(B_{\epsilon 3^{-k}}(y)) = \emptyset, \quad \forall x \neq y \in C(k).$$

It follows that

$$\text{Area}(f^{-1}(t) \cap B_{\epsilon 3^{-k}}(C(k))) \leq \omega_n \epsilon^n 3^{-nk}, \quad \forall t \in \mathbb{R}.$$

When it comes to case (ii), the following lemma is important.

Lemma 3.15. There exists positive constants $C = C(I(n+1,k))$ and $\epsilon_0 = \epsilon_0(I(n+1,k))$ such that for all $k \in \mathbb{N}$ and $0 < \epsilon \leq \epsilon_0$ we can find a Lipschitz map $F : I(n+1,k) \to I(n+1,k)$ satisfying

- F is homotopic to the identity map.
- $F(I^{n+1} \setminus B_{\epsilon 3^{-k}}(C(k))) \subset I(n+1,k)_n$.
- $|DF| \leq \frac{C}{\epsilon}$.

Define $\Phi_0 : \mathbb{R}P^p \to \mathcal{Z}_n(I(n+1,k); \mathbb{Z}_2)$ by

$$\Phi_0([a_0, a_1, \ldots, a_p]) = \partial\{x : a_0 + a_1 f(x) + \cdots + a_p f(x)^p \leq 0\}.$$

We claim that $\Phi_0 \in \mathcal{P}_p$. Since $\pi_1(\mathbb{R}P^p, 0) \cong \mathbb{Z}_2$, every homotopically nontrivial loop in $\mathbb{R}P^p$ is homotopic to $\gamma : S^1 \to \mathbb{R}P^p$ defined by

$$\gamma(e^{i\theta}) = [\cos(\pi\theta), \sin(\pi\theta), 0, \ldots, 0].$$

Then $\Phi_0 \circ \gamma : S^1 \to \mathcal{Z}_n(I(n+1,k); \mathbb{Z}_2)$ defined by

$$\Phi([\cos(\pi\theta), \sin(\pi\theta), 0, 0, \ldots]) = \partial\{\cos(\pi\theta) + \sin(\pi\theta) f < 0\} = \partial\{f < -\cot(\pi\theta)\}$$

is homotopically nontrivial. Since the generator $\lambda \in H^1(\mathbb{R}P^p; \mathbb{Z}_2)$ satisfies $\lambda(\gamma) = 1$, we have for any loop γ in $\mathbb{R}P^p$, $\lambda(\gamma) \neq 0$ iff $\Phi_0 \circ \gamma$ is homotopically nontrivial. This, together with $\lambda^p \neq 0$, gives $\Phi_0 \in \mathcal{P}_p$. If we let $\theta = [a_0, \ldots, a_p]$, then $\Phi_0(\theta)$ consists of at most p hyperplanes. Define $\Phi = F_\# \circ \Phi_0 : \mathbb{R}P^p \to \mathcal{Z}_n(I(n+1,k); \mathbb{Z}_2)$. Since F is homotopic to the identity map, we have $\Phi \in \mathcal{P}_p$.

Our goal is to bound Area($\Phi(\theta)$) by $Cp^{1/(n+1)}$. Then for each $p \in \mathbb{N}$ there exists a map $\Phi \in \mathcal{P}_p$ and a positive constant $C = C(M)$ such that

$$\omega_p(M, g) \leq \sup_{\theta \in \mathbb{R}P^p} \operatorname{Area}(\Phi(\theta)) \leq Cp^{\frac{1}{n+1}}.$$

As F is a Lipschitz map, we obtain that

$$\begin{aligned}\operatorname{Area}(\Phi|_{I(n+1,k)\setminus I(n+1,k)_n}) &= \operatorname{Area}(F_\# \circ (\Phi_0(\theta) \cap B_{\epsilon 3^{-k}}(C(k)))) \\ &\leq \left(\frac{C}{\epsilon}\right)^n \operatorname{Area}(\Phi_0(\theta) \cap B_{\epsilon 3^{-k}}(C(k))) \\ &\leq \left(\frac{C}{\epsilon}\right)^n p \cdot \omega_n (\epsilon 3^{-k})^n \\ &= Cp \cdot 3^{-kn}.\end{aligned}$$

Since we are using $\mathbb{Z}_2$ coefficients, the multiplicity is at most one and hence $\Phi|_{I(n+1,k)_n}$ contains at most n-dimensional faces in $I(n+1,k)_n$. This leads to the estimate

$$\operatorname{Area}(\Phi(\theta)) \leq (3^k)^{n+1} \cdot (3^{-k})^n + Cp \cdot 3^{-kn} \leq p^{\frac{1}{n+1}} + Cp \cdot p^{-\frac{n}{n+1}} \leq Cp^{\frac{1}{n+1}}$$

if we choose k such that $3^k \leq p^{1/(n+1)} \leq 3^{k+1}$. □

Next, we prove the following theorem by employing a Lusternik-Schnirelmann type argument.

Theorem 3.16. If $\omega_p(M, g) = \omega_{p+1}(M, g)$, then there exists infinitely number of distinct closed, C^∞-embedded, minimal hypersurfaces.

Proof. Suppose that there are only finitely many closed, C^∞ embedded, minimal hypersurfaces $\Sigma_1, \ldots, \Sigma_l$. Assume that there exists a $(p+1)$-dimensional cube complex X and $\Pi = \{\Psi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)\}$ a homotopy class of $(p+1)$-sweepouts such that

$$\omega_{p+1}(M, g) = \inf_{\Phi \in \Pi} \sup_{x \in X} \mathbf{M}(\Phi(x)).$$

Denote

$$\mathcal{S} = \{V \in \mathcal{V}_n(M) : \operatorname{spt} V = \sum_{i=1}^{l} m_i \Sigma_i \text{ with } ||V||(M) \leq \omega_{p+1} + 1\}$$

and

$$\mathcal{T} = \{T \in \mathcal{Z}_n(M; \mathbb{Z}_2) : T = 0 \text{ or } \operatorname{spt} T = \sum_{i=1}^{l} m_i [\Sigma_i] \text{ and } \mathbf{M}(T) \leq \omega_{p+1} + 1\}.$$

By the compactness theorem, one can check that $\forall \epsilon > 0$, $\exists \eta > 0$ such that

$$\mathbf{F}(|T|, \mathcal{S}) < \eta \Longrightarrow \mathcal{F}(T, \mathcal{T}) < \epsilon.$$

In other words, if $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$ is close to $\mathcal{S}$ in $\mathbf{F}$ metric, then T is close to $\mathcal{T}$ in the flat topology.

Lemma 3.17. There exists $\epsilon > 0$ such that

$$\mathcal{B}_\epsilon^{\mathcal{F}}(\mathcal{T}) = \{T \in \mathcal{Z}_n(M; \mathbb{Z}_2) : \mathcal{F}(T, \mathcal{T}) < \epsilon\}$$

has trivial fundamental group, i.e. any $\Phi : S^1 \to \mathcal{B}_\epsilon^{\mathcal{F}}(\mathcal{T})$ is homotopically trivial.

Pick $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ such that

$$\max_{x\in X} \mathbf{M}(\Phi(x)) = \omega_{p+1}(M,g).$$

Write $\lambda = \Phi^*(\bar{\lambda}) \in H^1(X;\mathbb{Z}_2)$ with $\lambda^{p+1} \neq 0$ in $H^{p+1}(X;\mathbb{Z}_2)$. Let $Z \subset X$ be a subspace defined as

$$Z := \{x \in X : \mathbf{F}(\Phi(x), \mathcal{S}) < \eta\}$$

and let $Y := X \setminus Z$. We claim that $\Phi|_Y$ is a p-sweepout. Let $i_1 : Z \hookrightarrow X$ and $i_2 : Y \hookrightarrow X$ denote the inclusion maps. By definition, we have $\Phi(Z) \subset \mathcal{B}^{\mathcal{F}}_{\epsilon}(\mathcal{T})$. Then Lemma 3.17 implies that for all $\mu : S^1 \to Z$, $\Phi \circ \mu(S^1)$ is homotopically trivial. Hence, $i_1^*\lambda = 0$ in $H^1(Z;\mathbb{Z}_2)$. The LES of cohomology for the pair (X,Z) with $\mathbb{Z}_2$ coefficients is given by

$$\cdots \longrightarrow H^1(X,Z;\mathbb{Z}_2) \xrightarrow{j^*} H^1(X;\mathbb{Z}_2) \xrightarrow{i^*} H^1(Z;\mathbb{Z}_2) \longrightarrow \cdots$$

By exactness, $\lambda = j^*\lambda_1$ for some $\lambda_1 \in H^1(X,Z;\mathbb{Z}_2)$. If $i_2^*(\lambda^p) = 0$, then the LES of cohomology for the pair (X,Y) with $\mathbb{Z}_2$ coefficients is given by

$$\cdots \longrightarrow H^p(X,Y;\mathbb{Z}_2) \xrightarrow{j^*} H^p(X;\mathbb{Z}_2) \xrightarrow{i^*} H^p(Y;\mathbb{Z}_2) \longrightarrow \cdots$$

By exactness, $\lambda^p = j^*\lambda_2$ for some $\lambda_2 \in H^p(X,Y;\mathbb{Z}_2)$. By considering the relative cup product

$$H^1(X,Z;\mathbb{Z}_2) \times H^p(X,Y;\mathbb{Z}_2) \xrightarrow{\smile} H^{p+1}(X, Y\cup Z;\mathbb{Z}_2) = 0,$$

we obtain $\lambda_1 \smile \lambda_2 = 0$, which contradicts with

$$j^*(\lambda_1 \smile \lambda_2) = j^*\lambda_1 \smile j^*\lambda_2 = \lambda^{p+1} \neq 0.$$

Hence, $i_2^*(\lambda^p) \neq 0$ and $\Phi|_Y$ is a p-sweepout.

