• Introduction to the Kähler–Ricci Flow.

## §1. Kähler manifolds.

- $M$: compact complex manifold of complex dimension $n$.
- $z=(z^1,\cdots,z^n)$: local holomorphic chart on $U\subseteq M$.
- Hermitian metric: $g=g_{i\bar{j}}\,dz^i\otimes d\bar{z}^{\bar{j}}$, s.t. $(g_{i\bar{j}})_{n\times n}$ is a positive definite Hermitian metric.
- Associated $g$ to a $(1,1)$-form $\omega$:

$$\omega=\sqrt{-1}\cdot g_{i\bar{j}}\,dz^i\wedge d\bar{z}^{\bar{j}}.$$

- $g$ is called a Kähler metric, if $d\omega=0$, or in local coord.s:

$$\partial_k g_{i\bar{j}}=\partial_i g_{k\bar{j}},\quad \forall\, i,j,k.$$

- $\omega$ is called a Kähler form.

- Example: $\mathbb{CP}^n=\left(\mathbb{C}^{n+1}\setminus\{0\}\right)/\sim$.

$\sim$: $(Z_0,\cdots,Z_n)\sim(Z_0',\cdots,Z_n')$ iff $\exists\,\lambda\in\mathbb{C}^*$, s.t. $Z_i=\lambda\cdot Z_i^*$, $\forall i$.

$[Z_0,\cdots,Z_n]\in\mathbb{CP}^n$: the equivalent class of $(Z_0,\cdots,Z_n)$.

- Fubini–Study metric: $\omega_{FS}:=\frac{\sqrt{-1}}{2\pi}\cdot\partial\bar{\partial}\log\left(|Z_0|^2+\cdots+|Z_n|^2\right).$

- $\omega$ defines a cohomology class $[\omega]$ inside:

$$H^{1,1}(M,\mathbb{R}) := \frac{\{\text{closed, real } (1,1)\text{-form on } M\}}{\sqrt{-1}\cdot\partial\bar\partial\, C^\infty(X,\mathbb{R})}.$$

- crucial formula: $R_{i\bar j} = \partial_i \partial_{\bar j} \log\det(g_{p\bar q})$.

- Ricci form: $\mathrm{Ric}(\omega) := \sqrt{-1}\cdot \sum_{i,j=1}^{n} R_{i\bar j}\, dz^i \wedge d\bar z^j$.

  then $\mathrm{Ric}(\omega)$ is a closed real $(1,1)$-form.

- First Chern class: $c_1(M) := [\mathrm{Ric}(\omega)]$.

  $c_1(M)$ do not depend on $\omega$: given two Kähler metrics $\omega$, $\tilde\omega$,

  $$\mathrm{Ric}(\omega) - \mathrm{Ric}(\tilde\omega) = \sqrt{-1}\,\partial\bar\partial \log\frac{\det g}{\det \tilde g} \Rightarrow [\mathrm{Ric}(\omega)] = [\mathrm{Ric}(\tilde\omega)].$$

- Canonical line bundle: $K_M := \Lambda^n (T^{1,0}M)^*$.

- $c_1(K_M) := -c_1(M) = -c_1(K_M^{-1})$.

  dual line bundle.

  anti-canonical line bundle.

## §2. Basics of Kähler-Ricci flow.

- $(M,\omega_0)$: compact Kähler manifold of complex dim. $n$.

- $\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t))$, $\omega(0) = \omega_0$.

- $\frac{d}{dt}[\omega(t)] = -c_1(M)$, $[\omega(0)] = [\omega_0]$

  $\Rightarrow$ $[\omega(t)] = [\omega_0] - t\cdot c_1(M)$.

  $\Rightarrow$ necessary condition is $t < T := \sup\{\tilde{t} > 0 : [\omega_0] - \tilde{t}\cdot c_1(M) > 0\}$. may be $+\infty$.

- We have a good characterization of the maximal existence time.

