• Perelman's Ricci flow theory

§1. Introduction.

• **Poincaré Conjecture** Every closed, smooth, simply connected 3-mfd is diffeomorphic to S^3. #

• **Rmk**: In dim. 3, every homeomorphism can be approximated by diffeomorphisms.

• More general, we have:

• **3-dim. Space Form Conjecture** Let M be a closed, connected 3-manifold, and suppose that the fundamental group of M is a free product of finite groups and infinite cylic groups. Then: M is diffeomorphic to spherical space forms, copies of $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$, or a connect sum of them. #

• The proof depends on the following long-time existence result for Ricci flow with surgery.

• **Theorem (Perelman).** • (M, g_0): closed Riem. 3-manifold.

• Suppose there is no embedded, locally separating $\mathbb{RP}^2$ contained in M.

- Then: there is a Ricci flow with surgery defined for all $t \in [0, +\infty)$, with initial metric (M, g_0).
- The set of discontinuity times for this Ricci flow with surgery is a discrete subset of $[0, +\infty)$.
- Topological change in the 3-manifold when we cross a surgery time:
    - do connect sum decomposition
    - removal of connected components diffeomorphic to:

      Spherical space forms, $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $\mathbb{RP}^3 \# \mathbb{RP}^3$. #

- For conjectures here, we don't need to analyze the nature of the flow as $t \to +\infty$.
- Theorem (Perelman). • M: closed 3-manifold.
- Assume $\pi_1(M)$ is a free product of finite groups and infinite cylic group.
- Then: for any Riem. metric g_0 on M, the Ricci flow with surgery start with g_0 in the above thm becomes extinct after some time $T < +\infty$, i.e., $M_t = \phi$ for all $t \geq T$. #
- Picture here:

Starting with (M, g_0), we have a seq. of Ricci flows:

$$(M_1, (g_1(t))_{t\in[0,T_1]}),\ (M_2, (g_2(t))_{t\in[T_1,T_2]}), \cdots$$

with $[0,+\infty) = [0,T_1]\cup[T_1,T_2]\cup\cdots\cup[T_k,T_{k+1}]\cup\cdots$, and

$$(M_i, g_i(T_i)) \xrightarrow{\text{surgery process}} (M_{i+1}, g_{i+1}(T_i))$$

If $M_k=\phi$, then M_0 is connect sum of

spherical space forms, $\mathbb{S}^2\times\mathbb{S}^1$, $\mathbb{S}^2\tilde{\times}\mathbb{S}^1$, $\mathbb{RP}^3\#\mathbb{RP}^3$.

• §2. Perelman's space-time geometry ($\mathcal{L}$-geometry).

• $(\mathcal{M}, G)$: n-dim. generalized Ricci flow (locally Ricci flow).

e.g.: $\mathcal{M} = M\times[0,T]$, $G(t) = g(t)$, $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t))$.

• **Def.** (Space-time curve). $t\uparrow$, $\tau = T-t\downarrow$.

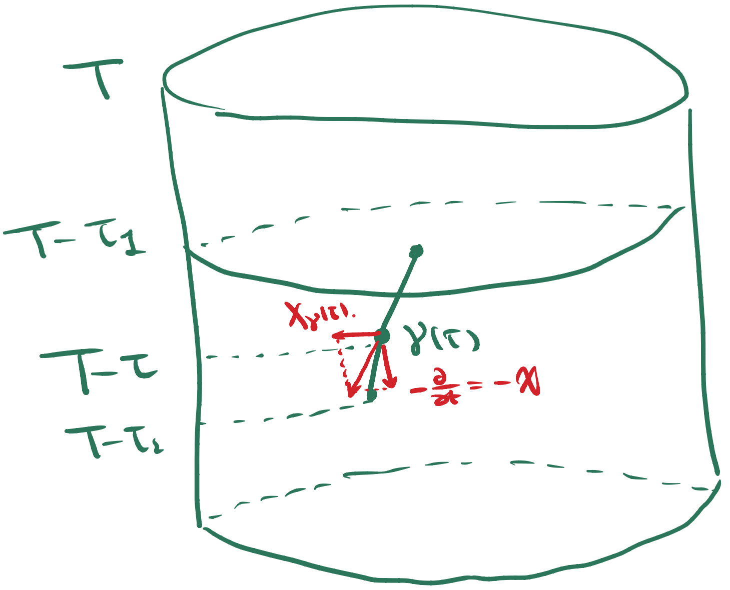
• $0\le\tau_1<\tau_2$, $\gamma: [\tau_1,\tau_2]\to\mathcal{M}$ be a continuous map, is called a space-time curve.

• γ is called parameterized by backward time, if:

$$\gamma(\tau)\in M_{T-\tau},\quad \tau\in[\tau_1,\tau_2].$$

• **Def.** • $\gamma: [\tau_1,\tau_2]\to\mathcal{M}$, a C^1-path parametrized by backward time.

• $\frac{d\gamma}{d\tau} = -\mathcal{X} + X_\gamma(\tau)$. $X_\gamma(\tau)\in\mathcal{H}T\mathcal{M}$ is the horizontal component.



or write:

$$\gamma(\tau)=\left(\tilde{\gamma}(\tau),\ T-\tau\right).$$

then:

$$\frac{d\gamma}{d\tau}=\Big(\underbrace{\dot{\tilde{\gamma}}(\tau)}_{X_{\gamma(\tau)}},\ \underbrace{-1}_{-\mathcal{X}}\Big).$$

• The $\mathcal{L}$-length of γ is defined by:

$$\mathcal{L}(\gamma)=\int_{\tau_1}^{\tau_2}\sqrt{\tau}\cdot\left(R(\gamma(\tau))+\left|X_{\gamma(\tau)}\right|^2_{G_{T-\tau}}\right)d\tau.$$

• Space parametrization: $s:=\sqrt{\tau}$, $A(s):=$ horizontal component of $\frac{d\gamma}{ds}$, then:

$$A(s)=2s\cdot X(s^2),\quad \mathcal{L}(\gamma)=\int_{\sqrt{\tau_1}}^{\sqrt{\tau_2}}\left(2\cdot R(\gamma(s))\cdot s^2+\frac{1}{2}\cdot|A(s)|^2_{G_{T-s^2}}\right)ds.$$

• $\mathcal{L}$-geodesics: critical points of $\mathcal{L}$-functional.

• Euler-Lagrange equation ($\mathcal{L}$-geodesic equation):

$$\nabla_X X-\frac{1}{2}\cdot\nabla R+\frac{1}{2\tau}\cdot X+2\,\mathrm{Ric}(X,\cdot)^*=0.$$

• Def. An $\mathcal{L}$-geodesic is said to be minimizing, if there is no curve parametrized by backward time, with the same endpoints and smaller $\mathcal{L}$-length.