As $\Phi|_Y \in \mathcal{P}_p$, we know that

$$\omega_p(M,g) \leq \max_{x\in Y} \mathbf{M}(\Phi(x)) \leq \omega_{p+1}(M,g) = \omega_p(M,g) \Longrightarrow \max_{x\in Y} \mathbf{M}(\Phi(x)) = \omega_p(M,g).$$

Assume that all varifolds in the critical set $\mathcal{C}(\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2))$ are stationary. Then all varifolds in the critical set $\mathcal{C}(\Phi : Y \to \mathcal{Z}_n(M;\mathbb{Z}_2))$ are stationary. At least one such varifold $V \in \mathcal{C}(\Phi : Y \to \mathcal{Z}_n(M;\mathbb{Z}_2))$ is almost minimizing in small annuli. Hence, $V \in \mathcal{S}$, which contradicts with the definition of Y. □

Now, we are ready to prove Yau's Conjecture in positive Ricci curvature case.

Theorem 3.18 (Marques-Neves 13 [14])**.** Let (M^{n+1}, g) be a compact Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. If the Ricci curvature of g is positive, then M contains an infinite number of distinct closed, C^∞-embedded minimal hypersurfaces.

Proof. By contradiction, suppose that the set $\mathcal{L}$ of all connected, closed, C^∞-embedded minimal hypersurfaces of M is finite. For every $p \geq 1$, we have

$$\omega_p(M) = ||V_p||(M),$$

for some V_p on M, where V_p is the varifold of a closed, C^∞-embedded minimal hypersurface, with possible multiplicities. We may write

$$V_p = n_1^{(p)}\Sigma_1^{(p)} + \cdots + n_{l_p}^{(p)}\Sigma_{l_p}^{(p)}$$

with $\Sigma_1^{(p)}, \dots, \Sigma_{l_p}^{(p)}$ all disjoint. As the Ricci curvature of g is positive, (M, g) satisfies the embedded Frankel property, i.e. any two closed, C^∞-embedded minimal hypersurfaces of M intersect each other. It follows that $l_p = 1$ for every $p \geq 1$.

Since $\mathcal{L}$ is finite, the previous theorem implies that $\{\omega_p\}$ is a strictly increasing sequence. Hence, we have

$$\#\{\omega_k(M) : k = 1, \dots, p\} = p.$$

Let $\delta := \min\{\text{Area}(\Sigma) : \Sigma \in \mathcal{L}\} > 0$. The upper bound for the volume spectrum gives $\omega_p(M) \leq Cp^{1/(n+1)}$, which implies that $n^{(p)} \in \{1, \dots, \lfloor Cp^{1/(n+1)}/\delta \rfloor\}$ and

$$\#\{\omega_k(M) : k = 1, \dots, p\} \leq C' p^{\frac{1}{n+1}}$$

for a constant $C' > 0$ independent of p. As p grows, we obtain a contradiction. □

3.4 Generic metrics case

In this section, we present a sketch of Irie-Marques-Neves's proof on Yau's Conjecture in generic case.

Theorem 3.19 (Irie-Marques-Neves 17 [8])**.** Let M^{n+1} be a closed manifold of dimension $3 \leq (n+1) \leq 7$. Then for a C^∞-generic Riemannian metric g on M, the union of all closed, C^∞-embedded minimal hypersurfaces is dense.

The main ingredients in the proof are the Weyl Law for volume spectrum (see Theorem 3.5) and the min-max theorem associated with p-width (see Theorem 3.13). The structure theory of White is also essential, which says a generic metric is bumpy, meaning that every closed minimal hypersurface is nondegenerate. To prepare for the proof, we shall first introduce the Manifold Structure Theorem of White [28, 29].

Definition. Let Σ, Σ_1 be minimal surfaces in M^m of dimension k. We denote by $N_\delta(\Sigma)$ the δ-neighborhood of Σ and $N(\Sigma)$ the normal bundle of Σ. Say Σ_1 is C^l close to Σ if $\Sigma_1 \subset N_\delta(\Sigma)$ and Σ_1 is the graph of a section $u : \Sigma_1 \to \mathbb{R}^{m-k} \in \Gamma(N(\Sigma))$ with $||u||_{C^l} \ll 1$.

Theorem 3.20 (Manifold Structure Theorem [29])**.** Let M^m be a smooth manifold and let $\Gamma^{(l+2)}$ be an open set of C^{l+2} Riemannian metrics on M. Consider the map

$$\{\text{All } C^l \text{ immersions } \Sigma^k \hookrightarrow M\} \times \Gamma^{(l+2)} \xrightarrow{H} C^{l+2}(M).$$

The set of pairs $\mathcal{M} = H^{-1}(0) = \{(\Sigma^k, g) : \exists i : \Sigma^k \hookrightarrow (M, g) \text{ an } C^l \text{ minimal immersion}\}$ is a C^2 separable Banach manifold. The projection map

$$\begin{aligned} \Pi : \mathcal{M} &\longrightarrow \Gamma^{(l+2)} \\ (\Sigma, g) &\longmapsto g \end{aligned}$$

is a C^2 Fredhom map with Fredholm index 0. Moreover, the kernel of $D\Pi|_{(\Sigma,g)}$ has dimension equal to the kernel of $DH|_{(\Sigma,g)}$, where

$$D\Pi|_{(\Sigma,g)} : T_{(\Sigma,g)}\mathcal{M} \longrightarrow C^{l+2}(\Sigma)$$

is the linear projection and

$$DH|_{(\Sigma,g)} : T_\Sigma\{\text{All } C^l \text{ immersions}\} \times T_g\Gamma^{(l+2)} \longrightarrow C^{l+2}(\Sigma)$$

is the Jacobi operator $L_\Sigma = -\Delta_\Sigma - \text{Ric}_M(\nu, \nu) - |A|^2$.

Theorem 3.21 (Sard-Smale [23]). The regular values of Π are generic in $\Gamma^{(l+2)}$ in the sense of Baire.

A direct corollary of this is the Bumpy Metrics Theorem of White [29, 30].

Theorem 3.22 (Bumpy Metrics Theorem [29]). A generic metric in the sense of Baire is bumpy.

Proof. Since $\mathcal{M}$ is separable and Π is proper, the regular values of Π are generic in $\Gamma^{(l+2)}$ by Theorem 3.21. This proves the theorem for any fixed Σ. Since there are only countably many diffeomorphism types of Σ, we are done. $\square$

Recall the definition of p-width. The following lemma will be used in the proof of Theorem 3.19 to derive a contradiction.

Lemma 3.23. The p-width $\omega_p(M, g)$ depends continuously on the metric g in the C^0-topology.

Proof. Suppose $g_i \to g$ in the C^0-topology. Given $\epsilon > 0$, pick $\Phi \in \mathcal{P}_p(M)$ such that

$$\sup_{x \in X}\{\mathbf{M}_g(\Phi(x))\} \leq \omega_p(M, g) + \epsilon,$$

where $\mathbf{M}_g(T)$ is the mass of T w.r.t. g. Since

$$\begin{aligned}
\omega_p(M, g_i) &\leq \sup_{x \in X}\{\mathbf{M}_{g_i}(\Phi(x))\} \\
&\leq \left(\sup_{v \neq 0} \frac{g_i(v, v)}{g(v, v)}\right)^{\frac{n}{2}} \sup_{x \in X}\{\mathbf{M}_g(\Phi(x))\} \\
&\leq \left(\sup_{v \neq 0} \frac{g_i(v, v)}{g(v, v)}\right)^{\frac{n}{2}} (\omega_p(M, g) + \epsilon),
\end{aligned}$$

we have $\limsup_{i \to \infty} \omega_p(M, g_i) \leq \omega_p(M, g)$ as $\epsilon \to 0$.

