• Hamilton's Ricci flow.

## §1. Ricci flow equation.

• $(M, g_0)$: compact Riem. mfd of dim. $n$. Hamilton introduced:

$$\begin{cases} \dfrac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(g(t)). \\ g(0) = g_0 . \end{cases}$$

• At $p \in M$, let $(x^1, \cdots, x^n)$ be harmonic coords, say $\Delta x^i = 0$ for all $i$,

we have:

$$\mathrm{Ric}_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g),$$

where $Q$ is a quadratic form in $g^{-1}$ and $\partial g$.

• Hence, the RF eqn is a heat eqn. for the Riem metric:

$$\frac{\partial g}{\partial t} = \Delta g + 2Q(g^{-1}, \partial g).$$

• Ricci flow on Einstein mfds:

• Assume $\mathrm{Ric}(g_0) = \lambda \cdot g_0$, $\lambda \in \mathbb{R}$.

• $g(t) = (1 - 2\lambda t) \cdot g_0$ is a solution of the RF:

$$\frac{\partial g(t)}{\partial t} = -2\lambda \cdot g_0 = -2 \cdot \mathrm{Ric}(g_0) = -2\,\mathrm{Ric}(g(t)).$$

• $\lambda \le 0$, $t \in [0, +\infty)$ ; $\lambda > 0$, $t \in \left[0, \frac{1}{2\lambda}\right)$.

• $(M, g_0) := (S^2, h_0)$, $h_0$: round metric of scalar curvature 1.

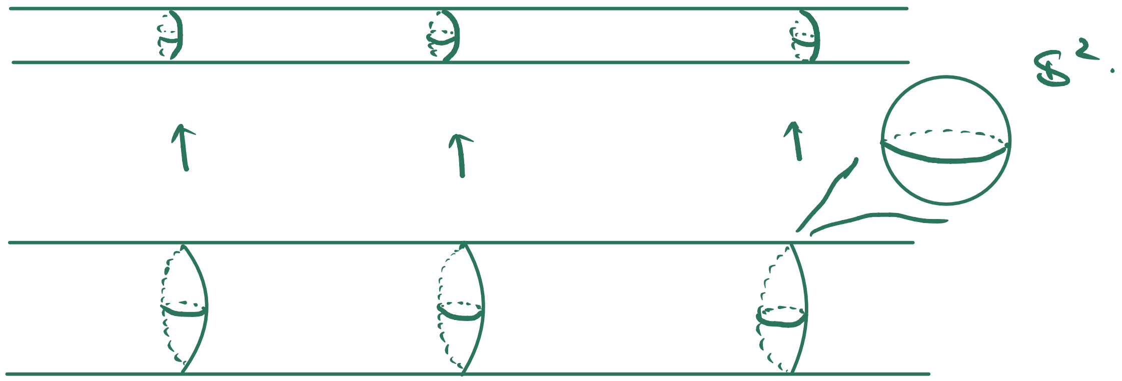
$h(t) := (1-t)\cdot h_0$, then

$$(\mathbb{S}^2, h(t)), \quad -\infty < t < 1$$

is a Ricci flow.

- Standard shrinking round cylinder:

$$(\mathbb{S}^2 \times \mathbb{R}, \ h(t) \times ds^2), \quad -\infty < t < 1.$$




- The RF eqn is weakly parabolic and is strictly parabolic modulo the diffeomorphism group of $M$.

- Thm. (Hamilton). (Short time existence and uniqueness).
  - $(M, g_0)$: compact $n$-dim. Riem. mfd.
  - $\exists !\, T_0 \in [0, +\infty]$, depending on $(M, g_0)$, s.t. $\exists !$ RF $(M, g(t))$, $0 \le t < T_0$, with $g(0) = g_0$.

- Hamilton's proof: Nash–Moser inverse function thm. (involved).

- DeTurck: Breaking the diffeomorphism invariance of the RF.