• Consider a family $\gamma(\tau,u)$ of $\mathcal{L}$-geodesics parametrized by u and $\tau\in[\tau_1,\tau_2]$, then: $Y(\tau) := \frac{\partial}{\partial u}\gamma(\tau,u)\Big|_{u=0}$ satisfies the $\mathcal{L}$-Jacobian equation:

$$Jac(Y) := \underset{\substack{\uparrow \\ \text{2nd order}}}{\nabla_X}\nabla_X Y + R(Y,X)X - \frac{1}{2}\nabla_Y(\nabla R) + \frac{1}{2\tau}\nabla_X Y$$
$$+ 2(\nabla_Y Ric)(X,\cdot)^* + 2Ric(\nabla_X Y,\cdot)^* = 0.$$

- If $\tau_1 > 0$, then $\exists!$ solution Y along γ solve this eq^n, with $Y(\tau_1) = 0$, $\nabla_X Y(\tau_1)$ is given; similar for τ_2. $\left(\tau_1 = 0, \ \lim_{\tau\to 0}\sqrt{\tau}\cdot\nabla_X Y\right)$.

Proposition. γ: minimizing $\mathcal{L}$-geodesic, $\tau \in [\tau_1, \tau_2]$.

- Y: vector field along γ s.t. $Y(\tau_1) = Y(\tau_2) = 0$
- $\gamma_u(\tau) = \gamma(\tau, u)$: family of curves parameterized by backward time.
- $\frac{\partial}{\partial u}\gamma_u(\tau)\Big|_{u=0} = Y(\tau)$.

Then:

$$\frac{d^2}{du^2}\mathcal{L}(\gamma_u)\Big|_{u=0} = -\int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\langle Jac(Y), Y\rangle\, d\tau \geq 0.$$

- This quantity vanishes iff Y is a $\mathcal{L}$-Jacobi field. #

- Fix $\tau_1 \geq 0$, $x \in \mathcal{M}$ with $t(x) = T - \tau_1$. Then $\forall Z \in T_x M_{T-\tau_1}$, $\exists$ a maximal $\mathcal{L}$-geodesic γ_Z, defined on some $[\tau_1, \tau_2]$, $\tau_2 \leq +\infty$, with:

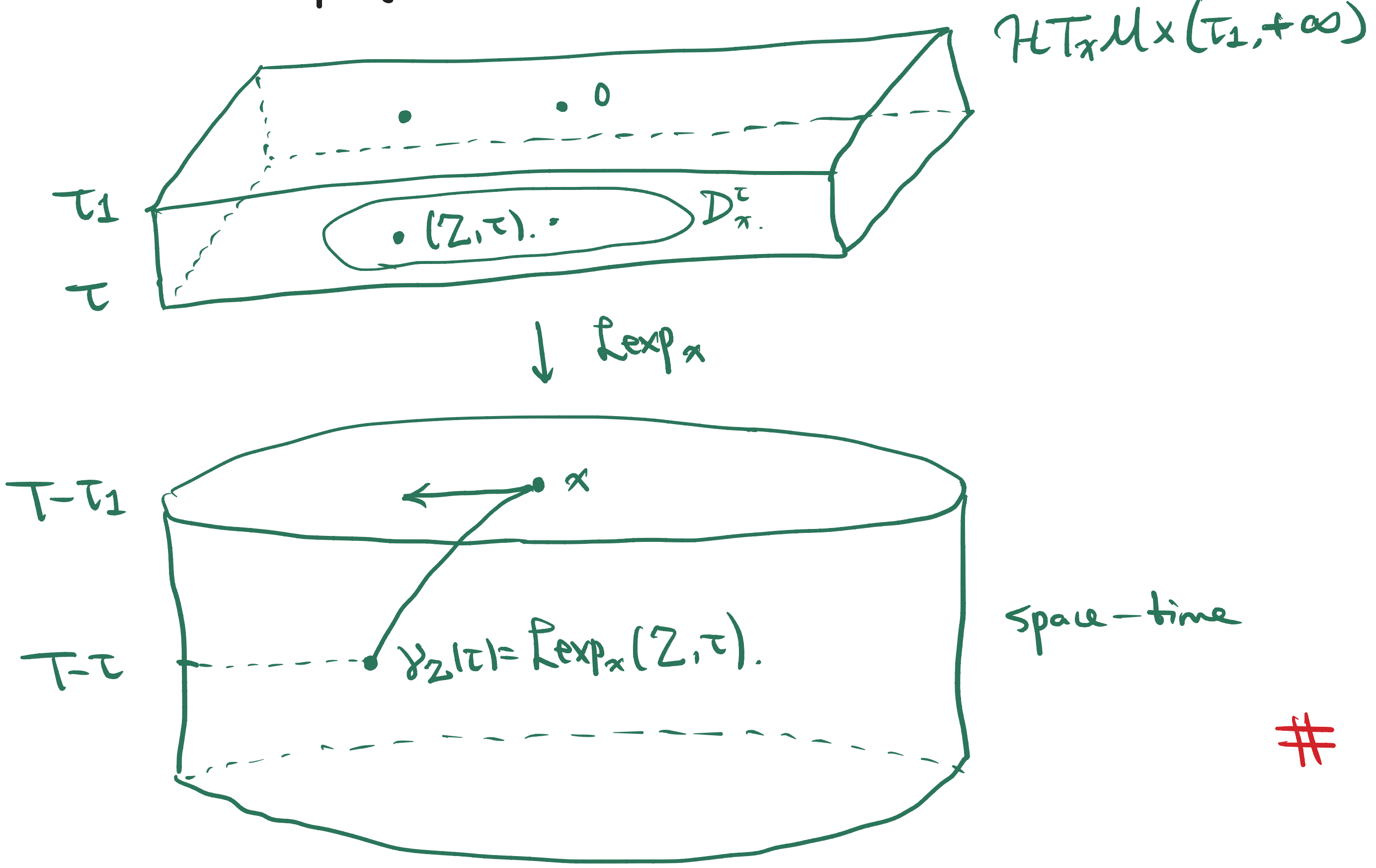
$$\gamma_Z(\tau_1) = x, \quad \sqrt{\tau_1}\cdot X(\tau_1) = Z \quad \left(\text{If } \tau_1 = 0, \ \lim_{\tau\to 0}\sqrt{\tau}\cdot X(\tau) = Z\right).$$

Def. • the domain of the def. of $\mathcal{L}\exp_x^{\tau}$, denoted by D_x, to be:

$(Z, \tau) \in T_x M_{T-\tau_1} \times (\tau_1, +\infty)$, s.t. τ is in the maximal domain of γ_Z.

• Then we define: $\mathcal{L}\exp_x : D_x \longrightarrow \mathcal{M}$ by:

$$\mathcal{L}\exp_x(Z, \tau) := \gamma_Z(\tau).$$



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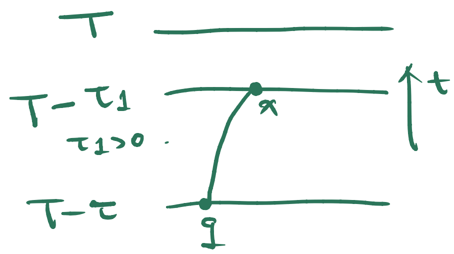
• $\tilde{\mathcal{U}}_x \subseteq D_x$: injectivity domain. $\mathcal{U}_x := \mathcal{L}\exp_x(\tilde{\mathcal{U}}_x) \subseteq \mathcal{M}$.