- **Thm** (Tian-Zhang, Cao, Tsuji).

  $\exists!$ maximal solution for $t \in [0, T)$.

- We have a long-time solution on $[0, +\infty)$ iff $X$ is a minimal model, i.e., $K_X$ is nef. ($K_X \cdot C \geq 0$, for any analytic curve $C$ on $M$).

- Kähler-Ricci flow on Calabi-Yau manifolds:

- **Thm** (Calabi conjecture, Yau 1978).

  If $c_1(M) = 0$, then $\forall$ Kähler metric $\omega_0$ on $M$,

  $\exists!$ Ricci flat Kähler metric in $[\omega_0]$.

- Yau used the continuity method (elliptic).

- **Thm** (Cao). Given any $\omega_0$, run the KRF, after suitable normalization, the KRF with smoothly deform to the unique Ricci-flat Calabi-Yau metric in the class $[\omega_0]$.

- Rmk: Cao also proved the result for $c_1(M) < 0$. (Yau, Aubin).

- Kähler-Ricci flow is parabolic Monge-Ampère flow.

- Choose $c_1(M) < 0$ as example.

  - Choose $\omega_0$ s.t. $[\omega_0] = -c_1(M) > 0$.

  - Solve: $\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) - \omega(t)$, $\omega(0) = \omega_0$,

  - Find volume form $\Omega$, s.t. $\sqrt{-1}\partial\bar\partial \log \Omega = \omega_0 \in -c_1(M)$.

  - Normaliz $\Omega$ s.t. $\int_M \Omega = \int_M \omega_0^n$. Then consider:

$$\frac{\partial \varphi(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar\partial \varphi(t))^n}{\Omega} - \varphi \quad , \quad \varphi|_{t=0} = 0$$

is a parabolic Monge-Ampère eqn, and

$$\omega(t) = \omega_0 + \sqrt{-1}\partial\bar\partial \varphi(t)$$

is the solution of our original KRF.

- Rmk: we have more tools to study $\varphi$ and $\frac{\partial \varphi}{\partial t}$ (than $\omega(t)$).

## §3. Some notions of positivity of line bundles.

- $(M, \omega)$: $n$-dim. compact Kähler mfd.

- $L$: line bundle on $M$.

- L is positive: if $c_1(L) > 0$, i.e., $\exists$ Hermitian metric $h$ on $L$, s.t.

$$R_h \text{ is a Kähler form.}$$

- $H^0(M,L)$: vector space of holomorphic sections of $L$,
- $h^0(M,L) := \dim H^0(M,L) < +\infty$.
- L is very ample: for any basis $\underline{s} = (s_0, \cdots, s_N)$ of $H^0(M,L)$, the map:

$$\iota_{\underline{s}} : M \to \mathbb{CP}^N, \quad \iota_{\underline{s}}(x) := [s_0(x), \cdots, s_N(x)]$$

is well-defined and embedding.

- L is ample: $\exists\, m_0 \in \mathbb{N}^+$ large, s.t. $L^m$ is very ample for all $m \geq m_0$.
- Kodaira Embedding Thm: L is ample $\Leftrightarrow$ L is positive.
- L is globally generated: $\forall x \in M$, $\exists s \in H^0(M,L)$, s.t. $s(x) \neq 0$.
- L is semi-ample: $\exists\, m_0 \in \mathbb{N}^+$, s.t. $L^{m_0}$ is globally generated.
- L is nef: for any smooth analytic curve $C$ on $M$,

$$L \cdot C := \int_C R_h \geq 0, \quad h: \text{Hermitian metric on } L.$$

- Relation: very ample $\Rightarrow$ ample $\Rightarrow$ semi-ample $\Rightarrow$ nef.
- there are other definitions of these notions.