## §2. Evolution of curvatures.

- Let $x=(x^1,\cdots,x^n)$ be local coord.s, the RF eqn is:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

- **Thm.** Set $B_{ijkl} = g^{pr} g^{qs} \cdot R_{ipjq} \cdot R_{krls}$, then:

  - $$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right)$$
  $$- g^{pq}\left(R_{pjkl} \cdot R_{qi} + R_{ipkl} \cdot R_{qj} + R_{ijpl} \cdot R_{qk} + R_{ijkp} \cdot R_{ql}\right)$$

  - $$\frac{\partial}{\partial t} R_{jk} = \Delta R_{jk} + 2 g^{pq} g^{rs} R_{pjkr} \cdot R_{qs} - 2 g^{pq} \cdot R_{jp} \cdot R_{qk}.$$

  - $$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2.$$

  - $$\frac{\partial}{\partial t} dvol(x,t) = -R(x,t) \cdot dvol(x,t).$$

- **Lower bound on scalar curvature.** From:

$$\frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{n} \cdot R^2,$$

if $R(\cdot,0) \geq R_{min}$, then for all $t \in [0,T)$,

$$R(\cdot,t) \geq \frac{n}{2} \cdot \frac{R_{min}}{\frac{n}{2} - R_{min} \cdot (t-t_0)}.$$

- **Evolving orthonormal frame.**

  - $\mathcal{F} := \{F_1, \cdots, F_n\}$, $g(0)$-orthonormal frame on $M$.

  - Evolving this frame by: $\dfrac{\partial F_a(t)}{\partial t} = Ric(F_a(t), \cdot)^*$ along RF.

- For all $t \in [0,T)$, the frame $\mathcal{F}(t)$ is local $g(t)$-orthonormal.
- Set $\mathcal{F}^* := \{F^1, \cdots, F^n\}$ be dual coframe to $\mathcal{F}$.
- In local coord. $(x^1, \cdots, x^n)$, $F_a = F_a^i \cdot \frac{\partial}{\partial x^i}$,

$$Rm = R_{abcd} F^a F^b F^c F^d, \quad R_{abcd} = R_{ijkl} \cdot F_a^i F_b^j F_c^k F_d^l.$$

then with $B_{abcd} = \sum_{e,f} R_{aebf} R_{cedf}$, we have:

$$\partial_t R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} - B_{abdc} - B_{adbc} + B_{acbd}).$$

- $\{\varphi^1, \cdots, \varphi^{\frac{n(n-1)}{2}}\}$: orthonormal basis of $\Lambda^2 T^* M$
- Write $\varphi^\alpha(F_a, F_b) = \varphi^\alpha_{ab}$, then rewrite $Rm$ as $\mathcal{T} = (\mathcal{T}_{\alpha\beta})$ so that:

$$R_{abcd} = \mathcal{T}_{\alpha\beta} \cdot \varphi^\alpha_{ab} \cdot \varphi^\beta_{cd}.$$

then:

$$\partial_t \mathcal{T}_{\alpha\beta} = \Delta \mathcal{T}_{\alpha\beta} + \mathcal{T}^2_{\alpha\beta} + \mathcal{T}^\#_{\alpha\beta},$$

where $\mathcal{T}^2_{\alpha\beta} = \mathcal{T}_{\alpha\gamma} \cdot \mathcal{T}_{\gamma\beta}$, $\mathcal{T}^\#_{\alpha\beta} = C_{\alpha\gamma\zeta} C_{\beta\delta\eta} \cdot \mathcal{T}_{\gamma\delta} \mathcal{T}_{\zeta\eta}$ is the Lie algebra square, with $C_{\alpha\beta\gamma} = \langle [\varphi^\alpha, \varphi^\beta], \varphi^\gamma \rangle$.

## §3. Maximum principle and Hamilton–Ivey pinching estimate.

- Maximum principle of scalar function: $(M, g)$: cpt Riem. mfd,
  - $h: M \times [0,T) \to \mathbb{R}$, $(\partial_t - \Delta_g) h = 0$.

  Then: $h(x,0) \geq 0$, for all $x \in M \Rightarrow h(x,t) \geq 0$, for all $(x,t)$.