• Assume $|Rm| \leq C$ globally : $\mathcal{U}_x$ open dense and has full measure, we have smooth inequalities on $\mathcal{U}_x$, and weak inequalities globally.

• **Def.** $L_x : \mathcal{U}_x \longrightarrow \mathbb{R}$, assigns to each $q \in \mathcal{U}_x$ the $\mathcal{L}$-length of the minimizing $\mathcal{L}$-geodesic from x to q.

Def. We define the ℓ-reduced length from x by:

$$\ell_x(q) := \frac{L_x(q)}{2\sqrt{\tau}} : \mathcal{U}_x \longrightarrow \mathbb{R}$$



where $\tau = T - t(q)$.

R.K. ℓ_x is easier to work with since it is scale invariant when $\tau_1 = 0$,

Per: $$K(x,t;y,s) \gtrsim \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \cdot e^{-\ell_{(x,t)}(y,s)} \sim \frac{d_s^2(x,y)}{t-s}$$

Theorem (Perelman). Assume $x \in M_T$ (i.e., $\tau_1 = 0$). Then for $q \in \mathcal{U}_x^{\bar\tau}$:

① $$\frac{\partial \ell_x}{\partial \tau}(q) + \Delta \ell_x^{\bar\tau}(q) - \frac{\boxed{\frac{n}{2}} - \ell_x^{\bar\tau}(q)}{\bar\tau} \leq 0.$$

② $$\frac{\partial \ell_x}{\partial \tau}(q) - \Delta \ell_x^{\bar\tau}(q) + \left|\nabla \ell_x^{\bar\tau}(q)\right|^2 - R(q) + \frac{n}{2\bar\tau} \geq 0$$

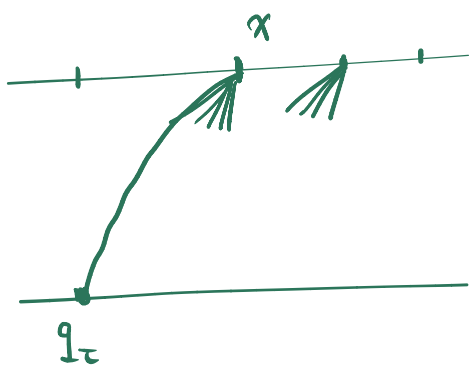
$\Updownarrow$

③ $$2\Delta \ell_x^{\bar\tau}(q) - \left|\nabla \ell_x^{\bar\tau}(q)\right|^2 + R(q) + \frac{\ell_x^{\bar\tau}(q) - n}{\bar\tau} \leq 0.$$ #

Globally, ① is true in the barrier sense. This leads to:

$\forall\ 0 < \tau < T$, $\exists\ q_\tau \in M \times \{T - \tau\}$, s.t.

$$\ell_x(q_\tau, \tau) \leq \frac{n}{2}.$$



· **Theorem (Non-collapsing)** · Given $\bar{\tau}_0 < +\infty$, $l_0 < +\infty$, $V > 0$

· $\exists$ const. $\kappa > 0$, depending on $\bar{\tau}_0$, l_0, V, n, s.t. the foll. holds:

· $(\mathcal{M}, G)$: generalized RF, n-dim, $0 < \tau_0 \leq \bar{\tau}_0$,

· $x \in M_T$, $0 < r \leq \sqrt{\tau_0}$ are given. Assume:

(1). $B(x, T, r) \subset M_T$ has compact closure

(2). $\exists$ an embedding $P(x, T; r, -r^2) \subset \mathcal{M}$ compatible

with time and vector field.

(3). $|Rm| \leq r^{-2}$, on $P(x, T; r, -r^2)$.

★(4). $\exists$ an open subset $\widetilde{W} \subset \widetilde{\mathcal{U}}_x(\tau_0) \subset T_x M_T$, s.t.: $\forall Z \in \widetilde{W}$,

let $\gamma_Z : [0, \tau_0] \to \mathcal{M}$ be $\mathcal{L}$-geodesic with initial value Z,

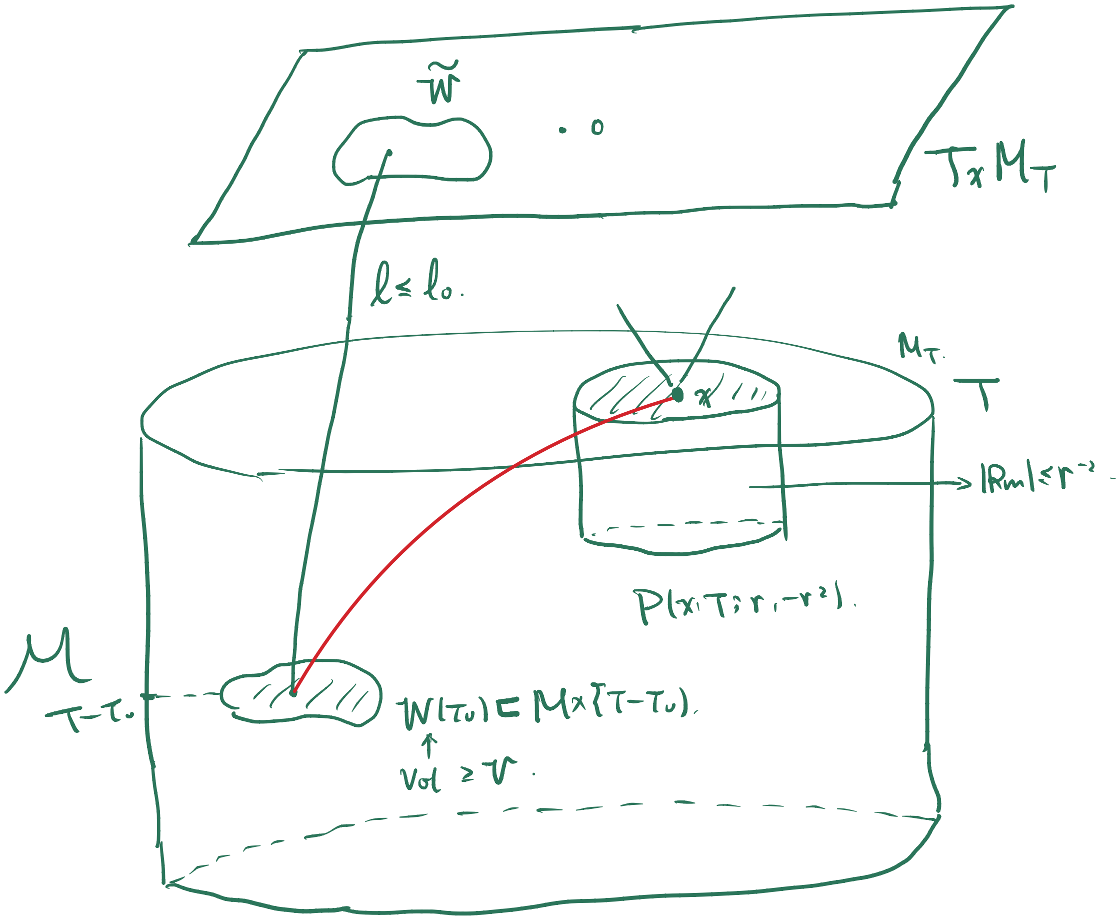
then: $l(\gamma_Z) \leq l_0$.