Conversely, let $\epsilon_i > 0$ satisfying $\lim_{i \to \infty} \epsilon_i \to 0$. Pick $\Phi_i \in \mathcal{P}_p(M)$ such that

$$\omega_p(M, g_i) \geq \sup_{x \in X_i}\{\mathbf{M}_{g_i}(\Phi(x))\} - \epsilon_i.$$

Since

$$\begin{aligned}
\omega_p(M, g_i) &\geq \sup_{x \in X_i}\{\mathbf{M}_{g_i}(\Phi(x))\} - \epsilon_i \\
&\geq \left(\sup_{v \neq 0} \frac{g(v, v)}{g_i(v, v)}\right)^{\frac{n}{2}} \sup_{x \in X_i}\{\mathbf{M}_g(\Phi(x))\} - \epsilon_i \\
&\leq \left(\sup_{v \neq 0} \frac{g(v, v)}{g_i(v, v)}\right)^{\frac{n}{2}} \omega_p(M, g) - \epsilon_i,
\end{aligned}$$

we have $\liminf_{i \to \infty} \omega_p(M, g_i) \geq \omega_p(M, g)$, which completes the proof. $\square$

Lemma 3.24. Let Σ be a closed, C^∞-embedded minimal hypersurface in (M^{n+1}, g). Then there exists a sequence of metrics g_i on M, $i \in \mathbb{N}$, converging to g in the C^∞-topology such that Σ is a nondegenerate minimal hypersurface in (M^{n+1}, g_i) for every i.

Proof of Theorem 3.19. Let $U \subset M$ be a nonempty open set. Define

$$\begin{aligned}\mathcal{M}_U = \{g : g \text{ is a smooth Riemannian metric such that there exists}\\ \text{a nondegenerate, closed, } C^\infty\text{-embedded minimal}\\ \text{hypersurface } \Sigma \subset (M, g) \text{ satisfying } \Sigma \cap U \neq \emptyset\}.\end{aligned}$$

It is sufficient to prove that $\mathcal{M}_U$ is open and dense in the C^∞-topology.

Let $g \in \mathcal{M}_U$ with some $\Sigma_g \subset (M, g)$ satisfying $\Sigma_g \cap U \neq \emptyset$. Since Σ_g is nondegenerate, the Inverse Function Theorem implies that for every g' close to g in the C^∞-topology, there exists a unique nondegenerate, closed, C^∞-embedded minimal hypersurface $\Sigma_{g'}$ close to Σ_g satisfying $\Sigma_{g'} \cap U \neq \emptyset$. This shows that $\mathcal{M}_U$ is open.

To see that $\mathcal{M}_U$ is dense, consider an arbitrary smooth Riemannian metric g and an arbitrary neighborhood $\mathcal{V}$ of g in the C^∞-topology. By Theorem 3.22, there exists $g' \in \mathcal{V}$ such that all closed, C^∞-immersed minimal hypersurfaces in (M, g') are nondegenerate. If $g' \in \mathcal{M}_U$, then we are done. Otherwise, suppose that all closed, C^∞-embedded minimal hypersurfaces in (M, g') are contained in $M \setminus U$. By Sharp's Compactness Theorem, we deduce that the set

$$\begin{aligned}\mathcal{C} = \{\sum_{j=1}^{N} m_j \operatorname{Vol}_{g'}(\Sigma_j) : N \in \mathbb{N}, \{m_j\}_{j=1}^{\mathbb{N}} \subset \mathbb{N}, \{\Sigma_j\}_{j=1}^{N} \text{ are disjoint, closed,}\\ C^\infty\text{-embedded minimal hypersurfaces in } (M, g')\}.\end{aligned}$$

is countable.

Now, choose $h : M \to \mathbb{R}^{\geq 0}$ a smooth function such that $\operatorname{supp} h \subset U$ and $h(x) > 0$ for some $x \in U$. If we perturb the metric slightly by letting $g'(t) = (1 + th)g'$ for $t \geq 0$, then there exists $t_0 > 0$ such that $g'(t) \in \mathcal{V}$ for $t \in [0, t_0]$ and $\operatorname{Vol}(M, g'(t_0)) > \operatorname{Vol}(M, g')$. Because of the Weyl Law for volume spectrum, it follows that $\omega_p(M, g'(t_0)) > \omega_p(M, g')$ for some $p \in \mathbb{N}$. This, together with the fact that $\mathcal{C}$ is countable and the p-width $\omega_p(M, g'(t))$ is continuous in t, guarantees that there exists a closed, C^∞-embedded minimal hypersurface $\Sigma_{g'(s)} \subset (M, g'(s))$ satisfying $\Sigma_{g'(s)} \cap U \neq \emptyset$, where $s \in [0, t_0]$. By Lemma 3.24, we may perturb $g'(s)$ slightly to g'' such that $g'' \in \mathcal{V} \cap \mathcal{M}_U$, which shows that $\mathcal{M}_U$ is dense. □

3.5 General case

In this section, we present a sketch of Song's proof on Yau's Conjecture in general case where the metric may not be generic.

Theorem 3.25 (A. Song 18 [25])**.** In any closed Riemannian manifold of dimension at least 3 and at most 7, there exist infinitely many distinct closed, C^∞-embedded minimal hypersurfaces.

The proof builds on the following result obtained by Marques and Neves.

Theorem 3.26 (Marques-Neves 13 [14])**.** Let (M^{n+1}, g) be a compact Riemannian manifold of dimension $3 \leq (n + 1) \leq 7$. Suppose that M satisfies the embedded Frankel property (any two closed, C^∞-embedded minimal hypersurfaces of M intersect each other). Then M contains an infinite number of distinct closed, C^∞-embedded minimal hypersurfaces.

In the proof, Song introduced a Weyl Law type formula called the cylindrical Weyl Law and developed the min-max theory on a non-compact manifold with cylindrical ends. This builds on Liokumovich-Marques-Neves's proof on the Weyl Law for volume spectrum and Li-Zhou's work on the free bounary min-max theory. The cylindrical Weyl Law and the min-max theory on a non-compact manifold with cylindrical ends turn out to be crucial for proving other interesting results, such as the generic scarring phenomenon of minimal hypersurfaces along a stable hypersurface (see Song-Zhou's work [26]) or a generalization of the Yau's Conjecture to some classes of complete non-compact manifolds (see Song's work [24]).

In addition, a geometric topology approach was employed in the proof to form a non-compact manifold with cylindrical ends. To this end, one need to first cut M along minimal hypersurfaces which are area minimizing at least on one side to obtain a new manifold U whose boundary, if not empty, has a contracting neighborhood. Then by attaching the cylinders to U along ∂U, one obtains the non-compact manifold with cylindrical ends $\mathcal{C}(U)$ and settles the stage for applying the cylindrical Weyl Law.

To prepare for the proof, we shall first introduce the p-width of a non-compact manifold and the cylindrical Weyl Law.

Definition. Let (N^{n+1}, g) be a complete non-compact manifold. Let $K_1 \subset K_2 \subset \cdots K_i \subset \cdots$ be an exhaustion of N by compact $(n+1)$-submanifolds with smooth boundary. The p-width of (N, g) is the number

$$\omega_p(N, g) = \lim_{i\to\infty} \omega_p(K_i, g) \in [0, \infty].$$

Remark. Since $\omega_p(K_i, g)$ is a nondecreasing sequence of nonnegative numbers, $\omega_p(N, g)$ is well-defined. Moreover, it is independent of the choices of the compact exhaustion $\{K_i\}$.

Let $(\mathcal{C}, h)$ be a complete $(n+1)$-dimensional manifold with cylindrical ends, i.e. outside a compact subset, the manifold is isometric to $\Sigma \times [0, \infty)$ endowed with a product metric $h_1 \oplus dt^2$ (here Σ is a smooth n-dimensional manifold).