## • Theorem (Shi's derivative estimates).

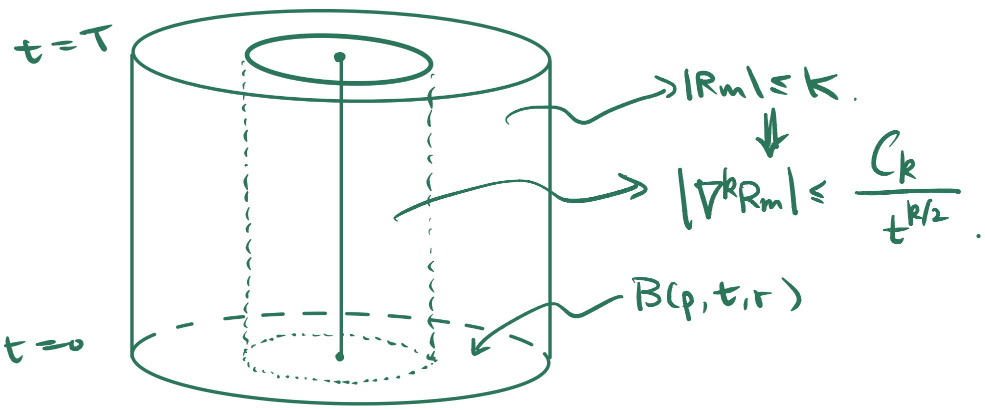
• $(M, g(t))$, $0 \le t \le T$, n-dim. RF. $p \in M$.

• $k = 1, 2, \cdots$, $r > 0$, $K < +\infty$, $\exists\, C_k = C_k(K, T, r, n)$, s.t., if

$$|Rm| \le K, \quad \text{on} \quad B(p, 0, r) \times [0, T]$$

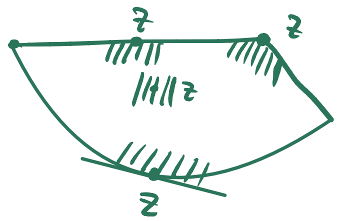
then:

$$|\nabla^k Rm| \le \frac{C_k}{t^{k/2}}, \quad \text{on} \quad B(p, 0, \tfrac{r}{2}) \times [0, T].$$




• $V$: finite dim. real vector space. $Z \subset V$: closed convex subset.

• $\forall\, z \in Z$, $T_z Z := \bigcap H$, $H$ is closed half-space of $V$, s.t. $z \in \partial H$, $Z \subset H$.




• **curvature preserving**: If $\psi$ is a vector field on $V$, we say $\psi$ preserves $Z$, if for every $z \in Z$, $\psi(z) \in T_z Z$.

## • Thm. (Maximum principle for tensors). (Hamilton).

• $M^n$: cpt Riem. mfd. $\mathcal{V} \to M$: tensor bundle.

- $g(t)$, $t\in[0,T]$ = smooth 1-parameter family metrics,
- $Z \subset \mathcal{V}\times[0,T]$ is a closed subset with $Z(t)$ is a convex subset of $\mathcal{V}\times\{t\}$, invariant under the parallel translation of $g(t)$.
- $\mathcal{F}$: a fiberwise vector field defined on a nbhd of $Z$ in $\mathcal{V}\times[0,T]$, that preserves the family $Z(t)$ for each $t\in[0,T]$.
- $\mathcal{T}(x,t)$, $t\in[0,T]$ = one-parameter family of sections of $\mathcal{V}$ that evolves by:

$$\frac{\partial \mathcal{T}}{\partial t} = \Delta_{g(t)}\mathcal{T} + \mathcal{F}(\mathcal{T}).$$

<u>Then</u>: if $\mathcal{T}(x,0)\in Z(0)$, for all $x\in M$,

$\Rightarrow$ if $\mathcal{T}(x,t)\in Z(t)$, for all $(x,t)\in M\times[0,T]$.