★(5). For each $\tau \in [0, \tau_0]$, let $W(\tau) := \mathcal{L}\exp_x^{\tau}(\widetilde{W})$, then:

$$\mathrm{Vol}_{g(T-\tau_0)}(W(\tau_0)) \geq V.$$

· **Then**:

$$\mathrm{Vol}_{g(T)}(B(x, T, r)) \geq \kappa \cdot r^n.$$



- Good example for general paradigm.

- <u>Theorem</u>. Fix $v_0 > 0$, $T_0 < +\infty$, $\exists\, \kappa = \kappa(v_0, T_0) > 0$, s.t.

- $(M, g(t))$, $0 \le t \le T \le T_0$, 3-dim RF on cpt mfd M.

- Assume $|Rm|(p,0) \le 1$. $Vol_{g(0)}\big(B(p,0,1)\big) \ge v_0$, for all $p \in M$.

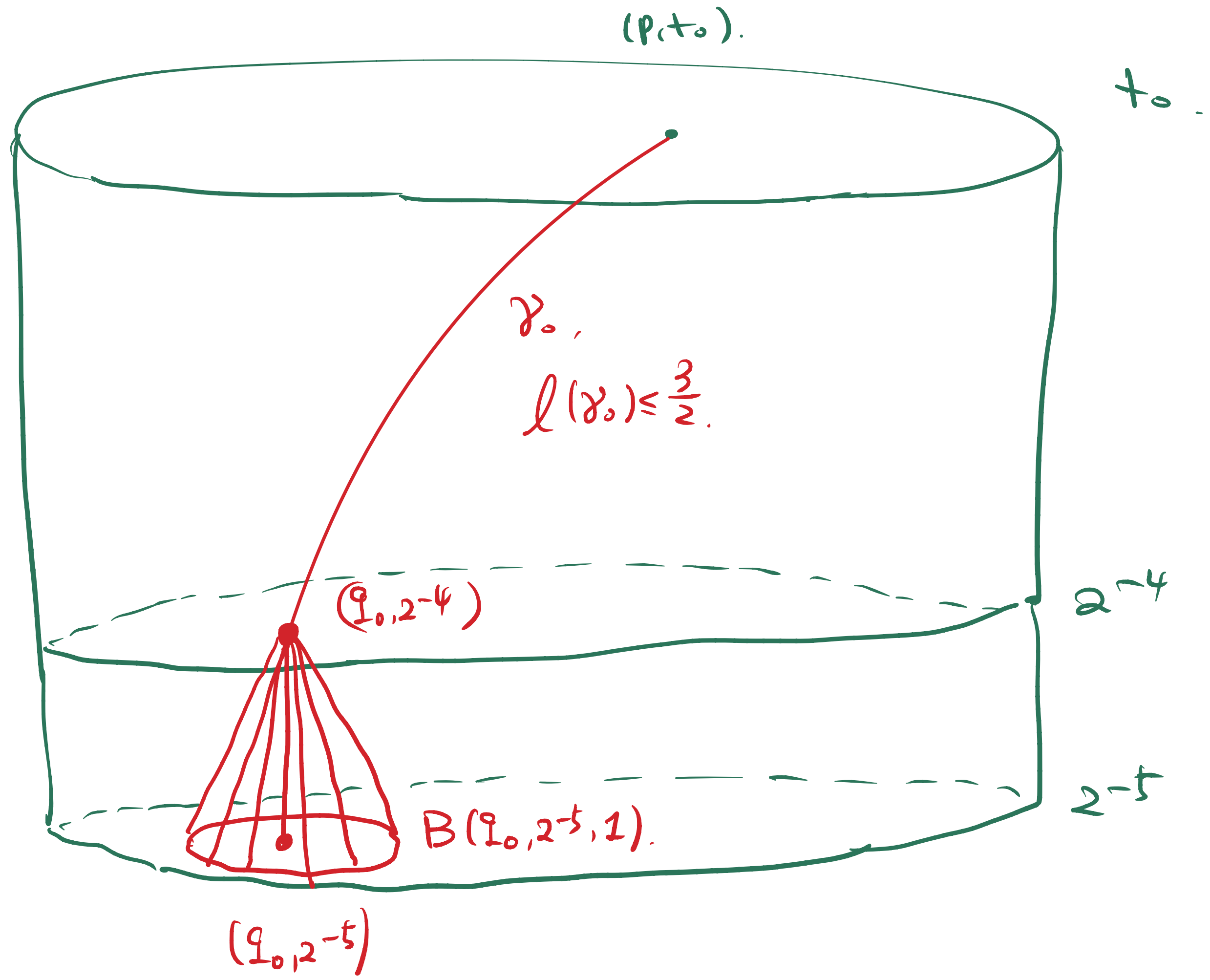
- <u>Then</u>: $\forall\, t_0 \le T$, $r > 0$ with $r^2 \le t_0$, $(p, t_0) \in M \times \{t_0\}$, if:

$$|Rm| \le r^{-2} \quad \text{on} \quad P(p, t_0; r, -r^2),$$

$\Leftarrow$ κ-non-collapsed

then:

$$Vol_{g(t_0)}\big(B(p, t_0, r)\big) \ge \kappa \cdot r^3.$$



· §3. Classification of gradient shrinking soliton.

- Fix $\kappa>0$. $\boxed{t\in(-\infty,0]\text{, complete, non-flat, } Rm\ge 0\text{. } \kappa\text{-non-collapsed.}}$
- $(M, g(t))$, $-\infty<t\le 0$, n-dim. κ-solution.
- Fix $p\in M$, $x=(p,0)\in M\times(-\infty,0]$.
- $\forall\,\tau>0$, $\exists$ a point $q_\tau\in M$ s.t. $\ell_x(q_\tau,\tau)\le\frac{n}{2}$.
- For $\bar\tau>0$, define

blow-down.

$$g_{\bar\tau}(t):=\frac{1}{\bar\tau}\cdot g(\bar\tau\cdot t),\quad -\infty<t\le 0.$$

$t=-1$ for $g_{\bar\tau}$
$\Updownarrow$
$t=-\bar\tau$ for g.

Theorem. Let $\{\bar{\tau}_k\}_{k=1}^{\infty}$ satisfies $\bar{\tau}_k \to +\infty$ as $k \to +\infty$.

- Passing to a subseq., we have:
  - For each $k \in \mathbb{N}^+$, consider $\left(M_k, g_k(t), (q_k, -1)\right)$, $-\infty < t \leq 0$,

$$M_k = M, \quad g_k(t) = g_{\bar{\tau}_k}(t), \quad q_k = q_{\bar{\tau}_k}$$

we have.