- In general. $L$ nef $\not\Rightarrow$ $L$ semi-ample.
- Abundance conjecture: $K_M$ nef $\Rightarrow$ $K_M$ semi-ample.
- $M$ is called a smooth minimal model if $K_M$ is nef.
- $L$ is big: $\exists\, m_0>0$, $c>0$, s.t. $h^0(M, L^m) \geq c\cdot m^n$, $\forall\, m\geq m_0$.
- $M$ is called general type if $K_M$ is big.
- Kawamata Base Point Free: $K_M$ nef & big $\Rightarrow$ $K_M$ semi-ample.
- Kodaira dim: $\kappa \in [-\infty, +\infty)$, s.t. $h^0(M, K_M^m) \leq C\cdot m^\kappa$, $\forall\, m\in \mathbb{N}^+$.

  If $H^0(M, K_M^m)=\phi$, set $\kappa=-\infty$.
- If $\kappa\neq-\infty$, then $\kappa\in\{0,1,\cdots,n\}$.
- general type (including $C_1(M)<0$): $\kappa=n$.

  Calabi-Yau ($C_1(M)=0$): $\kappa=0$.

  Fano-manifold ($C_1(M)>0$): $\kappa=-\infty$.

## §4. Kähler-Ricci flow on minimal model.

- $M$: smooth minimal model, consider the normalized KRF:

$$\frac{\partial\omega(t)}{\partial t} = -Ric(\omega(t)) - \omega(t), \quad \omega(0)=\omega_0.$$

- We usually need to assume Abundance Conjecture, i.e., $K_M$ semi-ample.
- The pluricanonical system $H^0(M, K_M^m)$ for large $m$ gives:

$$\Phi : M \longrightarrow B \subset \mathbb{CP}^N := \mathbb{P}H^0(M, K_M^m).$$

- $\dim_{\mathbb{C}}(B) = \mathrm{Kod}(M) =: \kappa.$
- Consider the case $1 \leq \kappa \leq n-1$, $B$ is algebraic variety.
- $S' \subset B$ : singularities of $B$ and critical value of $\Phi$.
- $\Phi$ is Calabi-Yau fibration:

$$\forall y \in B \setminus S', \quad M_y := \Phi^{-1}(y) \text{ is a smooth Calabi-Yau mfd.}$$

- $\chi := \Phi^* \omega_{FS}$ : semi-positive closed $(1,1)$-form on $M$.
- $\Omega$ : smooth volume form on $M$ s.t. $\sqrt{-1}\partial\bar{\partial} \log \Omega = \chi$.
- Define $F$ on $M \setminus \Phi^{-1}(S')$. by:

$$F := \Omega \Big/ \binom{n}{m} \cdot \chi^m \wedge \omega_0^{n-m}$$

then $F$ is constant along each $\Phi^{-1}(y)$, $y \in B \setminus S'$. Hence $F$ is a smooth function on $B \setminus S'$.

• Song–Tian: the Monge–Ampère equation:

$$\left(\chi+\sqrt{-1}\,\partial\bar\partial v\right)^{\kappa} = F\cdot e^{v}\cdot \chi^{\kappa}$$

has a unique solution $v \in PSH(\chi) \cap C^{0}(B) \cap C^{\infty}(B\setminus S')$.

• $\omega_B$ is canonical metric on $B$:

$$Ric(\omega_B) = -\omega_B + \omega_{WP}, \text{ on } B\setminus S'.$$

• Song–Tian also proved weak convergence $\omega(t) \to \omega_B$.

• Reference metric: $\hat\omega(t) = e^{-t}\cdot\omega_0 + (1-e^{-t})\cdot\chi$, then:

$$\frac{\partial\varphi}{\partial t} = \log\frac{e^{(n-\kappa)t}\cdot\left(\hat\omega(t)+\sqrt{-1}\,\partial\bar\partial\varphi\right)^{n}}{\Omega} - \varphi, \quad \varphi\big|_{t=0} = 0.$$

• parabolic Schwarz Lemma: $tr_{\omega(t)}\chi \leq C(n,\omega_0)$.