- We also have local version and strong version of this maximum principle.
- $n=3$, in an orthonormal basis in which

$$\mathcal{T} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \qquad \lambda \geq \mu \geq \nu$$

the vector field $\mathcal{F}(\mathcal{T})$ is given by:

$$\mathcal{F}(\mathcal{T}) = \mathcal{T}^2 + \mathcal{T}^{\#} = \begin{pmatrix} \lambda^2+\mu\nu & 0 & 0 \\ 0 & \mu^2+\lambda\nu & 0 \\ 0 & 0 & \nu^2+\lambda\mu \end{pmatrix}$$

- Nonnegative curvature is preserved:

- $(M, g(t))$, $0 \le t \le T$, RF on compact, connected 3-mfd.

- ① If $Rm(x,0) \ge 0$ for all $x \in M$, then:

$$Rm(x,t) \ge 0, \quad \text{for all } x \in M,\ t \in [0,T].$$

  - $\nu_x : Sym^2(\Lambda^2 T^*M) \to \mathbb{R}$ the function to the smallest eigenvalue.

$$Z := \bigcup_{x \in M} Z_x, \quad Z_x := \nu_x^{-1}([0,+\infty)), \text{ then } \mathcal{T} \in Z_x$$

$$\Rightarrow \nu_x(\mathcal{4}(\mathcal{T})) = \nu^2 + \lambda\mu \in Z_x \Rightarrow \mathcal{4} \text{ preserves } Z.$$

- ② If $Ric(x,0) \ge 0$ for all $x \in M$, then:

$$Ric(x,t) \ge 0 \quad \text{for all } x \in M,\ t \in [0,T].$$

  - $S_x : Sym^2(\Lambda^2 T^*M) \to \mathbb{R}$, $\mathcal{T} \mapsto \mu + \nu$.

$$Z := \bigcup_{z \in M} Z_x, \quad Z_x := S_x^{-1}([0,+\infty)), \text{ then } \mathcal{T} \in Z_x$$

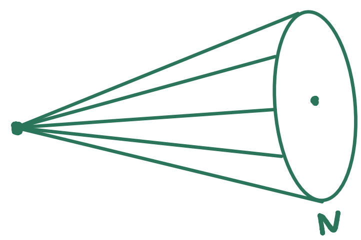
$$\Rightarrow S_x(\mathcal{4}(\mathcal{T})) = \mu^2 + \nu^2 + \lambda(\mu+\nu) \Rightarrow \mathcal{4} \text{ preserves } Z.$$

- Some rigidity results from strong maximum principle.

  - $(M, g(t))$, $0 \le t \le T$, 3-dim. RF with $Rm \ge 0$ everywhere.

  - ① If $R(p,T) = 0$ for some $p \in M$, then all $(M, g(t))$ are flat.

  - ② If $(M, g(T))$ is isometric to a non-empty open subset of a cone over a Riem. mfd, then $(M, g(t))$ is flat.




cone over N.

• ③ If $Rm(p,T)$ has a zero eigenvalue, say $\nu(p,T)=0$, then up to a covering, we have:

$$(M, g(t)) = (N^2, h(t)) \times (\mathbb{R}, ds^2).$$

If $(M, g(0))$ is non-flat, then $(N, h(t))$ is surface of positive curvature.

• Extending RFs.

• $(M, g(t))$, $0 \le t < T < +\infty$, be RF on compact mfd $M$.

• Then, either the flow extends to $[0, T')$ for some $T' > T$, or:

$$\limsup_{t \uparrow T} \sup_M |Rm| = +\infty.$$

• Theorem (Hamilton).

• $M$: compact connected 3-mfd. $(M, g(t))$, $0 \le t < T$, RF.

• If $Ric(x, 0) \ge 0$ for all $x \in M$, then:

  • either $Ric(x,t) > 0$, for all $(x,t) \in M \times [0,T)$.

  • or $Ric(x,t) = 0$, for all $(x,t) \in M \times [0,T)$.

• Assume $Ric(x,t) > 0$ for some $(x,t) \in M \times [0,T)$.

• Then the maximal existence time $T < +\infty$, and as $t \uparrow T$, $(M, g(t))$ are becoming round in the sense that:

$$\lim_{t \uparrow T} \frac{\max_{x \in M} \lambda(x,t)}{\min_{x \in M} \lambda(x,t)} = 1.$$

- Further, $\forall x \in M$, $\lambda(x,t) \to +\infty$ as $t \nearrow T$. If we rescale $(M, g(t))$ by $\lambda(x,t)$, then these mfds converge smoothly to compact round mfd.