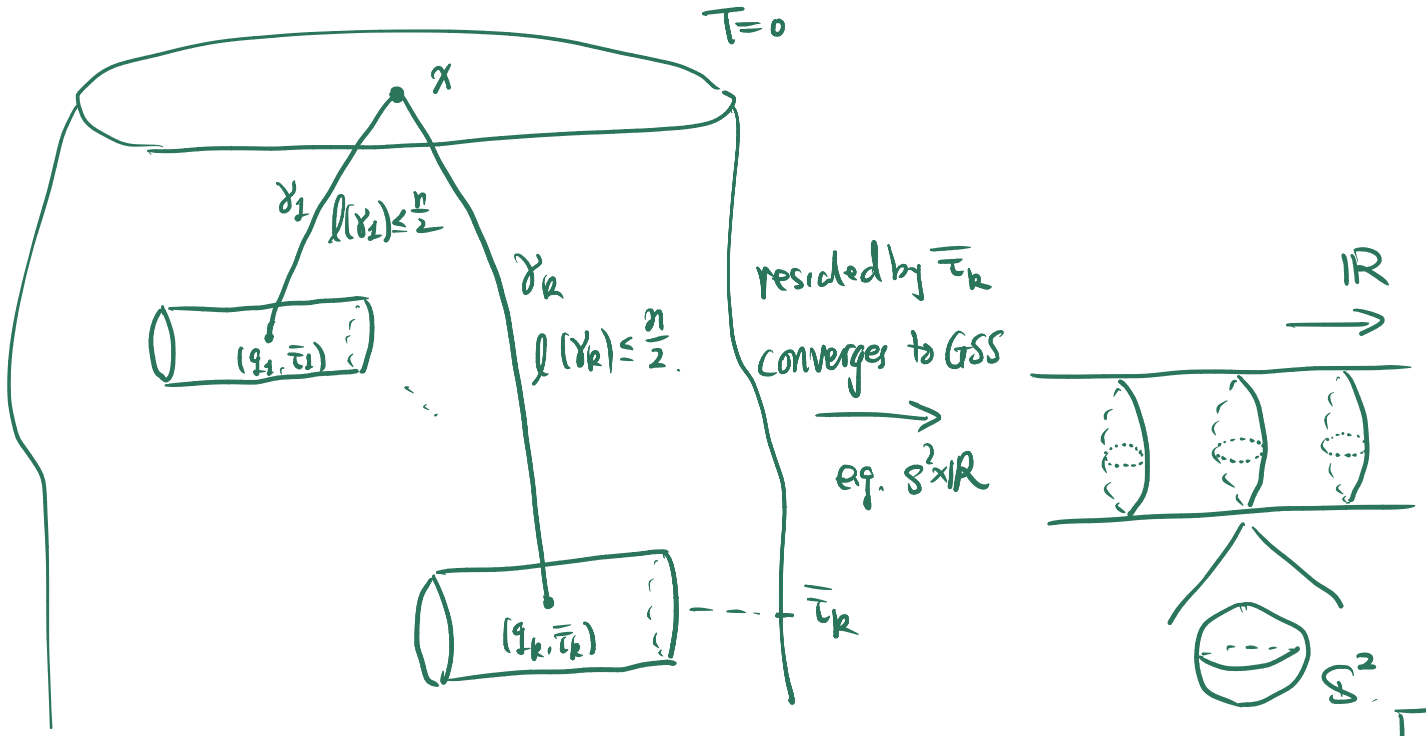
blow-down limit.

$$\left(M_k, g_k(t), (q_k,-1)\right) \xrightarrow[t\in(-\infty,0)]{C^\infty\text{-Cheeger-Gromov}} \left(M_\infty, g_\infty(t), (q_\infty,-1)\right).$$

- gradient shrinking soliton eqn.: $\exists\ f: M_\infty \times (-\infty, 0) \to \mathbb{R}$, smooth, s.t.

$$Ric_{g_\infty(t)} + Hess^{g_\infty(t)} f(t) + \frac{1}{2t} \cdot g_\infty(t) = 0.$$

- $(M_\infty, g_\infty(t))$ has non-negative curvature operator, is κ-non-collapsed, and satisfies $\partial R_{g_\infty}(x,t)/\partial t \geq 0$ for all $x \in M_\infty$, $t < 0$.



• In dim. 2,3, we have the following classification:

• **Theorem** • (M,g): complete, non-flat Riem. mfd of bounded, non-negative curvature of dim. 2 or 3.

• (M,g) is κ-non-collapsed, for some $\kappa>0$.

• Suppose $\exists$ a C^2-function $f: M \to \mathbb{R}$ s.t.

$$Ric_g + Hess^g(f) = \frac{1}{2} g,$$

• **Then**: $\exists$ a Ricci flow $(M, G(t))$, $-\infty < t < 0$, with $G(-1) = g$, and $(M, G(t))$ is isometric to $(M, |t| \cdot g)$ for every $t<0$.

• **In addition**, $(M, G(t))$ is one of the foll. three types:

(1). The flow $(M, G(t))$, $-\infty < t < 0$, is a shrinking family of compact, round (constant positive curvature) mfds. ($S^2, S^3, \mathbb{RP}^2, \mathbb{RP}^3$).

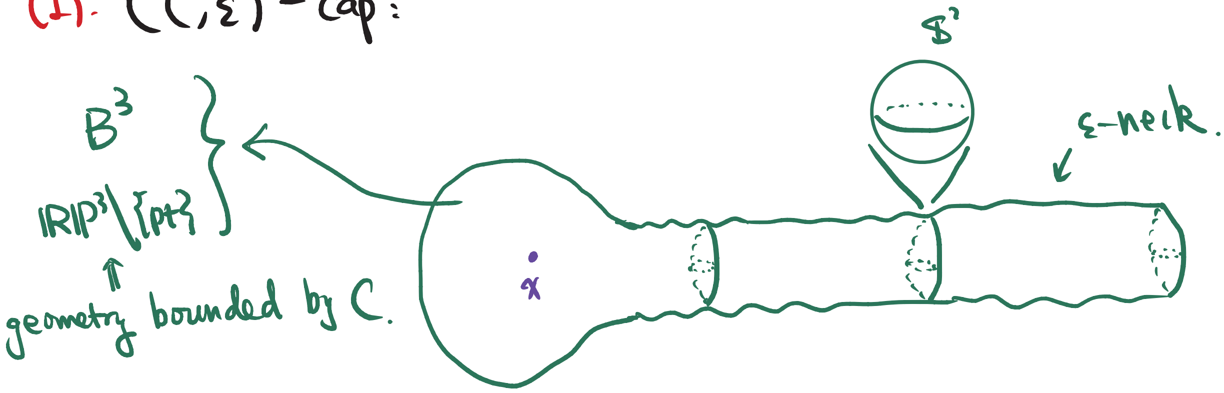
(2). The flow $(M, G(t))$, $-\infty < t < 0$, is a product of a shrinking family of 2-surfaces with $(\mathbb{R}, ds^2)$.

$S^2, \mathbb{RP}^2$.

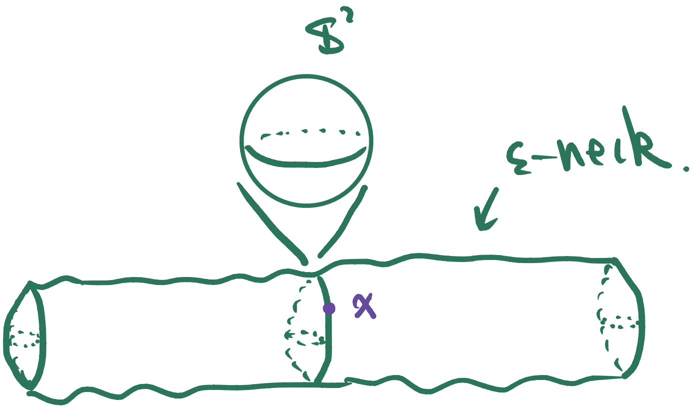
(3). Quotient of $S^2 \times \mathbb{R}$ in (2) under involution map ι.

