§8. Bochner formula. Splitting results.

§8.1. Connection on tensor bundles.

- (M^n, g): Riem. mfd.
- Recall Levi-Civita connection:

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad (X, Y) \mapsto \nabla_X Y.$$

- Define dual connection on cotangent bundle:

$$\nabla : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M), \quad (X, \omega) \mapsto \nabla_X \omega$$

via: $(\nabla_X \omega)(Y) = X[\omega(Y)] - \omega(\nabla_X Y)$.

- (r,s)-tensor bundles:

$$T^{r,s}M = \underbrace{T^*M \otimes \cdots \otimes T^*M}_{r} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{s}$$

- Connection on tensor bundles:

$$\nabla : \Gamma(TM) \times \Gamma(T^{r,s}M) \longrightarrow \Gamma(T^{r,s}M)$$
$$(X, T) \longmapsto \nabla_X T.$$

via: Y_i: 1-vector fields, ω^j: 1-forms,

$$(\nabla_X T)(Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s)$$
$$= X\left(T(Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s)\right)$$
$$- \sum_{1 \le i \le r} T(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_r, \omega^1, \cdots, \omega^s)$$

$$- \sum_{1 \le j \le s} T(Y_1, \cdots, Y_r, \omega^1, \cdots, \nabla_X \omega^j, \cdots, \omega^s).$$

- Also denote by:

$$\nabla : \Gamma(T^{r+1,s}M) \longrightarrow \Gamma(T^{r,s}M), \quad T \mapsto \nabla T,$$

via: $\nabla T(X, Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s)$

$$= (\nabla_X T)(Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s)$$

- Locally, in coord. chart $(U, x = (x^1, \cdots, x^n))$, write:

$$T = T_{i_1 \cdots i_r}^{j_1 \cdots j_s} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}$$

- Denote $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}} : \Gamma(T^{r,s}M) \longrightarrow \Gamma(T^{r,s}M)$.

- Denote by $\nabla_k T_{i_1 \cdots i_r}^{j_1 \cdots j_s}$ the coefficients of ∇T, namely:

$$\nabla T = \nabla_k T_{i_1 \cdots i_r}^{j_1 \cdots j_s} dx^k \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}$$

- Examples:

$$\nabla_k X^i = \frac{\partial X^i}{\partial x^k} + \Gamma_{kl}^i \cdot X^l, \quad \text{for } X = X^i \cdot \frac{\partial}{\partial x^i} \in \Gamma(TM).$$

$$\nabla_k \omega_i = \frac{\partial \omega_i}{\partial x^k} - \Gamma_{ik}^l \cdot \omega_l, \quad \text{for } \omega = \omega_j \cdot dx^j \in \Gamma(T^*M).$$

- In general:

$$\nabla_k T_{i_1 \cdots i_r}^{j_1 \cdots j_s} = \frac{\partial T_{i_1 \cdots i_r}^{j_1 \cdots j_s}}{\partial x^k}$$

$$- \sum_{l=1}^{r} \Gamma_{k i_l}^{p} \cdot T_{i_1 \cdots \underset{i_l\text{-position}}{p} \cdots i_r}^{j_1 \cdots j_s} + \sum_{l=1}^{s} \Gamma_{kq}^{j_l} \cdot \Gamma_{i_1 \cdots i_r}^{j_1 \cdots \overset{j_l\text{-position}}{q} \cdots j_s}$$

- For Riem. metric $g = g_{ij}\, dx^i \otimes dx^j$, we have:

$$\nabla_k g_{ij} = 0 = \nabla_k g^{ij}.$$

- Recall in local coord.,

$$R(X,Y)Z = X^i Y^j Z^l \cdot R_{ij}{}^{k}{}_{l} \cdot \frac{\partial}{\partial x^k}.$$

- Prop. 1 For vector field $X = X^k \frac{\partial}{\partial x^k}$, 1-form $\omega = \omega_k dx^k$, we have:

(i). $\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = R_{ij}{}^{k}{}_{l} \cdot X^l$.

(ii). $\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = -R_{ij}{}^{l}{}_{k} \cdot \omega_l$.

(iii). $[\nabla_i, \nabla_j] T_{i_1 \cdots i_r}^{j_1 \cdots j_s} = R_{ij}{}^{j_l}{}_{k} \cdot T_{i_1 \cdots i_r}^{j_1 \cdots \overset{l\text{-position}}{k} \cdots j_r} - R_{ij}{}^{k}{}_{i_l} \cdot T_{i_1 \cdots \underset{l\text{-position}}{k} \cdots i_r}^{j_1 \cdots j_s}$

- Recall some operators on smooth functions.

  - $df \in \Gamma(T^*M)$, differential of f, $df = \frac{\partial f}{\partial x^i} \cdot dx^i$.

  - $\nabla f \in \Gamma(TM)$, gradient field of f. $\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \cdot \frac{\partial}{\partial x^i}$.

  - $\langle \nabla f, X \rangle = X(f) = df(X), \quad \forall X \in \Gamma(TM)$.