- In particular, $M$ is diffeomorphic to a 3-dim. spherical space form.

- This thm confirms Poincaré conjecture if we assume $M$ admits a metric of positive curvature.

- Poincaré Conjecture Every closed, smooth, simply connected 3-mfd is diffeomorphic to $S^3$.

- Rmk: In dim. 3, every homeomorphism can be approximated by diffeomorphisms.

- In general, without $Ric(g_0) > 0$:
  - If the topology of $M^3$ is sufficiently complicated, then, no matter what $g_0$ is, the Ricci flow must develope singularities.
  - Even the topology of $M^3$ is simple, if $g_0$ is complicated, the Ricci flow also developes singularities.
  - Unlike the case of $Ric > 0$ ($M$ = singularities), here the singularities occur along proper subsets.

- Important results of Hamilton:

- **Theorem.** (Pinching toward positive curvature). (Hamilton)

  - $(M, g(t))$, $0 \le t < T$, RF on compact 3-mfd.
  - Assume $\nu(x,0) \ge -1$, for all $x \in M$.
  - Set $X(x,t) := \max\{-\nu(x,t), 0\}$. Then:

    (1). $R(x,t) \ge \dfrac{-6}{4t+1}$.

    (2). for all $(x,t)$ for which $X(x,t) > 0$,

$$R(x,t) \ge 2X(x,t) \cdot \left[\log X(x,t) + \log(1+t) - 3\right].$$

- The Harnack inequality.

- On a given Riem. mfd, we have:

  - **Li-Yau:** $(M^n, g)$: Riem. mfd with $Ric \ge 0$.

    Let $f: M \times [0,T) \to \mathbb{R}_+$ satisfies: $f > 0$, $(\partial_t - \Delta_g) f = 0$. Then:

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \ge 0, \quad \text{on } M \times (0,T).$$

  - **Hamilton:** $(M^n, g)$: Riem. mfd with $Sec \ge 0$ and $\nabla Ric = 0$,

    then:
$$\frac{\nabla^2 f_t}{f_t} - \frac{\nabla f_t \otimes \nabla f_t}{f_t^2} + \frac{g}{2t} \ge 0, \quad \text{on } M \times (0,T).$$

- Along Ricci flow, we have:

- **Theorem (Hamilton).**

  - $(M, g(t))$, $t \in (T_0, T_1)$, RF on complete mfd.

  - Assume for each $t$, $(M, g(t))$ has non-negative curvature operator and bounded curvature.

  - $X(x,t)$: time-dependent vector field on $M$. Then:

$$\frac{\partial R(x,t)}{\partial t} + \frac{R(x,t)}{t - T_0} + 2\langle X(x,t), \nabla R(x,t)\rangle + 2\mathrm{Ric}\left(X(x,t), X(x,t)\right) \geq 0.$$

- In particular, choose $X = 0$, we have:

$$\frac{\partial R(x,t)}{\partial t} + \frac{R(x,t)}{t - T_0} \geq 0.$$

- Or, choose $X = -\nabla R / 2R$, we have: $\forall\, t_1 < t_2 \in (T_0, T_1)$,

$$\log\left(\frac{R(x_2, t_2)}{R(x_1, t_1)}\right) \geq -\frac{d_{t_1}^2(x_1, x_2)}{2(t_2 - t_1)}.$$

- An important corollary is:

- **Cor.** • $(M, g(t))$, $-\infty < t \leq 0$, RF on complete mfd.

  - each $(M, g(t))$ has bounded and non-negative curvature operator.

  - Then: $\frac{\partial R(x,t)}{\partial t} \geq 0$.

## • §4. Compactness of RFs.

- We only consider complete Riem. mfds here.

- $(M_k, g_k(t))$, $-T \leq t \leq 0$, $n$-dim. RF,

- $(x_k, 0) \in M_k \times [-T, 0]$ : base-point.