Theorem 3.27 (Cylindrical Weyl Law, A. Song 18 [25])**.** Let $(\mathcal{C}, h)$ be an $(n+1)$-dimensional connected non-compact manifold with cylindrical ends as above. Let $\Sigma_1, \dots, \Sigma_l$ be the connected components of Σ and suppose that Σ_1 has the largest n-volume among these components:

$$|\Sigma_1| \geq \max\{|\Sigma_2|, \dots, |\Sigma_l|\}.$$

Then $\omega_p(\mathcal{C}) = \omega_p(\mathcal{C}, h)$ is finite for all p and the following holds:

1. $\omega_1(\mathcal{C}) \geq |\Sigma_1|$ and for all $p \in \{1, 2, \dots\}$,

$$\omega_{p+1}(\mathcal{C}) - \omega_p(\mathcal{C}) \geq |\Sigma_1|;$$

2. there exists a constant $C > 0$ depending on h such that for all $p \in \{1, 2, \dots\}$,

$$\omega_p(\mathcal{C}) \leq p|\Sigma_1| + Cp^{\frac{1}{n+1}}.$$

Proof. To begin with, we check that for all $i \in \{1, \dots, l\}$, $\omega_1(\Sigma_i \times [0, L]) = |\Sigma_i|$ for L large enough. Since the hypersurfaces $\{\Sigma_i \times \{r\}\}_{r\in[0,L]}$ give an explicit sweepout in $\mathcal{P}_1$, we have the upper bound

$$\omega_1(\Sigma_i \times [0, L]) \leq |\Sigma_i|.$$

For the lower bound, by applying the free boundary min-max theory we obtain a varifold V with $\operatorname{spt} V$ a smooth, almost properly embedded free boundary minimal hypersurface. By the maximum principle and the monotonicity formula, for L large enough we have the lower bound

$$\omega_1(\Sigma_i \times [0, L]) \geq |\Sigma_i|.$$

Next, consider the 1-width of $B_L = (\Sigma_1 \sqcup \cdots \sqcup \Sigma_l) \times [0, L]$. Applying the argument above to each component yields that

$$\omega_1(B_L) = \max\{|\Sigma_1|, \ldots, |\Sigma_l|\} = |\Sigma_1|.$$

It follows immediately that $\omega_1(\mathcal{C}) \geq |\Sigma_1|$. We show the rest of property 1 by using a Lusternik-Schnirelmann type argument. Given $\epsilon > 0$, fix $x_0 \in \mathcal{C}$ and choose R_p large enough such that

$$\omega_p(B_{R_p}(x_0)) \geq \omega_p(\mathcal{C}) - \epsilon.$$

Based on the fact that $B_{R_p}(x_0) \sqcup B'_L \subset \mathcal{C}$ where B'_L is isometric to B_L, the Lusternik-Schnirelmann inequality (see Lemma 3.6) gives

$$\omega_{p+1}(\mathcal{C}) \geq \omega_p(B_{R_p}(x_0)) + \omega_1(B_L) \geq \omega_p(\mathcal{C}) + |\Sigma_1| - \epsilon.$$

Since ϵ is arbitrary, we show property 1.

Finally, we show property 2 by using the gluing technique of Liokumovich-Marques-Neves, which enables us to combine the p-sweepouts over the same domain X of compact regions with disjoint interiors into one p-sweepout over X of their union. By assumption, we may write $\mathcal{C} = U \sqcup (\Sigma \times [0, \infty))$ where U is a compact submanifold with boundary. Fix $p \in \mathbb{N}$, and we know from Theorem 3.14 that there exists a p-sweepout $\Phi_1 : \mathbb{R}P^p \to \mathcal{Z}_n(U; \partial U; \mathbb{Z}_2)$ satisfying

$$\max_{x \in \mathbb{R}P^p} \mathbf{M}(\Phi_1(x)) \leq C p^{\frac{1}{n+1}}.$$

Recall that $B_L = (\Sigma_1 \sqcup \cdots \sqcup \Sigma_l) \times [0, L]$. Let $f : B_L \to \mathbb{R}$ be the Morse function defined by $f(x, t) := (j-1)L + t$ if $(x, t) \in \Sigma_j \times [0, L]$. Consider $\Phi_2 : \mathbb{R}P^p \to \mathcal{Z}_n(B_L; \partial B_L; \mathbb{Z}_2)$ defined by

$$\Phi_2([a_0, a_1, \ldots, a_p]) = \partial\{x : a_0 + a_1 f(x) + \cdots + a_p f(x)^p < 0\}.$$

Then Φ_2 is a p-sweepout satisfying

$$\max_{x \in \mathbb{R}P^p} \mathbf{M}(\Phi_2(x)) \leq p|\Sigma_1|.$$

By adding tiny tubes to connect the regions and gluing the p-sweepouts Φ_1 and Φ_2 together, we obtain a p-sweepout $\Phi : \mathbb{R}P^p \to \mathcal{Z}_n(U \sqcup B_L; \partial(U \sqcup B_L); \mathbb{Z}_2)$ satisfying

$$\begin{aligned}
\max_{x \in \mathbb{R}P^p} \mathbf{M}(\Phi(x)) &\leq \max_{x \in \mathbb{R}P^p} \mathbf{M}(\Phi_1(x)) + \max_{x \in \mathbb{R}P^p} \mathbf{M}(\Phi_2(x)) + C \\
&\leq C p^{\frac{1}{n+1}} + p|\Sigma_1| + C \\
&\leq C p^{\frac{1}{n+1}} + p|\Sigma_1|.
\end{aligned}$$

This completes the whole proof. □

Let (U, g) be a connected compact Riemannian manifold with boundary endowed with a smooth metric g. Suppose that ∂U is a minimal surface which admits a strictly mean convex foliation. In other words, we assume that there is a diffeomorphism

$$\Phi : \partial U \times [0, \hat{t}] \to U$$

where $\Phi(\partial U \times \{0\}) = \partial U$ is a minimal surface, and for all $t \in (0, \hat{t}]$, the leaf $\Phi(\partial U \times \{t\})$ has non-zero mean curvature vector pointing towards ∂U.

By attaching the cylinders $\partial U \times [0, \infty)$ to U via the identifying map $\varphi : \partial U \times \{0\} \to \partial U$, we obtain the following non-compact manifold with cylindrical ends:

$$\mathcal{C}(U) := U \cup_\varphi (\partial U \times [0, \infty)).$$

The metric h satisfies $h = g$ on U and $h = (g \llcorner \partial U) \oplus ds^2$.

Theorem 3.28 (A. Song 18 [25])**.** Let $(\mathcal{C}(U), h)$ be constructed as above. For all $p \in \{1, 2, \ldots\}$, there exist disjoint, connected, closed, C^∞-embedded minimal hypersurfaces $\Sigma_1, \ldots, \Sigma_N$ contained in $U \setminus \partial U$ and positive integers $m_1, \ldots, m_N$ such that

$$\omega_p(\mathcal{C}(U)) = \sum_{i=1}^{N} m_i |\Sigma_i|.$$

Besides, if Σ_j is one-sided, then the corresponding multiplicity m_j is even.

Proof. By varying the metric and resolving singularities around ∂U, we form the compact smooth approximations (U_ϵ, h_ϵ) of $(\mathcal{C}(U), h)$. Fix $p \in \mathbb{N}$. Applying the free boundary min-max theory developed by Li-Zhou gives a varifold V_ϵ with $\operatorname{spt} V_\epsilon = S_\epsilon = \sqcup_{i=1}^{N_\epsilon} m_{i,\epsilon} \Sigma_{i,\epsilon}$ a smooth, compact, almost properly embedded free boundary minimal hypersurface such that

$$\omega_p(U_\epsilon, h_\epsilon) = \mathbf{M}(V_\epsilon) = \sum_{i=1}^{N_\epsilon} m_{i,\epsilon} |\Sigma_{i,\epsilon}|.$$

Since the boundary $\Phi(\partial U \times \{\epsilon\})$ is strictly mean-concave, the monotonicity formula together with the maximum principle implies that S_ϵ must be compact in $U_\epsilon \setminus \Phi(\partial U \times \{\epsilon\})$.

As $\epsilon \to 0$, we have $\omega_p(U_\epsilon, h_\epsilon) \to \omega_p(\mathcal{C}(U), h)$. Then for a sequence $\epsilon_k \to 0$, the varifold V_{ϵ_k} converges in the varifold sense to a varifold V_∞ in $\mathcal{C}(U)$ of total mass $\mathbf{M}(V_\infty) = \omega_p(\mathcal{C}(U), h)$. By the index bound of Marques-Neves and Sharp's Compactness Theorem, the restriction of $\operatorname{spt} V_\infty = S_\infty$ to $\mathcal{C}(U) \setminus \partial U$ is a C^∞-embedded minimal hypersurface. The maximum principle by White implies that if $S_\infty \cap (\mathcal{C}(U) \setminus \overline{U}) \neq \emptyset$, S_∞ would be a connected component of some slice $\partial U \times \{\delta\}$, which contradicts with the strictly mean-concaveness of the foliation. As a consequence, S_∞ is contained in the compact set (U, g). Since S_∞ is a g-stationary integral varifold, the maximum principle by White implies that S_∞ is confined in U. This completes the proof that S_∞ is a C^∞-embedded minimal hypersurface in U. □

Proof of Theorem 3.25. Let (M^{n+1}, g) be any closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Suppose by contradiction that (M, g) contains finitely many closed, C^∞-embedded minimal hypersurfaces. Each one of them has either a contracting, expanding, or mixed neighborhood. Cut M along minimal hypersurfaces in a maximal way such that we obtain a new manifold "core" U whose boundary, if not empty, has a contracting neighborhood. By construction, the core satisfies the embedded Frankel property, i.e. all minimal hypersurfaces embedded in $\operatorname{int} U$ must intersect. By Theorem 3.26, (M, g) contains at least two disjoint minimal hypersurfaces. Hence, there is at least one nontrivial cut of M and the boundary ∂U is not empty.