§4. Canonical nbhds. (nbhd of singularities).

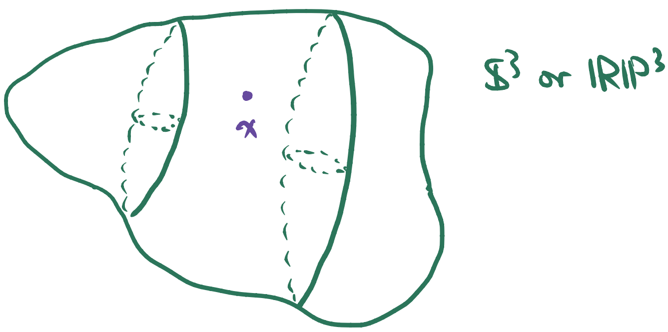
(1). (C,ε)-cap:



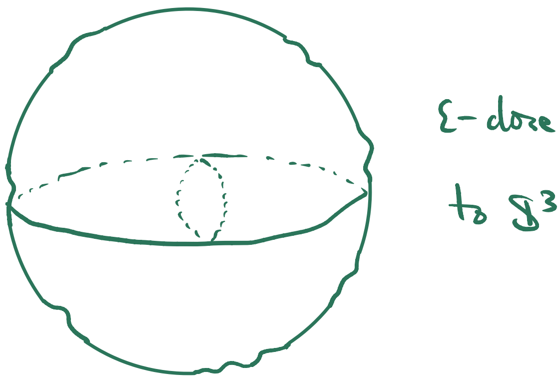
(2). ε-neck.



(3). C-component:



(4). ε-round



- (C,ε)-canonical assumption: For a function $r(t) \to 0^+$ as $t \to +\infty$, for any $(x,t) \in \mathcal{M}$ with $R(x,t) \geq r(t)^{-2}$, (x,t) has a (C,ε)-nbhd.

- Need to show: this assumption stays true if we construct the Ricci flow with surgery in suitable way.

• With (C,ε)-canonical nbhd assumption, we can apply:

• **Theorem. (Bounded curvature at bounded distance).**

• Fix $\varepsilon>0$, $C<+\infty$, $A<+\infty$.

• $\exists\ D_0<+\infty$, $D<+\infty$, depending on ε, C, A, s.t.:

• Assume $(\mathcal{M}, G)$ is a generalized RF with $t\in[0,+\infty)$, $x\in\mathcal{M}$, with:

(1). $(\mathcal{M}, G)$ has curvature pinched toward positive.

(2). For every $y\in\mathcal{M}$ with $R(y)\geq 4R(x)$ and $t(y)\leq t(x)$,

y has a (C,ε)-canonical nbhd.

contradiction argument to

• Then: if $R(x)\geq D_0$, then:

$$R(y)\leq D\cdot R(x), \text{ for all } y\in B(x,t,A\cdot R(x)^{-\frac{1}{2}}).$$

• Rmk: With this thm, we can apply Hamilton's compactness thm.

• About proof:

contradiction argument to extract a non-flat Gromov-Hausdorff limit on a cone.

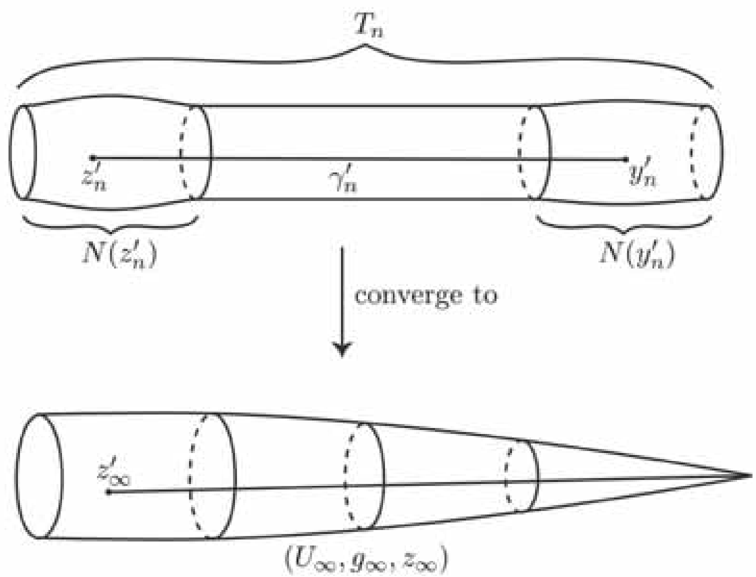


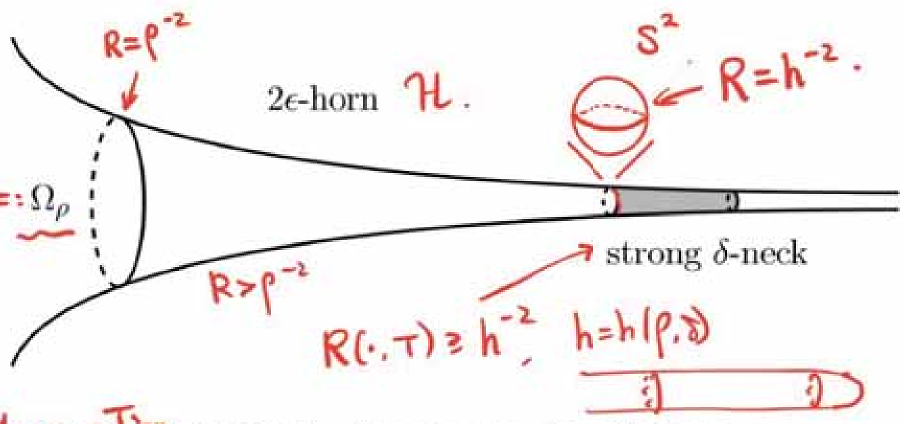
FIGURE 1. Limiting tube.

· Existence of ε-neck in non-trivial components.

· Define $\Omega := \{x \in M : \liminf_{t \nearrow T} R(x,t) < \infty\} \overset{\text{open}}{\subset} M$, then $g(t) \xrightarrow{C^\infty_{loc}(\Omega)} g(T)$

proper function in x.