  - $\mathrm{Hess}(f) = \nabla(df)$: $(2,0)$-tensor.

$$\mathrm{Hess}(f)(X,Y) = (\nabla_X df)(Y) = X[df(Y)] - df(\nabla_X Y)$$

$$= X(Yf) - (\nabla_X Y)f.$$

- $\mathrm{Hess}(f)(X,Y)$ is symmetric in X, Y:

$$\mathrm{Hess}(f)(X,Y) - \mathrm{Hess}(f)(Y,X)$$

$$= X(Yf) - Y(Xf) - (\nabla_X Y - \nabla_Y X)f \overset{\text{torsion free.}}{=} 0$$

- Locally, if we write $\mathrm{Hess}(f) = \nabla_i\nabla_j f \cdot dx^i \otimes dx^j$, where:

$$\nabla_i\nabla_j f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \cdot \frac{\partial f}{\partial x^k}.$$

- Laplacian: $f \in C^\infty(M)$, define:

$$\Delta f = \mathrm{tr}_g \mathrm{Hess}(f) = g^{ij} \cdot \nabla_i \nabla_j f$$

- In local coord.,

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \cdot \frac{\partial}{\partial x^k}\left(\sqrt{\det(g_{ij})} \cdot g^{kl} \cdot \frac{\partial f}{\partial x^l}\right).$$

• §8.2 Norms on tensors.

- Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of TM, with dual basis $\{e^i\}_{i=1}^n$ on T^*M.

- We introduce metric g on $T^{r,s}M$, s.t. the basis

$\{e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e_{j_1}\otimes\cdots\otimes e_{j_s}\}$ is orthonormal.

- $g_{ij}=g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$, $(g^{ij})_{n\times n}=(g_{ij})_{n\times n}^{-1}$, inverse matrix,

  $g^{ij}=g(dx^i,dx^j)$.

- In general,

$$\langle dx^{i_1}\otimes\cdots\otimes dx^{i_r}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}},\ dx^{k_1}\otimes\cdots\otimes dx^{k_r}\otimes\frac{\partial}{\partial x^{l_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{l_s}}\rangle$$

$$=g^{i_1k_1}\cdot\cdots\cdot g^{i_rk_r}\cdot g_{j_1l_1}\cdot\cdots\cdot g_{j_sl_s}.$$

- $|\nabla f|^2=|df|^2=g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}$.

  $|\nabla\nabla f|^2=g^{ik}g^{jl}\cdot\nabla_i\nabla_j f\cdot\nabla_k\nabla_l f$.

- **Lemma 2** (∇ compatible with inner product on tensor bundle).

$$\nabla_X\langle T_1,T_2\rangle=\langle\nabla_X T_1,T_2\rangle+\langle T_1,\nabla_X T_2\rangle.$$

§8.3. Bochner formula.

- **Prop.3** For $f\in C^\infty(M)$, we have

$$\frac{1}{2}\Delta|\nabla f|^2=|\mathrm{Hess}(f)|^2+\langle\nabla\Delta f,\nabla f\rangle+Ric(\nabla f,\nabla f).$$

- **Proof**: • Since $\nabla_i g_{kl}=0=\nabla_i g^{kl}$, $\forall i,k,l$, we compute:

$$\frac{1}{2}\Delta|\nabla f|^2=\frac{1}{2}\cdot g^{ij}\cdot\nabla_i\nabla_j\left(g^{kl}\cdot\nabla_k f\cdot\nabla_l f\right)$$

$$= \frac{1}{2} g^{ij} g^{kl} \cdot \nabla_i \left(\nabla_j \nabla_k f \cdot \nabla_l f + \nabla_k f \cdot \nabla_j \nabla_l f \right)$$

$$= \frac{1}{2} g^{ij} g^{kl} \cdot \Big(\nabla_i \nabla_j \nabla_k f \cdot \nabla_l f + \nabla_j \nabla_k f \cdot \nabla_i \nabla_l f$$

$$+ \nabla_i \nabla_k f \cdot \nabla_j \nabla_l f + \nabla_k f \cdot \nabla_i \nabla_j \nabla_l f \Big)$$

$$= g^{ij} g^{kl} \cdot \left(\nabla_i \nabla_j \nabla_k f \cdot \nabla_l f + \nabla_j \nabla_k f \cdot \nabla_i \nabla_l f \right)$$

$$= g^{ij} g^{kl} \cdot \nabla_i \nabla_j \nabla_k f \cdot \nabla_l f + |\mathrm{Hess}(f)|^2 .$$

- Next, $\nabla_j \nabla_k f = \nabla_k \nabla_j f$, hence:

$$\nabla_i \nabla_j \nabla_k f = \nabla_i \nabla_k \nabla_j f$$

$$= \nabla_k \nabla_i \nabla_j f - R_{ik}{}^{p}{}_{j} \cdot \nabla_p f$$

$$= \nabla_k \nabla_i \nabla_j f - g^{pq} \cdot R_{ikpj} \cdot \nabla_p f$$

- Hence:

$$g^{ij} g^{kl} \cdot \nabla_i \nabla_j \nabla_k f \cdot \nabla_l f = g^{ij} g^{kl} \cdot \left(\nabla_k \nabla_i \nabla_j f - g^{pq} \cdot R_{ikqj} \cdot \nabla_p f \right) \cdot \nabla_l f$$

$$= g^{kl} \cdot \nabla_k \left(g^{ij} \cdot \nabla_i \nabla_j f \right) \cdot \nabla_l f + g^{kl} g^{pq} \cdot R_{kq} \cdot \nabla_p f \cdot \nabla_l f$$

$$= \langle \nabla(\Delta f), \nabla f \rangle + \mathrm{Ric}(\nabla f, \nabla f).$$

Hence:

$$\frac{1}{2} \Delta |\nabla f|^2 = \langle \nabla(\Delta f), \nabla f \rangle + \mathrm{Ric}(\nabla f, \nabla f) + |\mathrm{Hess}(f)|^2 . \qquad \blacksquare$$

• §8.4. Splitting thms.