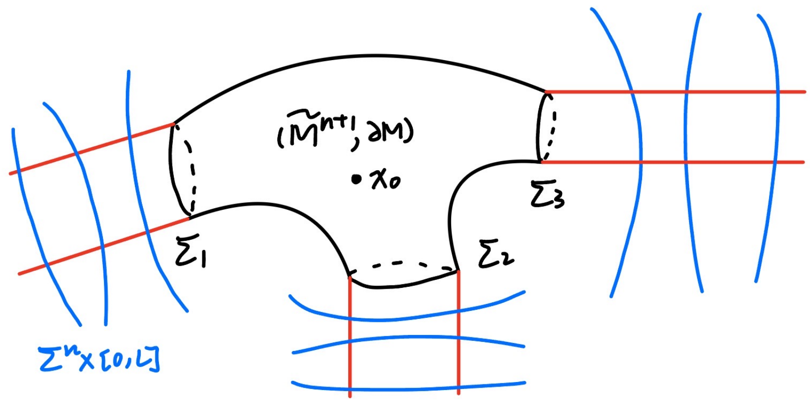
By attaching the cylinders to U along ∂U, we form the non-compact manifold with cylindrical ends $\mathcal{C}(U)$. Let Σ_1 be a component of ∂U with largest n-volume and WLOG assume that $|\Sigma_1| = 1$. By Theorem 3.28, each $\omega_p(\mathcal{C}(U))$ is realized as an integer multiple of closed, connected, C^∞-embedded minimal hypersurface in $\operatorname{int} U$. Since all the closed, C^∞-embedded minimal hypersurfaces in $\operatorname{int} U$ have their volume larger than that of Σ_1, the p-widths $\omega_p(\mathcal{C}(U))$ satisfies

- $\omega_p(\mathcal{C}(U)) > m_p |\Sigma_1|$;
- $\omega_{p+1}(\mathcal{C}(U)) \geq \omega_p(\mathcal{C}(U)) + |\Sigma_1| = \omega_p(\mathcal{C}(U)) + 1$.

By an arithmetic result, we obtain for a $\epsilon_0 > 0$ and all p large enough,

$$\omega_p > (1 + \epsilon_0) p,$$

which contradicts with the upper bound in Theorem 3.27. This completes the whole proof. □



4 Multiplicity One Conjecture

4.1 CMC/PMC min-max theorem

In [33, 34], Zhou-Zhu developed a min-max theory for CMC/PMC surfaces in any closed manifold M. In this section, we will state their main theorem and give an overview of the proof.

Definition. Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \le (n+1) \le 7$. Given $c \in \mathbb{R}$ or a smooth function $h : M \to \mathbb{R}$, we define the weighted area functionals for all $\Omega \in \mathcal{C}(M)$:

$$\mathcal{A}^c(\Omega) = \mathbf{M}(\partial\Omega) - c\mathcal{H}^{n+1}(\Omega);$$
$$\mathcal{A}^h(\Omega) = \mathbf{M}(\partial\Omega) - \int_\Omega h \,\mathrm{dvol}_g \,.$$

We have the following characterization of the perscribed mean curvature (PMC) hypersurfaces.

Lemma 4.1 (PMC)**.** If $\Sigma^n = \partial\Omega$ is a C^∞-embedded hypersurface, then Σ^n is stationary w.r.t. the functional $\mathcal{A}^h$ iff $H_\Sigma = h|_\Sigma$.

Proof. By a similar computation as in Section 1.1, we have the first variation formula for $\mathcal{A}^h$ along $X \in \mathfrak{X}(M)$:

$$\delta\mathcal{A}^h|_\Omega(X) = \int_{\partial\Omega} \mathrm{div}_{\partial\Omega} X d\mu_{\partial\Omega} - \int_{\partial\Omega} h\langle X, \nu\rangle d\mu_{\partial\Omega},$$

where ν is the outward unit normal on $\partial\Omega$. When the boundary $\partial\Omega = \Sigma$ is a C^∞-embedded hypersurface, the first variation becomes

$$\delta\mathcal{A}^h|_\Omega(X) = \int_\Sigma (H_\Sigma - h|_\Sigma\})\langle X, \nu\rangle d\mu_\Sigma.$$

From this we conclude that Σ is stationary iff $H_\Sigma = h|_\Sigma$. □

Lemma 4.2. Under the assumption above, the second variation formula for $\mathcal{A}^h$ along normal vector fields $X \in \mathfrak{X}(M)$, $X = \varphi\nu$ with $\varphi \in C^\infty(\Sigma)$ is given by

$$\delta^2\mathcal{A}^h|_\Omega(X, X) = \int_\Sigma \left[|\nabla\varphi|^2 - (|A^\Sigma|^2 + \mathrm{Ric}^M(\nu,\nu) - \partial_\nu h)\varphi^2\right] d\mu_\Sigma.$$

Definition. Let $U \subset M$ be an open set. Say that Σ is a stable h-hypersurface in U if

- $H_\Sigma = h|_\Sigma$,
- $\delta^2\mathcal{A}^h|_\Omega(\varphi\nu,\varphi\nu) \geq 0, \forall\varphi \in C^\infty(\Sigma)$ with $\operatorname{spt}\varphi \subset \Sigma \cap U$.

For stable h-hypersurfaces, we have the following variant of the famous Schoen-Simon-Yau and Schoen-Simon curvature estimates (see Theorem 3.1). The compactness statement follows in the standard way from the curvature estimates.

Corollary 4.3. Let $U \subset M$ be an open set. Given $\Lambda > 0$ and $h \in C^\infty(M)$, there exists a constant $C = C(U, g, \Lambda, h)$ such that if $\Sigma^n \hookrightarrow (U, g)$ is a smooth, 2-sided, stable h-hypersurface in U with $\partial\Sigma \cap U = \emptyset$ and $\mathrm{Area}(\Sigma) \leq \Lambda$, then

$$|A^\Sigma|^2(p) \leq \frac{C}{\mathrm{dist}_M^2(p, \partial U)}, \quad \forall p \in U.$$

Let $\{\Sigma_i\}$ be a sequence of smooth, 2-sided, stable h-hypersurfaces in U with $\partial\Sigma_i \cap U = \emptyset$ and $\sup_i \mathrm{Area}(\Sigma_i) < \infty$. Then up to a subsequence, Σ_i converges locally smoothly to a stable h-hypersurface Σ_∞ in U possibly with integer multiplicity.

Proposition 4.4 (1-sided Maximum Principle)**.** Let $H_\Sigma = c$ for a constant $c > 0$. If Σ_1 and Σ_2 are graphs over $\mathbb{R}^n$ with opposite orientations, then either $\Sigma_1 \cap \Sigma_2$ is contained in a $(n-1)$-dimensional submanifold or $\Sigma_1 \cap \Sigma_2 = \emptyset$.

In the following paragraph we shall introduce the theory of relative sweepouts, which sets the basis for stating the CMC/PMC min-max theorem. Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Let X be a k-dimensional cube complex with $Z \subset X$ a subcomplex. For each $\Phi_0 : (X, Z) \to (\mathcal{C}(M), \mathbf{F})$ continuous under the $\mathbf{F}$-metric, consider the relative homotopy class

$$\begin{aligned}\Pi(\Phi_0) = \{\Phi : (X, Z) \to (\mathcal{C}(M), \mathbf{F}) &\text{ continuous under the } \mathbf{F}\text{-metric such that} \\ \Phi|_Z = \Phi_0|_Z &\text{ and } \Phi \text{ is homotopic to } \Phi_0 \text{ rel } Z.\}\end{aligned}$$

with the fixed parameter space (X, Z) and define the $\mathcal{A}^h$-min-max value as

$$\mathbf{L}^h(\Pi) = \inf_{\Phi\in\Pi} \max_{x\in X} \mathcal{A}^h(\Phi(x)).$$

Now, we are ready to state the CMC/PMC min-max theorem confirmed by Zhou-Zhu [33, 34, 32].

Theorem 4.5 (CMC/PMC Min-max Theorem [33, 34])**.** Under the hypotheses above, if the non-triviality condition is satisfied, i.e.

$$\mathbf{L}^h(\Pi) > \max_{x\in Z} \mathcal{A}^h(\Phi_0(x)),$$

then there always exists a smooth, almost embedded (embedded outside the touching set) hypersurface $\Sigma^n = \partial\Omega$ for some $\Omega \in \mathcal{C}(M)$ such that

- $H_\Sigma = h|_\Sigma$;
- $\mathcal{A}^h(\Sigma) = \mathbf{L}^h(\Sigma)$;
- $\mathrm{index}(\Sigma) \leq k$.

Corollary 4.6. For every positive $c \in \mathbb{R}$, there always exists a smooth, closed, almost embedded (embedded outside the touching set) hypersurface Σ^n of $H_\Sigma \equiv c$.