• Song–Tian: $\exists\, C = C(n,\omega_0) < +\infty$, $\|R(t)\|_{L^{\infty}(M)} \leq C$.

• Many peopel has work here:

Fong, Zhang Zhou, Tosatti, Weinkove, Zhang. Z.L., ...

e.g.:

• Tosatti–Weinkove–Yang: $\|\omega(t)\|_{C^{0}(K,\hat\omega(t))} \leq C$, $K \subset\subset X\setminus S$.

• Fong, Lee, Chu: Higher-order estimate of $\omega(t)$.

• J-Song: $Diam(M, g(t)) \leq C$, $t\in[0,+\infty)$.

- Song-Tian's analytic Minimal Model Program:
- $X, Y$: projective varieties.
- rational map: $f: X \setminus V \to Y$, holomorphic map, $V$ is subvariety of $X$.

  identify two such map if they agree on $X \setminus W$, $W$ is subvariety of $X$.
- birational: $\exists$ rational maps $f: X \to Y$, $g: Y \to X$, s.t.

  $$f \circ g = \mathrm{id} \text{ as rational map.}$$

  In this case, we say $X$ and $Y$ are birational equivalent.
- Mori's Minimal Model Program: finding a "good" representative of a variety within its birational class.
- A "good" variety $X$ is one satisfying either:

  (i). $K_X$ is nef; or ← minimal model

  (ii). $\exists$ a holomorphic map $\pi: X \to Y$ to a lower dim. variety $Y$, s.t. the generic fiber $X_y = \pi^{-1}(y)$ is smooth Fano mfd, i.e., $K_{X_y} < 0$. ← Fano fiber space.
- (i) and (ii) are mutually exclusive.

- The rough ideal of MMP is to find a sequence of birational maps $f_1, \cdots, f_k$ and varieties $X_1, \cdots, X_k$,

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} X_2 \dashrightarrow \cdots\cdots \overset{f_k}{\dashrightarrow} X_k$$

so that $X_k$ is our "good" variety of either type (i) or (ii).

- Song-Tian: the Kähler-Ricci flow should construct the sequence of manifolds $X_1, \cdots, X_k$ of the MMP, and in the final step, after suitable normalization, converge to a canonical metric.
- Song-Tian also constructed weak solution for KRF through singularities by assuming MMP.
- General blow-up theory is largely open, except KRF on Fano mfds, in which case we have the Hamilton-Tian conjecture.

## §5. Hamilton-Tian Conjecture.

- $X$: Fano manifold of dim. $n$, i.e., $c_1(X) > 0$.
- Find $\omega_0 \in c_1(X)$ as our initial metric.

- The KRF:

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)), \quad \omega(0) = \omega_0,$$

has a solution on $[0,1)$, develops singularities at $t=1$.

- Renormalization: $s = -\ln(1-t)$, $\tilde{\omega}(s) = (1-t)^{-1}\cdot\omega(t)$, then:

$$\frac{\partial \tilde{\omega}(s)}{\partial s} = -\mathrm{Ric}(\tilde{\omega}(s)) + \tilde{\omega}(s), \quad \tilde{\omega}(0) = \omega_0, \quad s \in [0, +\infty).$$

- Hence we consider: for $t \in [0, +\infty)$,

$(*)_1$
$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) + \omega(t), \quad \omega(0) = \omega_0.$$

- Question: What is the limit of $(X, \omega(t))$ as $t \to +\infty$?

- Existence of KE metric:

  - $c_1(X) < 0$: Aubin, Yau.
  - $c_1(X) = 0$: Yau.

  In these two cases, we don't need extra assumption.

  But in the $c_1(X) > 0$ case, there are obstructions:

- Yau-Tian-Donaldson conjecture: A Fano manifold $X$ admits a Kähler-Einstein metric iff $X$ is K-stable.

  (Tian, CDS, ...).