• Thm4. (Laplacian comparison).

- (M^n, g): complete Riem. mfd with $Ric \geq -(n-1)k^2$, $k \geq 0$,
- $\underline{M}$: space form of sectional curvature $\equiv -k^2$ of dim. n.
- $p \in M$, $\underline{p} \in \underline{M}$. $\rho_M(\cdot) := d(p, \cdot)$, $\rho_{\underline{M}}(\cdot) := \underline{d}(\underline{p}, \cdot)$
- If $x \in M$, $y \in \underline{M}$ satisfy $\rho_M(x) = \rho_{\underline{M}}(y)$, and x is a smooth point of ρ_M, then:

$$\Delta \rho_M(x) \leq \underline{\Delta} \rho_{\underline{M}}(y).$$

Proof. • Fix $x \in \exp_p(\mathcal{U}_p)$, i.e., x is not a cut point of p.

- Let $l = d(x,p)$. $\gamma: [0, l] \to M$ be the minimizing geodesic joining p to x, i.e., $\gamma(0) = p$, $\gamma(l) = x$, γ is of unit speed.
- Denote $v = \dot\gamma(0)$, then $x = \exp_p(lv)$.
- Fix any $0 \neq X \in T_xM$, $X \perp \dot\gamma(l)$.
    - Since $x \notin Cut(p)$, $d(\exp_p)_{l \cdot v}: T_pM \to T_xM$ is isomorphism. Hence $\exists\, 0 \neq W \in \mathcal{U}_p$, s.t. $d(\exp_p)_{l \cdot v}(W) = X$.
    - Set $\tilde\gamma(t, u) = \exp_p\left(t \cdot (v + u \cdot W)\right)$, $J := \frac{1}{l} \cdot \frac{\partial}{\partial u}\Big|_{u=0} \tilde\gamma$

then J is Jacobi field along γ with:

$$J(0)=0, \quad J(\ell)=\frac{1}{\ell}\, d(\exp_p)_{\ell v}(\ell\cdot W)=X.$$

- Also note that $[J,\dot\gamma](t)\equiv 0$ along γ.

- Now, at the point x, $\nabla\rho_M=\dot\gamma(t)$,

$$\begin{aligned}
\mathrm{Hess}(\rho_M)(X,X) &= J(\ell)\big(J(\ell)\rho_M\big)-\big(\nabla_{J(\ell)}J(\ell)\big)\rho_M \\
&= J(\ell)\langle J(\ell),\dot\gamma(\ell)\rangle-\langle \nabla_{J(\ell)}J(\ell),\dot\gamma(\ell)\rangle \\
&= \langle J(\ell), \nabla_{J(\ell)}\dot\gamma(\ell)\rangle \\
&= \langle J(\ell), \nabla_{\dot\gamma(\ell)}J(\ell)\rangle
\end{aligned}$$

- Since $J(0)=0$, at x,

$$\begin{aligned}
\mathrm{Hess}(\rho_M)(X,X) &= \int_0^\ell \frac{d}{dt}\langle J,\nabla_{\dot\gamma}J\rangle\,dt \\
&= \int_0^\ell \big[\,|J'|^2+\langle J,\nabla_{\dot\gamma}\nabla_{\dot\gamma}J\rangle\big]\,dt \\
&= \int_0^\ell \big[\,|J'|^2-Rm(\dot\gamma,J,\dot\gamma,J)\big]\,dt.
\end{aligned}$$

- Hence we conclude: For any $0\neq X\in T_xM$, there is a Jacobi field $J\in\mathcal{J}_0^\perp$, s.t. $J(\ell)=X$, and:

$$\mathrm{Hess}(\rho_M)(X,X)=\int_0^l \left[|J'|^2-Rm(\dot\gamma,J,\dot\gamma,J)\right]dt.$$

- Find an orthonormal basis $\{e_1,\cdots,e_{n-1},\dot\gamma(l)\}$ of T_xM.
- For each $X=e_i$, $1\le i\le n-1$, find $J_i\in \mathcal{J}_0^\perp$, s.t. $J_i(l)=e_i$,

$$\mathrm{Hess}(\rho_M)(e_i,e_i)=\int_0^l \left[|J_i'|^2-Rm(\dot\gamma,J_i,\dot\gamma,J_i)\right]dt.$$

$$=I(J_i,J_i).$$

- Similarly, in $\underline{M}$, let $\underline{\gamma}$ be minimal geodesic from $\underline{p}$ to y, $\{\underline{e_1},\cdots,\underline{e_{n-1}},\underline{\dot\gamma}(l)\}$ be orthormal basis of $T_y\underline{M}$, and for each $\underline{e_i}$, $1\le i\le n-1$, find Jacobi field $\underline{J_i}\in\underline{\mathcal{J}_0}^\perp$, s.t. $\underline{J_i}(l)=\underline{e_i}$. Then:

$$\underline{\mathrm{Hess}}(\rho_{\underline{M}})(\underline{e_i},\underline{e_i})=I(\underline{J_i},\underline{J_i}).$$

- $\begin{cases}\underline{e_i}(t)\text{: parallel extend of } \underline{e_i} \text{ along } \underline{\gamma}.\\ e_i(t)\text{: parallel extend of } e_i \text{ along } \gamma.\end{cases}$
- $\exists$ a common function $f(t)$, s.t. $\underline{J_i}(t)=f(t)\cdot\underline{e_i}(t)$, $1\le i\le n-1$.
- Define $X_i(t):=f(t)\cdot e_i(t)$, $1\le i\le n-1$. Note $f(0)=0$.