$\{x \in \Omega : R(x,T) \leq \rho^{-2}\} =: \Omega_\rho$



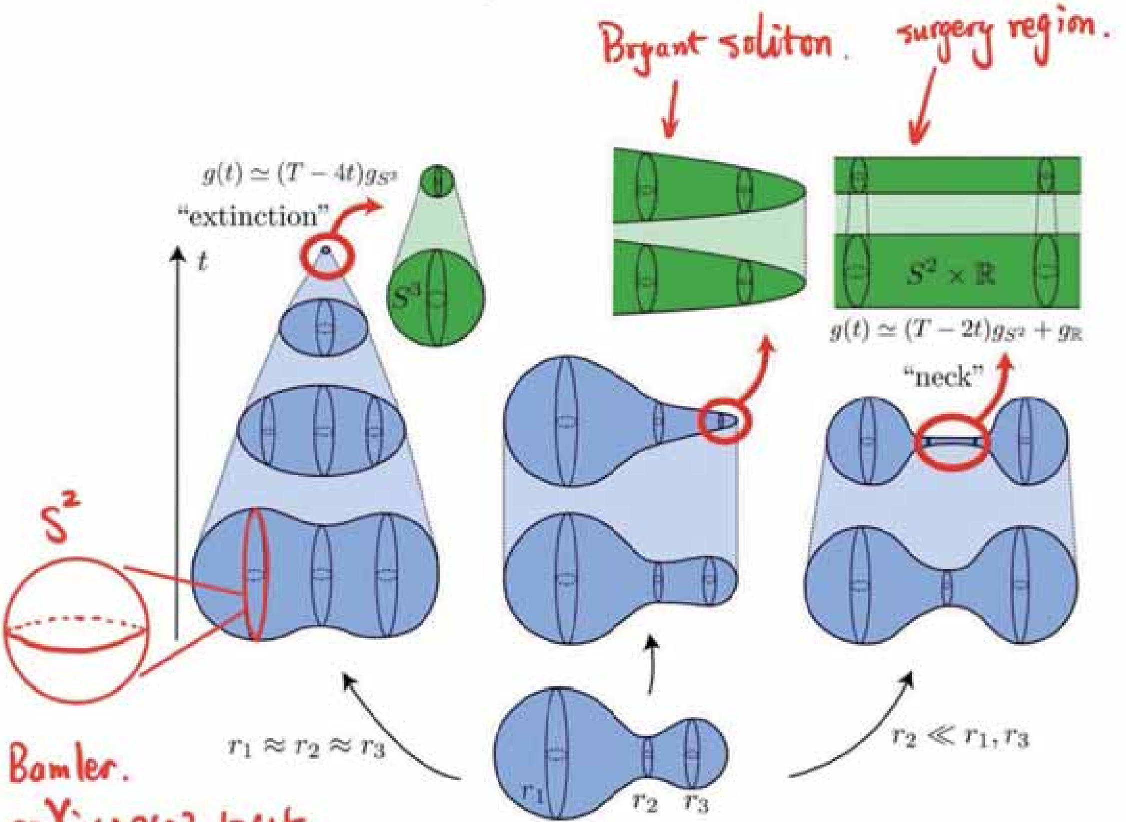
· The ends of $(\Omega, g(T))$:

finitely many.

Morgan-Tian FIGURE 4. δ-necks deep in a 2ε-horn.

· Typical example. Ricci flow with rotationally symmety

· Bryant soliton on $\mathbb{R}^3$: $g_{Bry} = f^2(r) \cdot g_{S^2} + dr^2$, $f(r) \sim r^{\frac{1}{2}}$.



Bamler.
arXiv: 2102.12615.

• Standard solution: (gluing piece).

· Fix a non-negatively curved, rotationally symmetric metric on $\mathbb{R}^3$, which is isometric near infinity to $S^2\times[0,\infty)$, with round metric of $R=1$ on S^2.

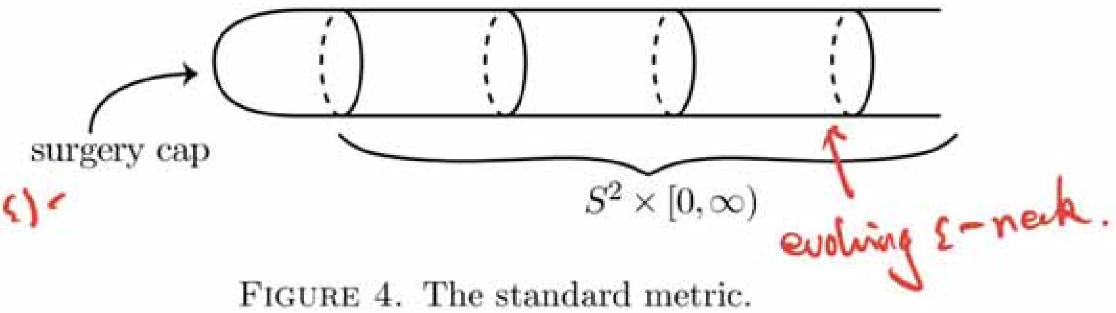


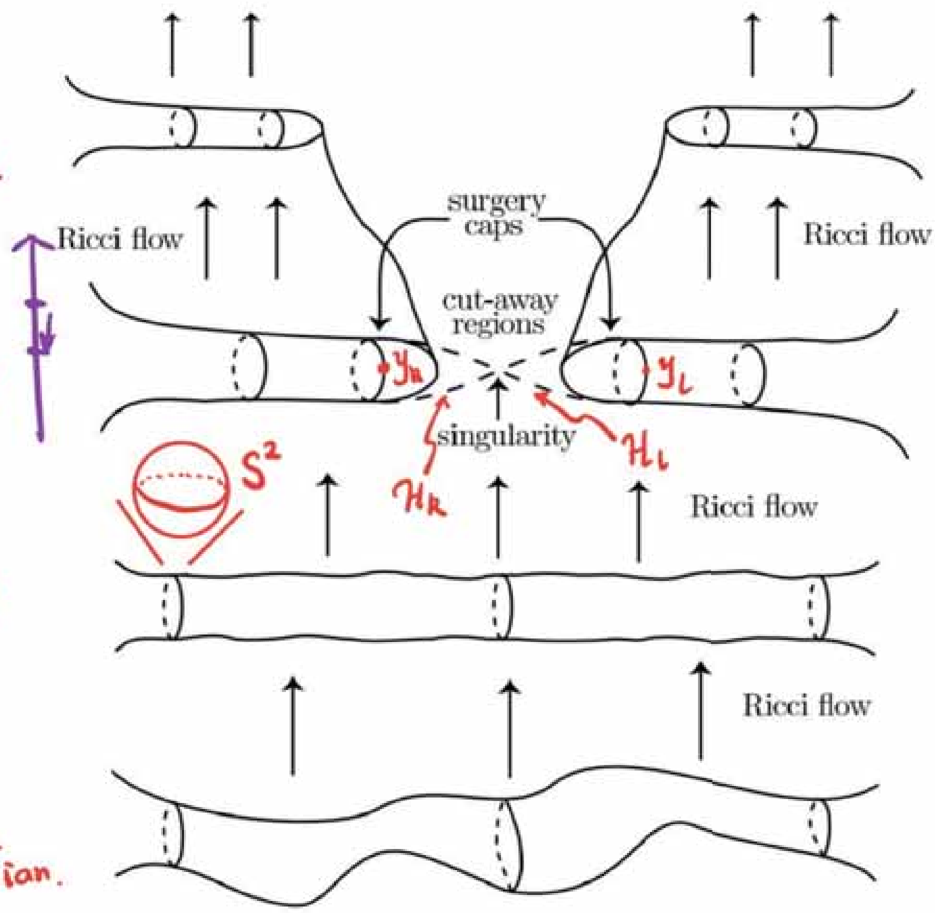
FIGURE 4. The standard metric.