- In general, we expect:

- Hamilton–Tian conjecture: For eqn $(*)_1$, as $t \to +\infty$, $(X, g(t))$ converges (at least along a subsequence) to a Kähler–Ricci soliton with mild singularities in the Gromov–Hausdorff sense.

- mild singularities: (i). singular set has codim $\geq 4$ in metric geometry.

  (ii). the limit is a normal variety.

- Resolution of Hamilton–Tian conjecture:

  - Perelman: $|R(t)| + \mathrm{Diam}(M, g(t)) \leq C$.

  - Tian–Zhu: assuming the existence of KE metric or soliton.

  - Tian–Zhang.Z.L.: in dim. $n \leq 3$, $\left( \int_X |\mathrm{Ric}(\omega(t))|^4 \, dt \leq C \right)$.

  - General case: Chen–Wang, R. Bamler.

## §5.1. Perelman's proof.

- $$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) + \omega(t) = \sqrt{-1} \cdot \partial\bar{\partial} u.$$

- What is $u$: $\omega(t) = \omega(0) + \sqrt{-1} \cdot \partial\bar{\partial} \phi(t)$, $u = \frac{\partial \phi}{\partial t}$

- Normalize $u$ so that $\int_X e^{-u} \cdot dg(t) = (2\pi)^n$.

- Evolution eqn. of $u$:

$$\partial_t u = \Delta u + u + a, \qquad a(t) = -(2\pi)^{-n} \cdot \int_X u \cdot e^{-u} \, dg(t).$$

- $a \geq -C$ : trivial.

- monotonicity of the entropy:

- Given $(M^n, g)$, $f \in C^1(M)$, $\tau > 0$, $u := (4\pi\tau)^{-\frac{n}{2}} \cdot e^{-f} \in P(M)$

  - $\mathcal{W}[g, f, \tau] := \int_M \left[\tau \cdot (|\nabla f|^2 + R) + f - n\right] \cdot u \, dg$.

  - $\mu[g, \tau] := \inf_f \mathcal{W}[g, f, \tau]$.

  - $\nu[g, \tau] := \inf_{0 < \tau' \leq \tau} \mu[g, \tau']$.

- $(M^n, g(t))$, $0 \leq t \leq T$, cpt $n$-dim. RF. $\square^* := -\partial_t - \Delta_{g(t)} + R$.

$u(t) = (4\pi\tau)^{-\frac{n}{2}} \cdot e^{-f} \in P(M)$, satisfies $\square^* u = 0$, along RF, then:

$$\frac{d}{dt} \mathcal{W}[g(t), f(t), T - t]$$

$$= \int_M 2\tau \cdot \left| Ric_{g(t)} + Hess^{g(t)} f - \frac{1}{2\tau} g(t) \right|^2 \cdot u \, dg(t) \geq 0.$$

• **Per**: the functions $t \mapsto \mu[g(t), T_0 - t]$, $t \mapsto \nu[g(t), T_0 - t]$

are non-decreasing functions.

• $R(t) = n - \Delta u$.

• **Upper bound of $a(t)$**:

$$-C = \nu[g(0), 1] \leq \mu[g(t), \tfrac{1}{2}]$$

$$\leq \int_X (2\pi)^{-n} \cdot [R + |\nabla u|^2 + u - 2n] \cdot e^{-u} \, dg(t)$$

$$= (2\pi)^{-n} \cdot \int_X [-\Delta u + |\nabla u|^2 + u - n] \cdot e^{-u} \, dg(t)$$

$$= -n + (2\pi)^{-n} \cdot \int_X u \cdot e^{-u} \, dg(t) = -n - a(t).$$

$$\Rightarrow a(t) \leq C.$$

• **Thm (Perelman)**. $R(g(t)) + \mathrm{Diam}(M, g(t)) \leq C$.