• By §5, Lemma 1, we have:

$$I(X_i, X_i) \geq I(J_i, J_i).$$

Hence:

$$\sum_{i\leq n-1} I(\underline{J}_i, \underline{J}_i) = \sum_{i \leq n-1} \int_0^{\ell} \left[|f'|^2 - f^2 \cdot \underline{Rm}(\underline{\dot\gamma}, \underline{e}_i, \underline{\dot\gamma}, \underline{e}_i) \right] dt$$

$$= (n-1) \cdot \int_0^{\ell} |f'|^2 dt - \int_0^{\ell} f^2 \cdot \underline{Ric}(\underline{\dot\gamma}, \underline{\dot\gamma})\, dt$$

$$\big\downarrow \; Ric \geq \underline{Ric}.$$

$$\geq (n-1) \cdot \int_0^{\ell} |f'|^2 dt - \int_0^{\ell} f^2 \cdot Ric(\dot\gamma, \dot\gamma)\, dt$$

$$= \sum_{i \leq n-1} I(X_i, X_i)$$

$$\geq \sum_{i \leq n-1} I(J_i, J_i).$$

• Finally, at x,

$$\Delta \rho_M(x) = \sum_{i\leq n-1} Hess(\rho_M)(e_i, e_i) = \sum_{i \leq n-1} I(J_i, J_i)$$

$$\leq \sum_{i \leq n-1} I(\underline{J}_i, \underline{J}_i) = \underline{\Delta} \rho_{\underline{M}}(y). \qquad \boxed{///}$$

• If $k=0$, then we obtain: if ρ_M is smooth at x (or if $x \notin Cut(p)$),

$$\Delta \rho_M(x) \leq \frac{n-1}{\rho_M(x)}.$$

Prop. 5 • (M, g): complete, $\mathrm{Ric} \geq 0$.

• $p \in M$. $\rho(x) := d(p, x)$, $x \in M$. Then:

$$\Delta \rho \leq \frac{n-1}{\rho}$$

globally on M, in the weak sense. That is, $\forall\, \varphi \in C_0^\infty(M)$, $\varphi \geq 0$,

$$\int_M \rho \cdot \Delta \varphi \, d\mathrm{vol} \leq \int_M \frac{n-1}{\rho} \cdot \varphi \, d\mathrm{vol}.$$

Proof. • Let $\Omega := \exp_p(\mathcal{U}_p)$, then $M = \Omega \cup \mathcal{C}_p$. Ω is an star-shaped open domain.

• In Ω, ρ is smooth, and pointwisely:

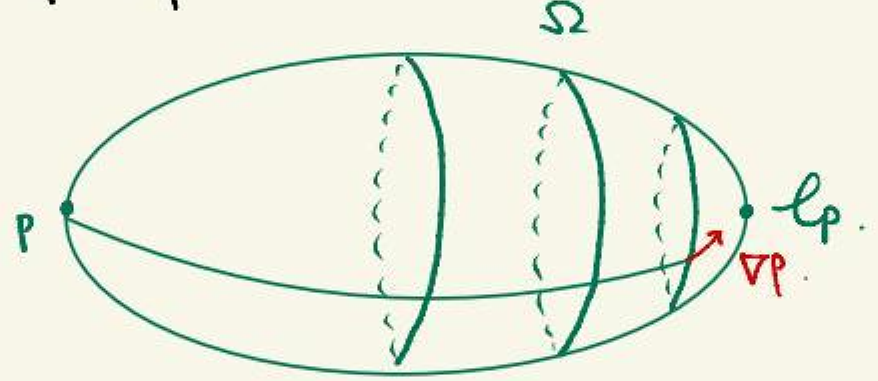
$$\Delta \rho \leq \frac{n-1}{\rho}$$

• Let $\varphi \in C_0^\infty(M)$, $\varphi \geq 0$. Since $\mathrm{Vol}(\mathcal{C}_p) = 0$, we have

$$\int_M \rho \cdot \Delta \varphi \, d\mathrm{vol} = \int_\Omega \rho \cdot \Delta \varphi \, d\mathrm{vol}.$$

• Since Ω is star-shaped, every geodesic ray emanating from p intersect $\partial \Omega = \mathcal{C}_p$ at most once.

• Find an increasing star-shaped domain Ω_ε, s.t. $\Omega_\varepsilon \Subset \Omega$, and $\lim_{\varepsilon \to 0} \Omega_\varepsilon = \Omega$, and Ω_ε increasing to Ω along the direction of $\nabla \rho$.



- Since Stoke's formula is also true for Lipschitz function, $\varphi \in C_0^\infty(M)$,

$$\int_M \rho \cdot \Delta \varphi \, d\mathrm{vol} = -\int_M \nabla \rho \cdot \nabla \varphi \, d\mathrm{vol} = \lim_{\varepsilon \to 0} -\int_{\Omega_\varepsilon} \nabla \rho \cdot \nabla \varphi \, d\mathrm{vol}.$$

since $|\nabla \rho| \leq 1$ a.e., and $|\nabla \varphi| \leq C$ on M.