• §5. Ricci flow with surgery.

· surgery process: M_{i+1}.

surgery time $\longrightarrow$ T_i

Ricci flow M_i



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• A 3-dim. Ricci flow with surgery consists a sequence of Ricci flows:

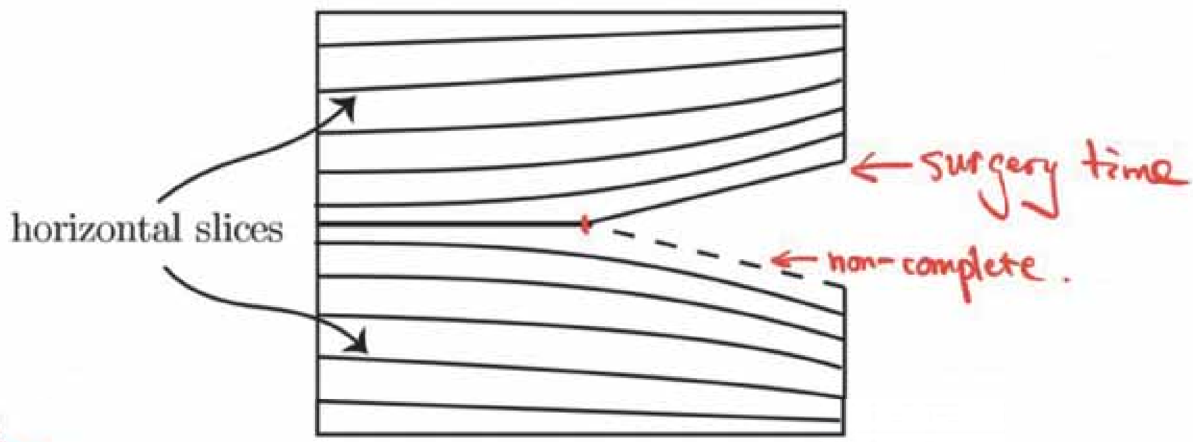
$$\left(M_1, (g_1(t))_{t\in[0,T_1]}\right), \quad \left(M_2, (g_2(t))_{t\in[T_1,T_2]}\right), \cdots\cdots$$

with $[0,\infty) = [0,T_1]\cup[T_1,T_2]\cup\cdots$, and

$$\left(M_i, g_i(T_i)\right) \xrightarrow{\text{surgery process}} \left(M_{i+1}, g_{i+1}(T_i)\right)$$

• Find $\mathcal{H}_1, \cdots, \mathcal{H}_j$ the 2ε-horns as above, $h_i \ll \rho_i \ll r_i \to 0$.

find $y_k \in \mathcal{H}_k$ with $R(y_k) = h_i^{-2}$, cut the cross-sectional 2-spheres through y_k, glue in cap-shaped 3-disks to this cutting surface.



Morgan-Tian.

FIGURE 5. Model for singularities in space-time.

• This is a generalized Ricci flow.

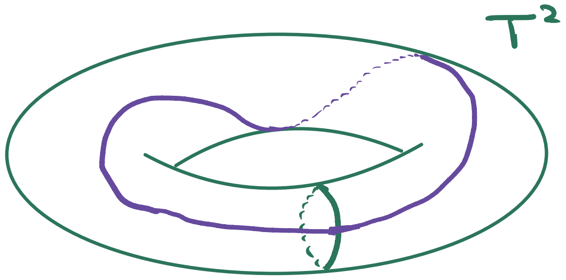
• Inductive conditions:

(1) curvature pinched toward positive.

(2) κ_i-non-collapsed on scale $\leq r_i$ of $(M_i, g_i(t))$, $T_{i-1}\leq t\leq T_i$.

(3) (C,ε)-canonical nbhd assumption.

- **Why S^2 is important:**
  - surgery along S^2 produces the connect sum decomposition.
  - Dehn surgery along tori can destroy the topology.



- **Perelman:** Let (M, g) (orientable.) be a closed 3-mfd, if r_i ($\downarrow 0$) are chosen sufficiently small, then:

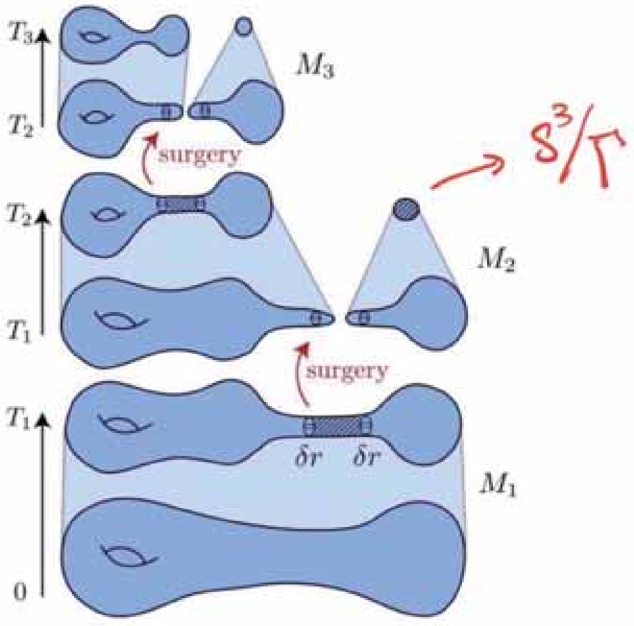
(1). the surgery times T_i do not accumulate.

(2). a Ricci flow with surgery with initial condition

$$(M_1, g_1(0)) = (M, g)$$ can be constructed.

- **Change in topology:** for any i,

$$M_1 \approx M_i \#_{j=1}^{k} (S^3/\Gamma_j) \# m\,(S^2 \times S^1).$$



Bamler.
arXiv: 2102.12615.

Figure 3: A schematic depiction of a Ricci flow with surgery. The almost-singular parts $M_{\text{almost-sing}}$, i.e. the parts that are discarded under each surgery construction, are hatched.

- Perelman: If M_1 is simply connected, then the flow has to go extinct, i.e., $M_i = \phi$ for i large.

$\Rightarrow$ Poincaré Conjecture:

Any closed, simply connected 3-manifold is diffeomorphic to S^3.

- Idea of finite time extinction. (back to Hamilton)
  - Define $W_2(t): [0, \infty) \to \mathbb{R}_{\geq 0}$ by:

    $W_2(t)$ = minimal area of all homotopically non-trivial 2-sphere.
  - Then in the weak sense,

$$\frac{dW_2(t)}{dt} \leq -4\pi + \frac{3}{(4t+1)} \cdot W_2(t).$$

pass surgery

$\Rightarrow W_2(t) < 0$ for t large, a contradiction.

- Perelman: Analysis of the long time solution solves the Geometrization Conjecture.

- Ricci flow through singularities in dim. 3.
  - the surgery is not unique, depends on:
    - surgery scale.
    - surgery spheres.
  - Conjecture (Perelman). As $r_i \to 0$ for each i, we will obtain some kind of canonical Ricci flow through singularities.
  - Kleiner-Lott, Bamler-Kleiner proved this conjecture, and use this to prove the Generalized Smale Conjecture.