Example 4.7. Consider the special case when $X = [0,1]$ and $Z = \{0,1\}$. Given a Morse function $f : M \to \mathbb{R}$, define Φ_0 by $\Phi_0(x) = f^{-1}([0,x])$. Note that $\Phi_0(0) = \emptyset$, $\Phi_0(1) = M$, and Φ_0 is continuous under the $\mathbf{F}$-metric. Under the assumptions that $\sup_M |h| = c < \infty$ and $\int_M h \,\mathrm{dvol}_g \geq 0$, the $\mathbf{L}^h$-min-max value satisfies $\mathbf{L}^h(\Pi) > 0$. We shall present a heuristic proof here using the lower bound for the isoperimetric profiles for small volumes (see Lemma 2.4).

Proof. Let $C_0 > 0$ and $V_0 > 0$ be the constants in Lemma 2.4, and fix $0 < V \leq V_0$ such that $V^{\frac{-1}{n+1}} > 2c/C_0$. Consider any smooth 1-parameter family $\{\Omega_x : x \in [0,1]\}$ satisfying $\Omega_0 = \emptyset$ and $\Omega_1 = M$. By the Intermediate Value Theorem, there exists $x_0 \in (0,1)$ such that $\mathrm{Vol}(\Omega_{x_0}) = V$. By the isoperimetric profiles for small volumes, we have

$$\max_{x\in[0,1]} \mathcal{A}^h(\Omega_x) \geq \mathcal{A}^h(\Omega_{x_0}) \geq C_0 V^{\frac{n}{n+1}} - cV \geq cV > 0.$$

Since this holds for any sweepout, we conclude that $\mathbf{L}^h(\Pi) > 0$. □

Definition. The critical set of $\{\Phi_j\}$ is given by

$$\begin{aligned} C(\{\Phi_j\}) := \{(\Omega_\infty, V_\infty) \in \mathcal{C}(M) \times \mathcal{V}_n(M) : \Omega_\infty = \lim_{i\to\infty} \Phi_{j_i}(x_i), V_\infty = \lim_{i\to\infty} |\partial\Phi_{j_i}(x_i)|, \\ \text{and } \mathcal{A}^h(\Phi_{j_i}(x_i)) \to \mathbf{L}^h(\Pi)\}. \end{aligned}$$

Similarly as in Section 2.2, we shall construct the tightening map adapted to the $\mathcal{A}^h$ functional and prove that after applying the tightening map to a critical sequence, every element in the critical set has c-bounded first variation, where $c = \sup_M |h|$. This variational property is a generalization of bounded mean curvature, and is loose enough to be satisfied by the min–max limit V (after tightening) while providing enough control to develop the regularity theory. In particular, varifolds with c-bounded first variation satisfy a uniform monotonicity formula, and any blowup is stationary.

Proposition 4.8 (Tightening)**.** Assume $\mathbf{L}^h(\Pi) > 0$. For any critical sequence $\{\Phi_j^*\}$ for Π, there exists another critical sequence $\{\Phi_j\}$ for Π such that $C(\{\Phi_j\}) \subset C(\{\Phi_j^*\})$ and each pair $(\Omega, V) \in C(\{\Phi_j\})$ is $\mathcal{A}^h$-stationary, i.e. $\forall X \in \mathfrak{X}(M)$,

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \mathcal{A}^h(\Phi_t^x(\Omega), \Phi_t^x(V)) \\ &= \int_{G_n(M)} \mathrm{div}_S \, dV(x,S) - \int_{\partial\Omega} h\langle X, \nu\rangle d\mu_{\partial\Omega}. \end{aligned}$$

Corollary 4.9. Under the hypotheses above, V has c-bounded first variation.

Proof. This comes from the following estimate:

$$|\delta V(X)| \leq |\int_{\partial\Omega} h\langle X, \nu\rangle d\mu_{\partial\Omega}| \leq c\int_M |X| d\mu_V,$$

where $c = \sup_M |h|$. □

We proceed to introduce the notion of h-almost minimizing varifolds, and construct h-replacements for any h-almost minimizing varifold after solving a natural constrained minimization problem.

Definition. Given $\epsilon, \delta > 0$ and an open set $U \subset M^{n+1}$, define

$$\begin{aligned} \mathcal{A}^h(U;\epsilon,\delta) := \{\Omega \in \mathcal{C}(M) \text{ such that if } \Omega = \Omega_0, \ldots, \Omega_m \in \mathcal{C}(M) \text{ satisfying} \\ 1.\,\mathrm{spt}(\Omega_i - \Omega) \subset\subset U; \\ 2.\mathcal{F}(\Omega_i, \Omega_{i+1}) \leq \delta; \\ 3.\mathcal{A}^h(\Omega_i) \leq \mathcal{A}^h(\Omega) + \delta, \\ \text{then } \mathcal{A}^h(\Omega_m) \geq \mathcal{A}^h(\Omega) - \epsilon\}. \end{aligned}$$

Definition. Say a varifold $V \in \mathcal{V}_n(M)$ is h-almost minimizing in U if there exists $\epsilon_i \to 0$, $\delta_i \to 0$, and $\Omega_i \in \mathcal{A}^h(U;\epsilon_i,\delta_i)$ such that $\mathbf{F}(|\partial\Omega_i|, V) \leq \epsilon_i$ for every i.

Definition. A varifold $V \in \mathcal{V}_n(M)$ is h-almost minimizing in small annuli if $\forall p \in M$, $\exists r_{\mathrm{am}}(p) > 0$ such that V is h-almost minimizing in $\mathcal{A}_{s,r}(p) = B_r(p) \setminus B_s(p)$ for all $0 < s < r < r_{\mathrm{am}}(p)$.

Theorem 4.10 (Existence of h-almost minimizing varifold)**.** Assume $\mathbf{L}^h(\Pi) > 0$ and let $\{\Phi_j\}$ be a pull-tight minimizing sequence of sweepouts for Π. Then there exists a nontrivial pair $(\Omega, V) \in C(\{\Phi_j\})$ such that

1. V has c-bounded first variation;
2. V is h-almost minimizing in small annuli.

Proposition 4.11 (Existence and properties of h-replacements)**.** Let $V \in \mathcal{V}_n(M)$ be h-almost minimizing in an open set $U \subset M$ and let $K \subset\subset U$ be a compact subset of U. Then there exists $V^* \in \mathcal{V}_n(M)$ called an h-replacement of V in K such that, with $c = \sup_M |h|$,

1. $V \llcorner (M \setminus K) = V^* \llcorner (M \setminus K)$;
2. $-c\,\mathrm{Vol}(K) \leq ||V||(M) - ||V^*||(M) \leq c\,\mathrm{Vol}(K)$;
3. V^* is also h-almost minimizing in U;
4. $V^* = \lim_{i\to\infty} |\partial\Omega_i^*|$ for some $\Omega_i^* \in \mathcal{A}^h(U;\epsilon_i,\delta_i)$ with $\epsilon_i, \delta_i \to 0$ and Ω_i^* locally minimizes $\mathcal{A}^h$ in $\mathrm{int}\,K$ for all i.
5. if V has c-bounded first variation in M, so does V^*.

Proposition 4.12 (Regularity of h-replacement)**.** Let $3 \leq (n+1) \leq 7$. Under the same hypotheses as Proposition , if $\Sigma = \mathrm{spt}\,||V^*|| \cap \mathrm{int}\,K$, then Σ is a smooth, almost embedded, stable h-hypersurface.

Theorem 4.13 (Main regularity)**.** Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. Given a smooth function $h : M \to \mathbb{R}$, set $c = \sup_M |h|$. Assume that $V \in \mathcal{V}_n(M)$ has c-bounded first variation in M and is h-almost minimizing in small annuli. Then V is induced by Σ, where Σ is a closed, almost embedded h-hypersurface.

4.2 Free boundary min-max theorem

In [10], Li-Zhou developed the min-max theory for free boundary minimal hypersurfaces in the general Almgren-Pitts setting. In this section, we will state their main theorem without giving a proof.

Definition. Let $(M^{n+1}, \partial M, g)$ be a Riemannian manifold with boundary. A hypersurface $\Sigma \hookrightarrow (M^{n+1}, \partial M)$ is called properly embedded if

- $\mathrm{int}\,\Sigma \subset \mathrm{int}\,M$;
- $\partial\Sigma \subset \partial M$.

Define the collection of tangential vector fields as

$$\mathfrak{X}^t(M) := \{X \in \mathfrak{X}(M) : X(p) \in T_p(\partial M), \forall p \in \partial M\}.$$

Any compactly supported $X \in \mathfrak{X}^t(M)$ generates a smooth one parameter family of diffeomorphisms Φ_t^X such that $\Phi_t^X(\Sigma)$ is a family of properly embedded hypersurfaces in M. By the first variation formula, we have $\forall X \in \mathfrak{X}^t(M)$,

$$\begin{aligned}\delta\Sigma(X) :=& \frac{d}{dt}\bigg|_{t=0} \operatorname{Area}(\Phi_t^X(\Sigma))\\ =& \int_\Sigma \operatorname{div}_\Sigma X d\mathcal{H}^n\\ =& -\int_\Sigma \langle X, H\rangle d\mathcal{H}^n + \int_{\partial\Sigma} \langle X, \mu\rangle d\sigma^{n-1},\end{aligned}$$

where H is the mean curvature vector of Σ and μ is the outward unit co-normal of $\partial\Sigma$.