- By Green formula,

$$-\int_{\Omega_\varepsilon} \nabla \rho \cdot \nabla \varphi \, d\mathrm{vol} = \int_{\Omega_\varepsilon} \Delta \rho \cdot \varphi \, d\mathrm{vol} - \int_{\partial \Omega_\varepsilon} \varphi \cdot \frac{\partial \rho}{\partial \nu} \, d\mathrm{vol}$$

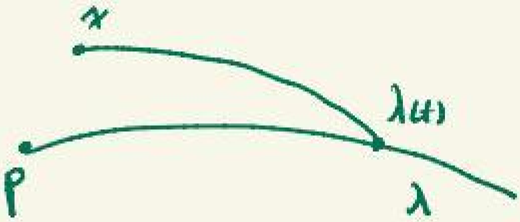
where ν is outer normal direction of $\partial \Omega_\varepsilon$, hence $\frac{\partial \rho}{\partial \nu} \geq 0$. hence:

$$-\int_{\Omega_\varepsilon} \nabla \rho \cdot \nabla \varphi \, d\mathrm{vol} \leq \int_{\Omega_\varepsilon} \Delta \rho \cdot \varphi \, d\mathrm{vol} \leq \int_{\Omega_\varepsilon} \frac{n-1}{\rho} \cdot \varphi \, d\mathrm{vol}.$$

- Hence we conclude:

$$\int_M \rho\Delta\varphi \, d\mathrm{vol} \leq \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\frac{n-1}{\rho}\varphi \, d\mathrm{vol} = \int_{\Omega}\frac{n-1}{\rho}\varphi \, d\mathrm{vol} = \int_M \frac{n-1}{\rho}\varphi \, d\mathrm{vol}. \qquad \blacksquare$$

- **Busemann function B^+** : $p \in M$.
  - $\lambda : [0,+\infty) \to M$: minimizing geodesic ray with $\lambda(0) = p$.
  - $B_t^+(x) := d(\lambda(t), x) - t$, $t \in [0,+\infty)$



  - 1-Lipschitz: $\forall x, y \in M$,

$$\left| B_t^+(x) - B_t^+(y) \right| \leq d(x,y).$$

  - Since λ is minimizing, $B_t^+(p) \equiv 0$, $\forall t \geq 0$. Hence:

$$B_t^+(x) \geq -d(p,x), \quad \forall x \in M.$$

  - For fixed x, B_t^+ is decreasing in t: $\forall\ t_2 > t_1 > 0$,

$$B_{t_2}^+(x) - B_{t_1}^+(x) = d(\lambda(t_2), x) - d(\lambda(t_1), x) - (t_2 - t_1)$$

$$\leq d(\lambda(t_2), \lambda(t_1)) - (t_2 - t_1) = 0.$$

  - Hence for each $x \in M$, $\lim_{t\to+\infty} B_t^+(x)$ exists, denote:

$$B^+(x) := \lim_{t\to+\infty} B_t^+(x).$$

  - Clearly, $B^+(x) \geq -d(p,x)$, $B^+(\cdot)$ is 1-Lipschitz,

$$B^+(\lambda(s)) = -s, \quad \forall\ s \geq 0.$$

• If $\lambda : (-\infty, +\infty) \to M$ is a minimizing geodesic line, we similarly can define B^-:

$$B^-(x) := \lim_{t \to -\infty} \left(d(x, \lambda(t)) + t \right).$$

By triangle inequality, we have:

- $B^+ + B^- \equiv 0$, on λ
- $B^+ + B^- \geq 0$, on M.

• **Theorem 6** (Cheeger-Gromoll).

• (M, g): n-dim. complete Riem. mfd, $Ric \geq 0$.

• If M contains a minimizing geodesic line, then isometrically,

$$(M, g) \cong (N, h) \times (\mathbb{R}, ds^2)$$

where N is a Riem. mfd of dim. $n-1$.

• Proof. • Let $\lambda : (-\infty, +\infty) \to M$ be a geodesic line. $p := \lambda(0) \in M$.

• Let $\varphi \in C_0^\infty(M)$, $\varphi \geq 0$. By Prop. 5, $\forall t \geq 0$,

$$\int_M B_t^+ \Delta\varphi = \int_M \varphi \cdot \Delta B_t^+ = \int_M \varphi \cdot \Delta d(\gamma(t), \cdot) \leq \int_M \frac{n-1}{d(\gamma(t), \cdot)} \cdot \varphi,$$

• Letting $t \to +\infty$, we obtain:

$$\int_M B^+ \cdot \Delta\varphi \leq 0,$$

that is, $\Delta B^+ \leq 0$ in the weak sense.

- Similarly, $\Delta B^- \leq 0$ in the weak sense.

- We obtain:
  - $B^+ + B^- \geq 0$, on M.
  - $B^+ + B^- \equiv 0$, on λ.
  - $\Delta(B^+ + B^-) \leq 0$, on M.

- **Maximum principle**: $\cdot$ (M,g): connected, $f: M \to \mathbb{R}$ continuous.
  - If $\Delta f \geq 0$ in the weak sense, then f is locally constant near any local maximum.

- Hence, by this Maximum principle, $B^+ + B^- \equiv 0$ on M. Hence:

$$\Delta B^+ + \Delta B^- = \Delta(B^+ + B^-) \equiv 0, \text{ in the weak sense}$$

$$\Rightarrow \Delta B^+ = 0, \quad \Delta B^- = 0, \text{ in the weak sense}$$

By standard elliptic regularity, B^+ and B^- are smooth.