- **Condition (1)**: $\forall\, \delta > 0$, $A < +\infty$, $\exists\, C = C(\delta, A) < +\infty$, s.t.

$$|Rm_{g_k}| \leq C(\delta, A), \quad \text{on} \quad B_{g_k}(x_k, 0, A) \times (-T+\delta, -\delta).$$

- **Condition (2)**: $\exists\, \kappa > 0$, $r_0 > 0$, s.t. for all $k$ large,

$$\mathrm{Vol}_{g_k}\big(B(x_k, 0, r_0)\big) \geq \kappa \cdot r_0^n.$$

- Then, after passing to a subsequence, for $-T < t < 0$,

$$\big(M_k, g_k(t), (x_k, 0)\big) \xrightarrow{C^\infty\text{-Cheeger-Gromov}} \big(M_\infty, g_\infty(t), (x_\infty, 0)\big).$$

- Each $(M_\infty, g_\infty(t))$ is complete.

- Without Condition (1), we can only have weaker convergence (e.g., GH, $\mathbb{F}$-limit) to limits with complicated singularities.

## §5. 4-dim. RF with positive curvature.

• **Theorem** (Hamilton). A compact 4-manifold with positive curvature operator is diffeomorphic to the sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.

• **PIC condition**: A $n$-dim. ($\geq 4$) Riem. mfd $(M, g)$ is said to have positive isotropic curvature, if for every orthonormal 4-frame, the Rm curvature satisfies:

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} \geq 2R_{1234}.$$

• **Micallef-Moore**: use minimal surface theory showed: every compact simply connected mfd with PIC is homeomorphic to $\mathbb{S}^n$.

• In dim 4, we have

• **Theorem** (Hamilton, Chen-Zhu, Chen-Tang-Zhu).

A compact 4-manifold with no essential incompressible ($\pi_1$ injects.) space-form and with a metric of PIC is diffeomorphic to $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{S}^3\times\mathbb{S}^1$, or $\mathbb{S}^3\tilde{\times}\mathbb{S}^1$, or a connect sum of them.

• Here, Hamilton proposed the notion of Ricci flow with surgery.

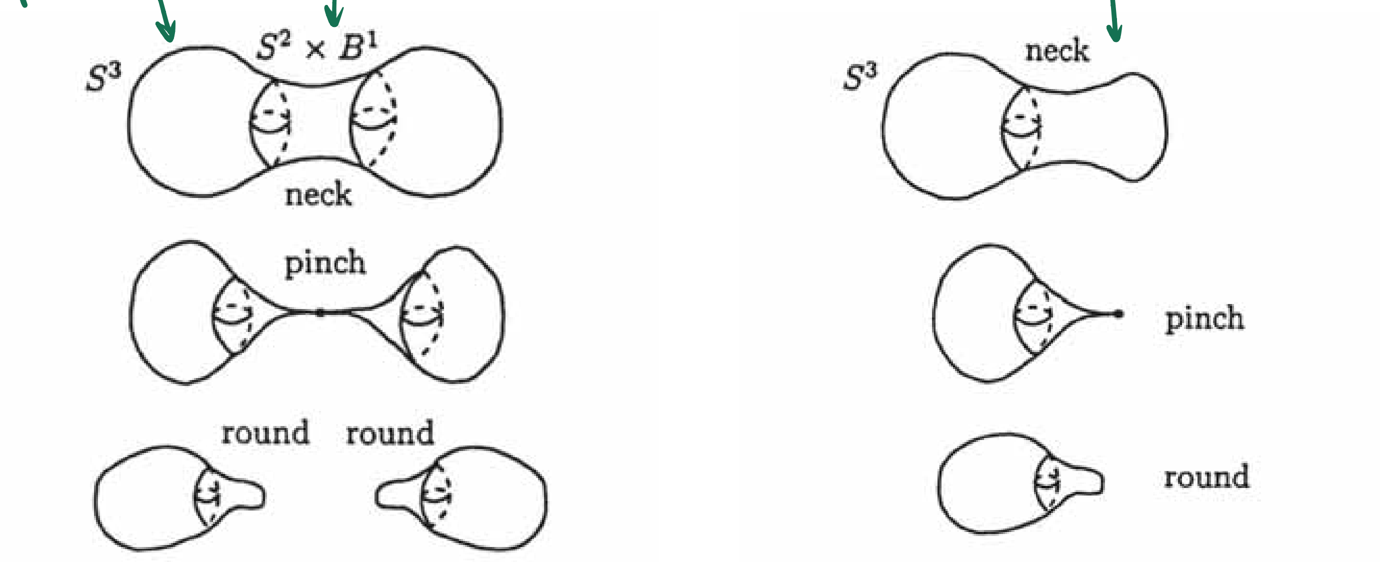
• **Rmk**: Hamilton's proof is incomplete, e.g., how to prevent the surgery times from accumulation.