Definition. A properly embedded minimal hypersurface $\Sigma^n \hookrightarrow (M^{n+1}, \partial M)$ is called a free boundary minimal hypersurface (FBMH) if the mean curvature of Σ vanishes and Σ meets ∂M orthogonally along $\partial\Sigma$.

Proposition 4.14. A FBMH $\Sigma^n \hookrightarrow (M, \partial M)$ is a stationary point of the area functional.

Recall that in Section 1.3, we have introduced the space of mod-2 flat chains $\mathcal{Z}_k(M; \mathbb{Z}_2)$. To set up the free boundary min-max theory, we restrict our attention to the space of mod-2 flat chains relative to boundary $\mathcal{Z}_k(M; \partial M; \mathbb{Z}_2)$. An analogous result for Almgren's Isomorphism Theorem is stated below:

Theorem 4.15. $\mathcal{Z}_n(M; \partial M; \mathbb{Z}_2)$ is weakly homotopic to $\mathbb{R}P^\infty$.

Let $(M^{n+1}, \partial M, g)$ be a compact Riemannian manifold with boundary of dimension $3 \le (n+1) \le 7$. Let X be a k-dimensional cube complex. For each $\Phi_0 : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ continuous under the $\mathbf{F}$-metric, consider the homotopy class

$$\begin{aligned}\Pi(\Phi_0) = \{\Phi : X \to \mathcal{Z}_n(M; \partial M; \mathbb{Z}_2) &\text{ continuous under the } \mathbf{F}\text{-metric}\\ &\text{such that } \Phi \text{ is homotopic to } \Phi_0\}\end{aligned}$$

with the fixed parameter space X and define the free boundary min-max value as

$$\mathbf{L}(\Pi) = \inf_{\Phi\in\Pi} \max_{x\in X} \mathbf{M}(\Phi(x)).$$

Now, we are ready to state the free boundary min-max theorem confirmed by Li-Zhou [10].

Theorem 4.16. Under the hypotheses above, if $\mathbf{L}(\Pi) > 0$, then there exists a disjoint collection of smooth, almost properly embedded FBMHs $\{\Sigma_i\}$ such that

$$\mathbf{L}(\Pi) = \sum_{i=1}^{l} m_i \operatorname{Area}(\Sigma_i).$$

Remark. Here, the almost properly embedded FBMHs are those FBMHs that may have non-empty touching sets, i.e. $\operatorname{int}(\Sigma) \cap \partial M \neq \emptyset$.

4.3 Multiplicity One Conjecture

Note that the Almgren-Pitts min-max theory works for families of cycles within a homotopy class, while the definition of the volume spectrum concerns all families via the cohomological condition. To link them together, Marques-Neves systematically studied the Morse index for minimal hypersurfaces produced by the Almgren-Pitts theory [13]. In particular, they proved the following version of the min-max theorem.

Theorem 4.17. Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. For each $k \in \mathbb{N}$, there exists a disjoint collection of connected, closed, C^∞-embedded minimal hypersurfaces $\{\Sigma_i^k : i = 1, \cdots, l_k\}$ with integer multiplicities $\{m_i^k : i = 1, \cdots, l_k\} \subset \mathbb{N}$, such that

$$\omega_k(M, g) = \sum_{i=1}^{l_k} m_i^k \cdot \mathrm{Area}(\Sigma_i^k) \quad \text{and} \quad \sum_{i=1}^{l_k} \mathrm{index}(\Sigma_i^k) \leq k.$$

The possible existence of multiplicities greater than 1 formed a major obstacle in applications of the Almgren-Pitts theory since the 1980s. In addition to the possible repeated occurrence of minimal hypersurfaces when applying Theorem 4.17 to $\{\omega_k\}_{k\in\mathbb{N}}$, min-max varifolds with higher multiplicities cannot fit into the program of Marques-Neves [15] to obtain Morse index lower bounds (see also [12]). The following famous conjecture was formulated by Marques [4] and Neves [16]; see also [15].

Conjecture (Multiplicity One Conjecture)**.** For a bumpy metric on M^{n+1}, $3 \leq (n+1) \leq 7$, there exists a collection $\{\Sigma_i^k\}$ as in Theorem 4.17, such that every component Σ_i^k is 2-sided and of multiplicity one.

This conjecture was confirmed by Zhou in [32].

Theorem 4.18. Multiplicity One Conjecture is true.

Theorem 4.18 together with the program on Morse index lower bounds developed by Marques-Neves [15] imply that for bumpy metrics, there exists a closed minimal hypersurface of Morse index k and area $\omega_k(M, g)$ for each $k \in \mathbb{N}$. The above works together established a satisfactory global Morse theory for the area functional. Later, Marques-Montezuma-Neves proved Morse inequalities for the area functional [12], and hence established a local Morse theory as well.

By Sharp's Compactness Theorem, the same conclusions in Theorem 4.18 hold true for metrics with positive Ricci curvature.

Sketch of proof of Theorem 4.18. The key idea of the proof is to approximate the area functional by the weighted $\mathcal{A}^h$-functional used in the PMC min-max theory (see Section 4.1). There are two crucial parts in the proof. First, we show that given a bumpy metric the volume spectrum $\omega_k(M)$ can be realized by the area of some minimal hypersurfaces coming from relative min-max constructions using sweepouts of boundaries. Next, we observe that, still assuming bumpiness, if one approximates Area by a sequence $\{\mathcal{A}^{\epsilon_k h}\}_{k\in\mathbb{N}}$ where $\epsilon_k \to 0$, and if $h : M \to \mathbb{R}$ is carefully chosen, then the limit min-max minimal hypersurfaces (of min-max PMC hypersurfaces associated with $\mathcal{A}^{\epsilon_k h}$) are all 2-sided and have multiplicity one.

Part 1: Given a bumpy metric, for each $k \in \mathbb{N}$ by [13], there exists a free homotopy class Π of maps $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$, where X is a fixed k-dimensional parameter space, such that

$$\mathbf{L} = \inf_{\Phi\in\Pi} \max_{x\in X} \mathrm{Area}(\Phi(x)) = \omega_k(M).$$

Choose $\Phi_0 \in \Pi$ so that $\max_{x \in X} \operatorname{Area}(\Phi_0(x))$ is very close to $\mathbf{L}$. Since $\mathcal{C}(M)$ forms a double cover of $\mathcal{Z}_n(M, \mathbb{Z}_2)$ via the boundary map (see Lemma 1.23), we can lift Φ_0 to $\widetilde{\Phi}_0 : \widetilde{X} \to \mathcal{C}(M)$, where $\pi : \widetilde{X} \to X$ is also a double cover. Together they satisfy the following diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi"'] & \mathcal{C}(M) \arrow[d, "\partial"'] \\ X \arrow[r, "\Phi"] & \mathcal{Z}_n(M;\mathbb{Z}_2) \end{tikzcd}$$

Next, denote

$$\mathcal{S} = \{\Sigma \subset M : \Sigma \text{ is a closed, } C^\infty\text{-embedded minimal hypersurface with} \\ \operatorname{Area}(\Sigma) \leq \mathbf{L} + 1 \text{ and } \operatorname{index}(\Sigma) \leq k\}.$$

and

$$Y = \{x \in X : \mathbf{F}(\Phi_0(x), \mathcal{S}) < \epsilon\}.$$

Set $Z = \overline{X \setminus Y}$. As $\mathcal{S}$ is a finite set by [20], Y is topologically trivial, and hence $\widetilde{Y} = \pi^{-1}(Y)$ is a disjoint union of two homeomorphic copies of Y, that is, $\widetilde{Y} = Y^+ \sqcup Y^-$ with $Y \simeq Y^+ \simeq Y^-$. On the other hand, since no element in $\Phi_0(Z)$ is close to being regular, we can deform $\Phi_0|_Z$ based on Pitts's combinatorial argument [17], so that

$$\max_{x \in Z} \operatorname{Area}(\Phi_0(x)) < \mathbf{L}.$$

Now consider the $(\widetilde{X}, \widetilde{Z})$-relative homotopy class of maps generated by $\widetilde{\Phi}_0$: $\widetilde{\Pi} = \{\Psi : \widetilde{X} \to \mathcal{C}(M) : \Psi|_{\widetilde{Z}} = \widetilde{\Phi}_0|_{\widetilde{Z}}\}$.