- But $|\nabla \rho| = 1$, a.e., hence $|\nabla B^+| \equiv 1$.

- Claim: ∇B^+ is a parallel field on M, i.e., $\nabla(\nabla B^+) = 0$.

• By the Bochner formula, Prop.3,

$$0 = \Delta |\nabla B^+|^2$$

$$= 2 \cdot |\mathrm{Hess}(B^+)|^2 + \langle \nabla(\overset{0}{\overset{\shortparallel}{\Delta B^+}}), \nabla B^+ \rangle + \overset{\geq 0}{\mathrm{Ric}(\nabla B^+, \nabla B^+)}$$

$$\geq 2 \cdot |\mathrm{Hess}(B^+)|^2$$

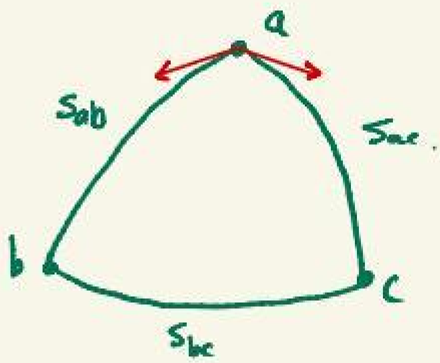
• De Rham decomposition gives:

$$(M, g) \cong (N, h) \times (\mathbb{R}, ds^2).$$

$\square$

• Triangle in (M, g): • $\Delta(a, b, c)$, $a, b, c \in M$, three pts.

• three slides S_{ab}, S_{bc}, S_{ac} are minimal geodesics:



• $\angle a$: the angle at a between the rays S_{ab} and S_{ac}.

• Thm 7. (Toponogov's length comparison).

• Suppose (M, g) is a Riem. mfd of non-negative sectional curvature.

• Suppose that $\Delta = \Delta(a, b, c)$ is a triangle in M, and $\Delta' = \Delta'(a', b', c')$ be a Euclidean triangle in $\mathbb{R}^2$.

(i) • Suppose $S_{ab} = S_{a'b'}$, $S_{ac} = S_{a'c'}$, $S_{bc} = S_{b'c'}$, then:

$$\angle a' \leq \angle a, \quad \angle b' \leq \angle b, \quad \angle c' \leq \angle c.$$

• Let $\alpha \in (0, S_{ab}]$, $\beta \in (0, S_{ac}]$. Let $x \in S_{ab}$, $x' \in S_{a'b'}$,

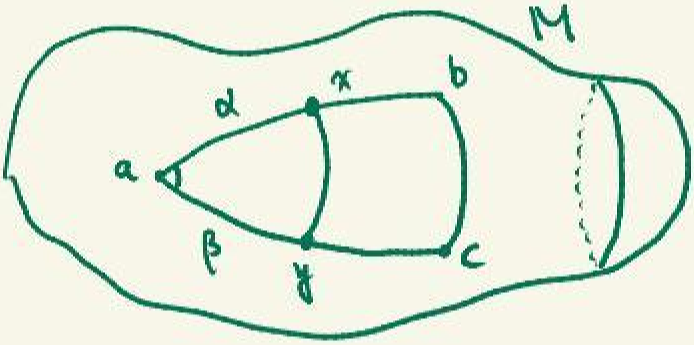
$y \in S_{ac}$, $y' \in S_{a'c'}$, such that:

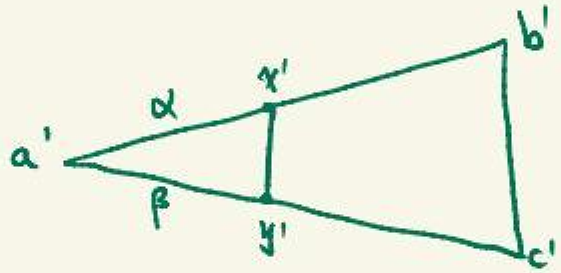
$$d(x,a) = \alpha, \quad d(y,a) = \beta, \quad d(x',a') = \alpha, \quad d(y',a') = \beta.$$

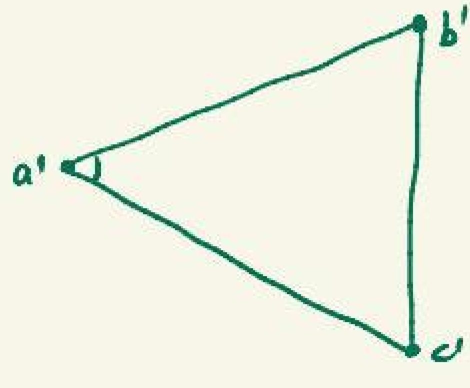
Then: $d(x,y) \geq d(x',y')$.

(ii). Suppose that $|S_{ab}| = |S_{a'b'}|$, $|S_{ac}| = |S_{a'c'}|$, $\angle a = \angle a'$, then:

$$|S_{b'c'}| \geq |S_{bc}|.$$







• Def. 8. (Cheeger-Gromov convergence)

• Let (M_k, g_k, x_k), $k \in \mathbb{N} \cup \{\infty\}$ be a sequence of complete, connected n-dim. Riem. mfd.

• We say (M_k, g_k, x_k) converge to $(M_\infty, g_\infty, x_\infty)$ in the

Cheeger-Gromov sense, if:

(i). $\exists$ exhausting open subsets $U_1 \subset U_2 \subset \cdots \subset M_\infty$, s.t. $\bigcup_{k=1}^{\infty} U_k = M_\infty$.