• ① Getting a lower bound on Ricci potential $u(t)$

• ② Compute: $(\partial_t - \Delta) \dfrac{|\nabla u|^2}{u + 2B}$ & $(\partial_t - \Delta) \dfrac{-\Delta u}{u + 2B}$,

the use the maximum principle to obtain:

- $|\nabla u|^2 \leq C \cdot (u + 2B)$.

- $-\Delta u \leq C \cdot (u+2B) \Rightarrow R(t) \leq C \cdot (u+2B)$.

- ③. From our normalization $(2\pi)^n = \int_X e^{-u} dg(t)$, $\mathrm{Vol}_{g(t)}(X) = \mathrm{Vol}_{g(0)}(X)$.
  $\uparrow$ const.

$$\min_X u(\cdot, t) \leq C.$$

From Step ②, $\left|\nabla (u+2B)^{\frac{1}{2}}\right| \leq C$, hence:

$$\mathrm{Diam}(M, g(t)) \leq C$$

$$\Rightarrow \max_X u(\cdot, t) \leq C$$

- ④ Contradiction argument to prove $\mathrm{Diam}(M, g(t)) \leq C$.

  - Fix $p \in M$.

  - Assume $\mathrm{Diam}(M, g(t_i)) \to +\infty$ as $i \to +\infty$.

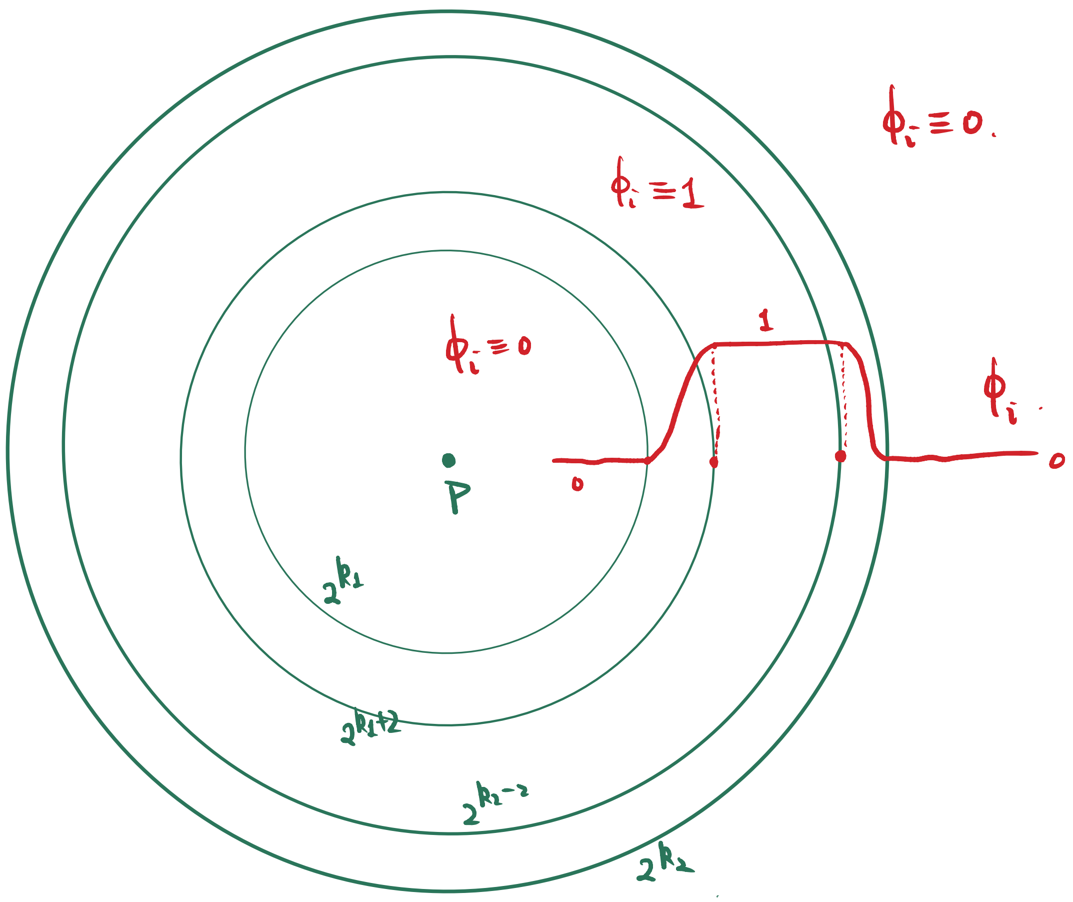
  - Find sequences $k_1^i < k_2^i$, tending to $+\infty$, s.t.