• Intuitive solution in dim. 3 of Hamilton:

positive curvature

slightly negative curvature.

pinch off a little sphere.




dumbell.

# §6. Geometrization. (Naive introduction).

• In dim. 2:

| genus: | 0 | 1 | 2 | 3 ... |
|---|---|---|---|---|
| $M^2$. |  | | | |
| | $\mathbb{S}^2$ | $\mathbb{T}^2$ | | |
| universal cover | $\mathbb{S}^2$ | $\mathbb{R}^2$ | $\mathbb{H}^2$ | |
| $K$ | $\equiv +1$ | $\equiv 0$ | $\equiv -1$. | |

• The topology and geometry are determined by each other.

• First approach: View $(M^2, g)$ as a Riemann surface, then from complex analysis, the universal covering of $M$ is conformal to either $\mathbb{S}^2$, or $\mathbb{R}^2$, or the hyperbolic disk $D$.

$\Rightarrow$ the topology of $M$ is determined by $\pi_1(M)$.

• Second approach (Elliptic theory). Given $(M^2, g)$, consider a new metric $\tilde{g} = e^{\varphi} \cdot g$, $\varphi \in C^\infty(M)$. Then $\tilde{g}$ has constant curvature $\mu \in \mathbb{R}$ iff:

$$-\Delta_g \varphi + K(g) = \mu \cdot e^{\varphi}.$$

For $\mu > 0$, $= 0$, $< 0$, this eqn. has a solution $\Rightarrow$ uniformization.

Nirenberg problem: Nirenberg, Kazdan-Warner, Aubin, ....

- Third approach (Ricci flow). (Hamilton, Chow, Chen-Lu-Tian).

Given $(M^2, g_0)$, the flow:

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(g(t)), \quad g(0) = g_0$$

has a global solution after normalization, and converge to a metric $g_\infty$ of constant curvature.

- In dim 3, the situation is much more complicated.

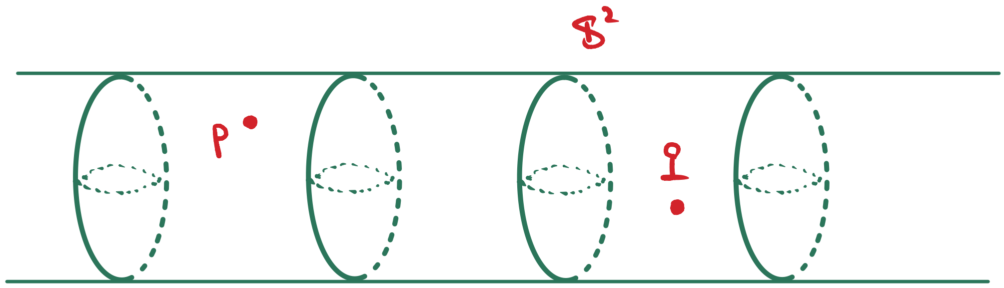
- $(M^n, g)$: Riem. mfd.
  - $(M, g)$ is called isotropic, if at each $p \in M$, every isometry of $T_pM$ is realised by an isometry of $M$.
  - $(M, g)$ is called homogeneous, if for every $p, q \in M$, there is an isometry of $M$ sending $p$ to $q$.