(ii). for each k, the closure $\overline{U}_k$ is compact.

(iii). for each k, $\exists$ open subsets $V_k \subseteq M_k$, s.t., $\forall D < +\infty$, for k large enough, $B(x_k, D) \subseteq V_k$.

(iv). for each k, $\exists$ diffeomorphism $\varphi_k : U_k \to V_k$, such that $\varphi_k(x_\infty) = x_k$, and for any compact subset $K \subset\subset M_\infty$,

$$\varphi_k^* g_k \longrightarrow g_\infty \quad \text{in } C^\infty(K).$$

• Theorem 9 (Splitting at infinity).

• (M, g): complete, connected mfd of non-negative sectional curvature.

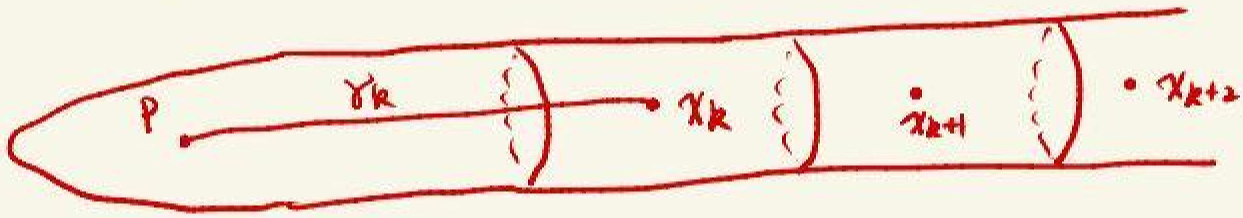
• $\{x_k\}$: a seq. of points going to infinity.

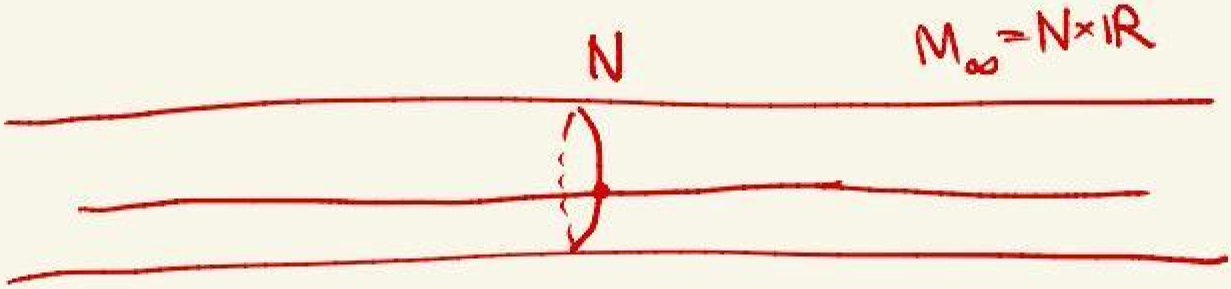
• $\{\lambda_k\}$: rescaling factors, $\lambda_k > 0$, s.t. the based Riem. mfds $(M, \lambda_k g, x_k)$ have a Cheeger-Gromov limit $(M_\infty, g_\infty, x_\infty)$.

• Suppose $\exists$ a point $p \in M$, s.t. $\lambda_k \cdot d^2(p, x_k) \to +\infty$ as $k \to +\infty$.

• <u>Then</u>: passing to a subseq., minimizing geodesic arcs γ_k from x_k to p converge to a minimizing geodesic ray in M_∞. This minimizing geodesic ray is part of a minimizing geodesic line ℓ in M_∞.

• <u>In particular</u>. $\exists$ a Riem. product decomposition $M_\infty = N \times \mathbb{R}$, with the property that ℓ is $\{x\} \times \mathbb{R}$ for some $x \in N$.





<u>Proof</u>: • Let $d_k := d(p, x_k)$.

• γ_k: minimizing geodesic arcs from p to x_k. (of unit speed)

• Passing to a subseq., we can assume that $\dot{\gamma}_k(0)$, the tangent directions at p of γ_k converges.

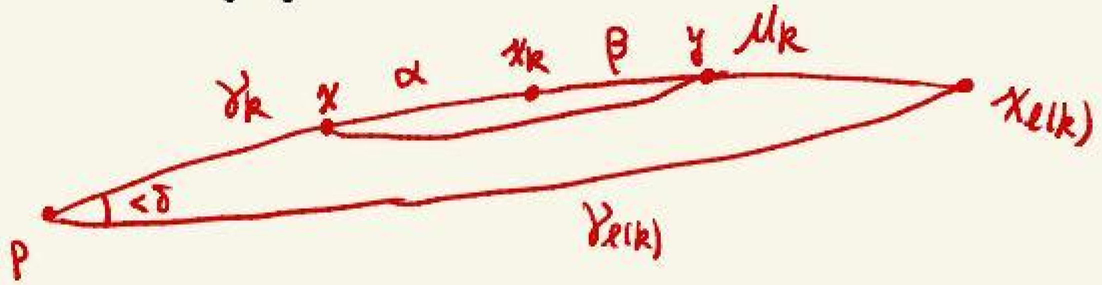
$\Rightarrow \forall\, 0 < \delta \ll 1$, $\exists\, N_0$ sufficiently large, s.t. for all $k, \ell \geq N_0$, the angle between γ_k and γ_ℓ at p is less than δ.