Lemma 4.19. The min-max value $\widetilde{\mathbf{L}}$ of $\widetilde{\Pi}$ satisfies

$$\widetilde{\mathbf{L}} := \inf_{\Psi \in \widetilde{\Pi}} \max_{x \in \widetilde{X}} \operatorname{Area}(\partial \Psi(x)) \geq \mathbf{L} = \omega_k(M).$$

Hence we have the nontriviality condition $\widetilde{\mathbf{L}} > \max_{x \in Z} \operatorname{Area}(\Phi_0(x))$.

Proof. If the conclusion were false, then since

$$\max_{x \in \widetilde{Z}} \operatorname{Area}(\partial \widetilde{\Phi}_0(x)) = \max_{x \in Z} \operatorname{Area}(\Phi_0(x)) < \mathbf{L},$$

one can deform $\widetilde{\Phi}_0$ on $\widetilde{Y}$ so that the maximum area is less than $\mathbf{L}$. However, as Y^+ and Y^- are disjoint, the deformations on Y^+ (or on Y^-) can be passed to the quotient to give deformations of $\Phi_0|_Y$ in $\mathcal{Z}_n(M, \mathbb{Z}_2)$. As all the maps are fixed on Z, we then obtain deformations of Φ_0 after which the maximum area is less than $\mathbf{L}$, which is a contradiction. $\square$

Part 2: The main conclusion follows from the result below.

Theorem 4.20 (X. Zhou 19 [32])**.** In the above notation, if g is bumpy, $\widetilde{\mathbf{L}}$ can be realized as the area of a multiplicity one, closed, C^∞-embedded, 2-sided, minimal hypersurface.

To derive Theorem 4.18, first note that by the choice of Φ_0, we know $\widetilde{\mathbf{L}}$ is very close to $\mathbf{L}$. By the bumpiness of g, the values of $\widetilde{\mathbf{L}}$ should stabilize to $\mathbf{L}$ when they are close enough.

Proof of Theorem 4.20. To simplify notions, we will drop all the tilde's in this part. Given a smooth function $h : M \to \mathbb{R}$, and $\epsilon > 0$, we can approximate $\mathbf{L}$ by the min-max values for the $\mathcal{A}^{\epsilon h}$-functional:

$$\mathbf{L}^{\epsilon h} = \inf_{\Psi \in \Pi} \max_{x \in X} \mathcal{A}^{\epsilon h}(\Psi(x)),$$

that is, $\mathbf{L}^{\epsilon h} \to \mathbf{L}$ as $\epsilon \to 0$. Note that we require $\Psi|_Z = \Phi_0|_Z$ for all $\Psi \in \Pi$. By the fact $\mathbf{L} > \max_{x\in Z} \operatorname{Area}(\partial\Phi_0(x))$, and that the term $\epsilon \int_\Omega h dM$ in $\mathcal{A}^{\epsilon h}(\Omega)$ is uniformly small, we have, for ϵ small enough,

$$\mathbf{L}^{\epsilon h} > \max_{x \in Z} \mathcal{A}^{\epsilon h}(\Psi(x)). \tag{4.21}$$

For a generic choice of h, applying the multi-parameter PMC min-max theory [32] (based on the one parameter version in Section 4.1), we obtain a smooth, almost embedded hypersurface $\Sigma_\epsilon = \partial\Omega_\epsilon$ for some $\Omega_\epsilon \in \mathcal{C}(M)$ such that

- $H_{\Sigma_\epsilon} = \epsilon h|_{\Sigma_\epsilon}$;
- $\mathcal{A}^{\epsilon h}(\Omega_\epsilon) = \mathbf{L}^{\epsilon h}$;
- the Morse index (w.r.t. $\mathcal{A}^{\epsilon h}$) $\operatorname{index}(\Sigma_\epsilon) \leq k$.

Letting $\epsilon \to 0$, by the above and compactness theorem for PMCs with bounded index [32], up to taking a subsequence, Σ_ϵ converge locally smoothly away from a finite set $\mathcal{W}$ to a closed embedded minimal hypersurface Σ_0 with an integer multiplicity $m \in \mathbb{N}$. Therefore $\mathbf{L} = m \operatorname{Area}(\Sigma_0)$, and it remains to prove that Σ_0 is 2-sided (which is skipped here) and $m = 1$.

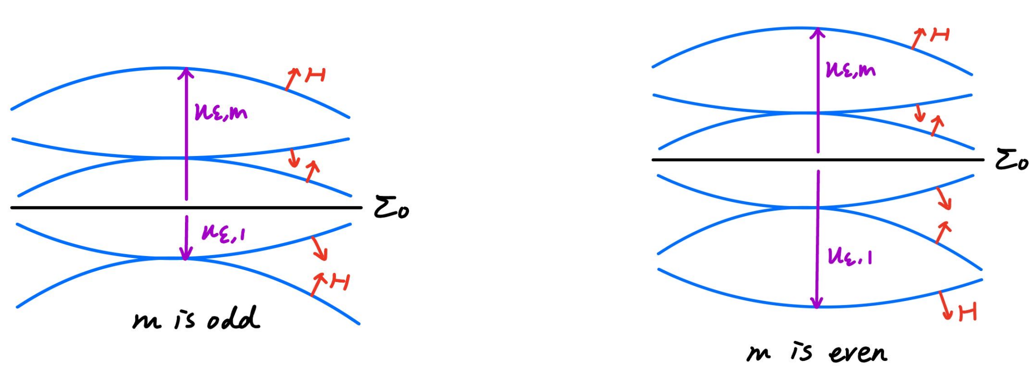
The convergence implies that Σ_ϵ locally decomposes as an m-sheeted graph over $\Sigma_0 \setminus \mathcal{W}$, with graphing functions: $u_\epsilon^1 \leq u_\epsilon^2 \leq \cdots \leq u_\epsilon^m$. And by Proposition 4.4, the outward unit normal of Ω_ϵ will alternate orientations along these sheets. The proof proceeds depending on whether m is odd or even.

Claim 1. If $m \geq 3$ is odd, then Σ_0 is degenerate, hence a contradiction.

Proof. Since m is odd, the top and the bottom sheets have the same orientation, so by subtracting the PMC equations of the two sheets, we have

$$L(u_\epsilon^m - u_\epsilon^1) + o(u_\epsilon^m - u_\epsilon^1) = 0,$$

where L is the Jacobi operator associated with $\delta^2\Sigma_0$. After renormalizations, the height differences $u_\epsilon^m - u_\epsilon^1$ will converge subsequentially to a positive Jacobi field of $\Sigma_0 \setminus \mathcal{W}$, which extends to Σ_0 by standard trick. $\square$



Claim 2. If m is even, there exists a solution of $L\varphi = 2h|_{\Sigma_0}$ which doesn't change sign.

Proof. Now the top and the bottom sheets have opposite orientations. Thus

$$L(u_\epsilon^m - u_\epsilon^1) + o(u_\epsilon^m - u_\epsilon^1) = \pm\epsilon(h(x, u_\epsilon^1) + h(x, u_\epsilon^m)).$$

Using the renormalization procedure again and noting that $u_\epsilon^m - u_\epsilon^1 > 0$, we get either a positive Jacobi field (which cannot happen) or a positive function φ satisfying $L\varphi = 2h|_{\Sigma_0}$ or $L\varphi = -2h|_{\Sigma_0}$. $\square$

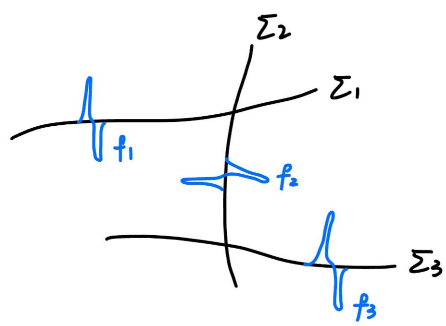
The following key lemma says that Claim 2 cannot hold for a suitably chosen h. Hence the proof of Theorem 4.20 is complete.

Lemma 4.22. For a suitably chosen h, the solutions of $L\varphi = 2h|_\Sigma$ on a closed, C^∞-embedded minimal hypersurface Σ with $\text{Area}(\Sigma) \leq C$ and $\text{index}(\Sigma) \leq k$ must change sign.

Proof. By Sharp's Compactness Theorem, the set of minimal hypersurfaces with Area $\leq C$ and index $\leq k$ is finite, which we denote by $\{\Sigma_1, \Sigma_2, \cdots, \Sigma_N\}$. Take pairwise disjoint neighborhoods $U_j^\pm \subset \Sigma_j$ and a smooth function f defined on $\bigcup U_j^\pm$ with compact support such that

1. $f|_{U_j^+}$ is non-negative and is positive at some point;
2. $f|_{U_j^-}$ is non-positive and is negative at some point.

Next extend Lf to some $h_0 \in C^\infty(M)$ and take a generic h as close to h_0 as we want. Then any solution φ of $L\varphi = 2h|_{\Sigma_j}$ would be close to $2f$ for each Σ_j, and hence must change sign. $\square$



$\square$

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