$$\begin{cases} \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i, k_2^i)\right) < \frac{1}{i}, \\ \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i, k_2^i)\right) \leq 2^{10n} \cdot \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i+2, k_2^i-2)\right) \end{cases}$$

$\mathrm{Vol}_{g(t)}(X) = \mathrm{Vol}_{g(0)}(X)$
$= \mathrm{const}.$

where $B_{g(t)}(k_1, k_2) := \{ z \in X : 2^{k_1} \leq d_t(z, p) \leq 2^{k_2} \}$.

and: $\displaystyle\int_{B_{g(t_i)}(k_1^i,k_2^i)} R(t_i)\, dg(t_i) \leq C \cdot \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i, k_2^i)\right).$




- Find such $\phi_i$, then find $C_i \in \mathbb{R}$, s.t.

$$u_i(x) := e^{C_i} \cdot \phi_i\left(d_{t_i}(x,p)\right) \quad \text{satisfies:} \quad (2\pi)^{-n} \int_X u_i^2 \, dg(t_i) = 1.$$

- Then:

$$(2\pi)^n = e^{2C_i} \cdot \int_X \phi_i^2 \, dg(t_i) \leq e^{2C_i} \cdot \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i, k_2^i)\right) \leq e^{2C_i} \cdot \frac{1}{i}$$

$$\Rightarrow \lim_{i \to \infty} C_i = +\infty.$$

- Again from the lower bound of the $\mu$-entropy:

$$-C = \nu[g(0), 1] \leq \mu[g(t_i), \tfrac{1}{2}]$$

$$\leq \mathcal{W}[g(t_i), 2\ln u_i, \tfrac{1}{2}]$$

$$= (2\pi)^{-n} \cdot e^{2C_i} \cdot \int_{B_{g(t_i)}(k_1^i, k_2^i)} \left[4\left|\phi_i'(d_{t_i}(p,\cdot))\right|^2 - 2\phi_i^2 \cdot \ln\phi_i\right] dg(t_i)$$

$$+ (2\pi)^{-n} \cdot \int_{B_{g(t_i)}(k_1^i, k_2^i)} R \cdot u_i^2 \, dg(t_i) - 2n - 2C_i$$

$$\leq \tilde{C} - 2C_i$$

where:

$$\int_{B_{g(t_i)}(k_1^i, k_2^i)} R \cdot u_i^2 \, dg(t_i) \leq e^{2C_i} \cdot \int_{B_{g(t_i)}(k_1^i, k_2^i)} R \, dg(t_i)$$

$$\leq e^{2C_i} \cdot C \cdot \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i, k_2^i)\right)$$

$$\leq e^{2C_i} \cdot C \cdot 2^{10n} \cdot \mathrm{Vol}_{g(t_i)}\left(B_{g(t_i)}(k_1^i + 2, k_2^i - 2)\right)$$

$$\leq C \cdot 2^{10n} \cdot \int_X u_i^2 \, dg(t_i) = C \cdot 2^{10n} \cdot (2\pi)^n.$$

- Hence we conclude:

$$-C \leq \tilde{C} - 2C_i \longrightarrow -\infty, \quad \text{as } i \to +\infty,$$

a contradiction!