- $\forall k$, choose $l(k)$ large, s.t.

$$d_{l(k)} \geq d_k \cdot \left(1+\delta^{-1}\right),$$

fix such $l(k)$.

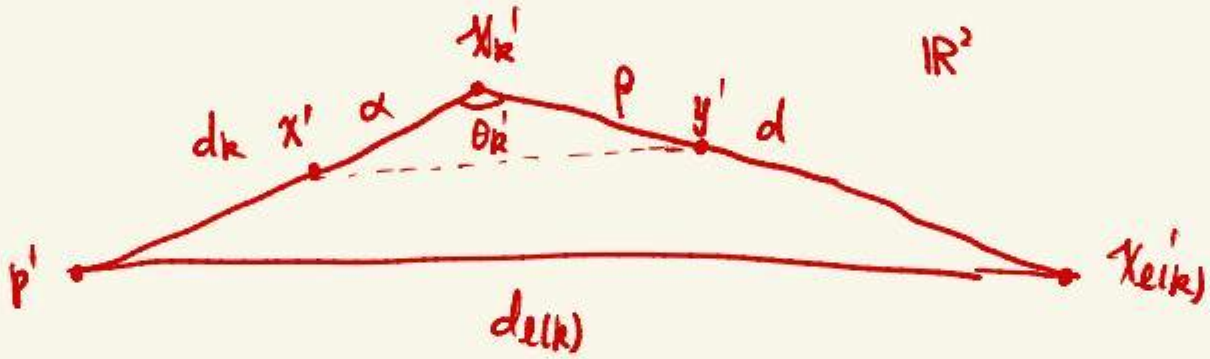
- μ_k : minimizing geodesic from x_k to $x_{l(k)}$



- $d := d(x_k, x_{l(k)})$.

- Euclidean triangle $\Delta(x_k', p', x_{l(k)}') \subseteq \mathbb{R}^2$, with:

$$|s_{p'x_k'}| = d_k, \quad |s_{p'x_{l(k)}'}| = d_{l(k)}, \quad |x_k' x_{l(k)}'| = d.$$



Then from Toponogov comparison, we have:

$$\angle p' \leq \angle p \leq \delta,$$

hence:

$$d_{l(k)} - d_k \underset{①}{\leq} d \underset{②}{\overset{\text{cosine's law}}{\leq}} \sqrt{d_k^2 + d_{l(k)}^2 - 2d_k d_{l(k)} \cdot \cos(\delta)}.$$

- Let $\theta_k = \angle x_k'$, then:

$$d_k^2 + d^2 - 2d_k d \cdot \cos\theta_k = d_{\ell(k)}^2$$

- For any $\alpha \in (0, d_k)$, $\beta \in (0, d)$, let $x \in \gamma_k$, $y \in \mathcal{U}_k$, s.t.

$$d(x, x_k) = \alpha, \quad d(y, x_k) = \beta.$$

$x' \in S_{p'x_k'}$, $y' \in S_{x_k' x_{\ell(k)}'}$

then from Toponogov's comparison thm, we have:

$$d(x, y) \geq d'(x', y') = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cdot \cos(\theta_k)}.$$

- Now, we have:

$$\cos(\theta_k) = \frac{d_k^2 + d^2 - d_{\ell(k)}^2}{2d_k d} \overset{②}{\leq} \frac{2d_k^2 - 2d_k d_{\ell(k)} \cdot \cos(\delta)}{2d_k d}$$

$$= \frac{d_k}{d} - \frac{d_{\ell(k)}}{d} \cdot \cos(\delta)$$

$$\leq \delta - (1-\delta) \cdot \cos(\delta).$$

- ① $\Rightarrow d \geq d_{\ell(k)} - d_k \geq \delta^{-1} \cdot d_k$

$\Rightarrow \frac{d_{\ell(k)}}{d} \geq \frac{d - d_k}{d} \geq 1 - \delta$

- Since $\delta \to 0$ as $k \to +\infty$, it follows that, $\forall\, \delta > 0$, $\forall\, k$ sufficiently large, $1 + \cos(\theta_k) \leq \delta$

- <u>Recall</u>: we are assuming $(M, \lambda_k g, x_k)$ in the sense of

Cheeger-Gromov to $(M_\infty, g_\infty, x_\infty)$, and $d_{\lambda_k g}(p, x_k) \to +\infty$

as $k \to +\infty$. Hence:

$$l_{\lambda_k g}(\gamma_k),\ l_{\lambda_k g}(\mu_k) \to +\infty.$$

Thus, passing to a subsequence, we can assume $\{\gamma_k\}$, $\{\mu_k\}$ converge

to minimizing geodesic arcs, which we denote by $\tilde{\gamma}$ and $\tilde{\mu}$, resp'ly.

in M_∞, emanating from x_∞.

- Since $\theta_k \to \pi$, as $1 + \cos\theta_k \to 0$, the angle between $\tilde{\gamma}$ and $\tilde{\mu}$ is π, hence the union of $\tilde{\gamma}$ and $\tilde{\mu}$ is a geodesic, say l, which is a minimizing geodesic.

- ① $\exists$ a minimizing geodesic line.
  ② $\mathrm{Sec}_{g_\infty} \geq 0$
  $\Rightarrow M = N \times \mathbb{R}$,
  $g = g_N + ds^2$,
  $l = \{x\} \times \mathbb{R}$, $x \in N$.

Toponogov, Cheeger-Gromoll

(Lemma 2.10)

VIII