- A complete isotropic mfd is also homogeneous and has constant sectional curvature, e.g.,

$$S^n, \quad \mathbb{R}^n, \quad \mathbb{H}^n, \text{ and their quotients.}$$

• In dim 2., isotropic metric $\Longleftrightarrow$ topology.

• In dim 3., $(\mathbb{S}^2 \times \mathbb{R}, h_{\mathbb{S}^2} \times ds^2)$ is homogeneous, but not isotropic.




• Eight geometric 3-manifolds.

$\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $PSL_2(\mathbb{R})$, Sol.

• Thurston's Geometrization Conjecture.

Any closed, orientable, prime (not $\mathbb{S}^3$, every seperating $\mathbb{S}^2$ bounds a 3-ball.) 3-manifold M contains a disjoint union of embedded incompressible 2-tori and Klein bottles, such that each connected component of the complement admits a complete, locally homogeneous Riem. metric of finite volume.

• Rmk: Classification of complete, finite volume hyperbolic 3-manifold remains open.

• Simple remark: For a Riem. n-mfd $(M^n, g)$, the Riemannian curvature $R_{ijkl}$ is skew-symmetric in $(ij)$ and $(kl)$, and satisfies the Bianchi identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

Here are $d(n) = n^2 \cdot (n^2-1)/12$ eqns.

- A metric has constant sectional curvature $\kappa$ if

$$R_{ijkl} = \kappa \cdot \left( g_{ik} g_{jl} - g_{jk} g_{il} \right)$$

- In dim. 3, $K \equiv \lambda \Leftrightarrow Ric \equiv 2\lambda$.

- So naively, the condition of $K \equiv \kappa$ can be regarded locally as $d(n)$ eqns for $\underbrace{n\cdot(n+1)/2}_{1+2+\cdots+n}$ variables making up of $g_{ij}$.

| dim | variables | eqns |
|---|---|---|
| $n$ | $n\cdot(n+1)/2$ | $d(n)$ |
| 2 | 1 | 1 |
| 3 | 6 | 6 |
| 4 | 10 | 20 |
| 5 | 15 | 50 |
| ⋮ | ⋮ | ⋮ |

$K \equiv$ const is "determined" PDE.

⇓

geometrization in dim. 2 and 3.

- The Einstein equation: $Ric = \lambda \cdot g$ is "determined" eqn, hence it is reasonable to hope to find Einstein metrics in a fairly general context.

- Smale's h-cobordism theorem. $(n \geq 5)$.

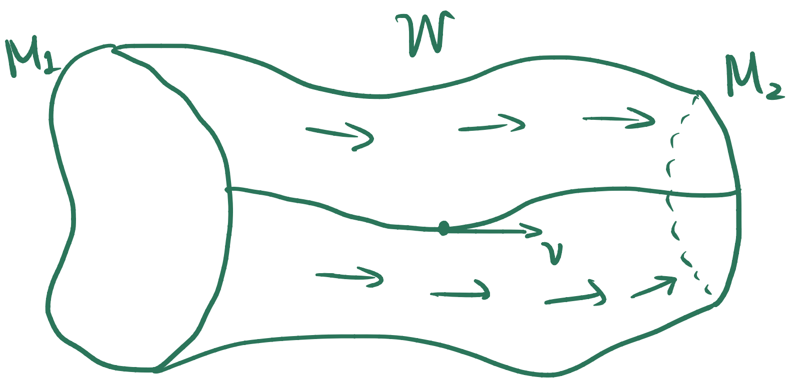
  - Two n-manifolds $M_1$, $M_2$ is called h-cobordant, if $\exists$ a $(n+1)$-manifold $W$ with boundary:

    $$\partial W = M_1 \cup M_2$$

    such that the inclusions $M_i \hookrightarrow W$ are homotopy equivalence.

  - If $n \geq 5$, if $M_1$ and $M_2$ are simply connected, h-cobordant n-mfds, then they are diffeomorphic.

  - The proof is by constructing a nowhere vanishing vector field $v$ on $W$, s.t. the flow lines of $v$ go from $M_1$ to $M_2$.




- The smooth h-cobordism thm fails in dim $n \leq 4$.