$\Rightarrow \operatorname{Diam}(M, g(t)) \leq C$

$\Rightarrow \max_X u(\cdot, t) \leq C$

$\Rightarrow \max_X R(t) \leq C.$

- Combining $\max_X R(t) \leq C$, by Perelman's volume non-collapsing thm, $\forall\, 0<r<1$, $\forall\, p \in X$,

$$\operatorname{Vol}_{g(t)}(B_{g(t)}(p, r)) \geq C^{-1} \cdot r^{2n}.$$

But $\operatorname{Vol}_{g(t)}(X) = \operatorname{Vol}_{g(0)}(X)$ is a constant, hence for any seq. $t_i \to +\infty$, passing to a subseq.,

$$(X, d_{g(t_i)}) \xrightarrow{\text{Gromov–Hausdorff}} (X_\infty, d_\infty).$$

Hence the Hamilton–Tian conjecture remains the regularity theory of this GH convergence and the limit.

- In Bamler's approach, under uniform scalar curvature bound, we need to prove distance distortion estimate, forward and backward pseudolocality, and Gaussian heat kernel bounds, and extend Cheeger-Colding theory to this set-up.

## 6. Recent works of Bamler.

- Given a sequence of pointed, n-dim. Ricci flows

$$\left(M_i^n,\ (g_i(t))_{t\in[-T_i,0]},\ (x_i,0)\right),$$

with $\lim_{i\to\infty} T_i = T_\infty \in [0,+\infty]$, denote

$$d\nu_{x_i,0;t} := K(x_i,0;\cdot,t)\,dg_t \in \mathcal{P}(M_i).$$

heat kernel.

then we have:

$$\left(M_i,\ (g_i(t))_{t\in[-T_i,0]},\ d\nu_{x_i,0;t}\right) \xrightarrow[i\to\infty]{\mathbb{F}} \left(\mathcal{X},\ (\nu_{x_\infty;t})_{t\in[-T_\infty,0]}\right)$$

metric flow, parabolic version of metric space.

- non-collapsing condition: $\mathcal{N}_{x_i,0}(r_0^2) \geq -Y_0$.

  - Nash entropy $\mathcal{N}_{x_0,t_0}(\tau)$: $K(x_0,t_0;\cdot,t_0-\tau) =: (4\pi\tau)^{-\frac{n}{2}} e^{-f}$,

$$\mathcal{N}_{x_0,t_0}(\tau) := \int_M f\, d\nu_{x_0,t_0;t_0-\tau} - \frac{n}{2}.$$

normalized volume

  - $$e^{\mathcal{N}_{x_0,t_0}(r^2)} \approx \frac{\mathrm{Vol}_{t_0}(B(x_0,t_0,r))}{r^n}$$

- Structure of the limiting metric flow.

  - $\exists$ a regular-singular decomposition, such that: $\mathcal{X} = \mathcal{R} \sqcup \mathcal{S}$.

(1). $\mathcal{R}$ is a smooth Ricci flow spacetime, $(\mathcal{X}_t, d_t) = \overline{(\mathcal{R}_t, d_{g_t})}$.

(2). $\dim_{\mathcal{M}^*} \mathcal{S} \le (n+2)-4$. (optimal).

time has dim. 2.


(3). Tangent flows are (singular) gradient shrinking solitons.

(4). $\exists$ a filtration $\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n-2} = \mathcal{S}$ similar to the theory of Cheeger, Colding, Naber, Tian.

• Further questions:

with surgery?

(1). Can we construct 4-dim. Ricci flow through singularities?

• 4-dim smooth Poincaré Conjecture. (Bamler. arXiv: 2102.12615.)

topological: Freedman.

• 11/8 - Conjecture.

• Every closed, spin 4-mfd $M$ satisfies

$$b_2(M) \ge \frac{11}{8} \cdot |\sigma(M)|.$$

• Missing piece in the classification of closed, simply connected, smooth 4-mfd up to homeomorphy.

(Donaldson, Freedman, Kirby).