Introduction to Complex Geometry

Selected Topics 3: Constant scalar curvature Kähler metrics and Bergman kernel

SHI, Yalong
(Nanjing University)

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Outline

1. Constant scalar curvature Kähler metrics
2. Bergman kernel and its asymptotic expansion
3. cscK metrics and Chow stability

§1 Constant scalar curvature Kähler metrics

cscK metrics

Definition

The scalar curvature of a Kähler metric ω_g is defined to be

$$s_g := s(\omega_g) := g^{\bar{j}i} R_{i\bar{j}}.$$

It is easy to see that s_g is one-half of the Riemannian scalar curvature. If s_g is a constant, we say ω_g is a "constant scalar curvature Kähler metric" ("cscK metric" for short.)

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- When $s_g \equiv c$, then c is uniquely determined by $c_1(X)$ and $[\omega_g]$:

$$c = \frac{1}{V}\int_X s_g dV = \frac{1}{\int_X \omega_g^n}\int_X n\, Ric(\omega_g) \wedge \omega_g^{n-1} = \frac{2n\pi c_1(X)\cdot[\omega_g]^{n-1}}{[\omega_g]^n} =: \underline{s}.$$

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- As a PDE, we need to find $\varphi \in C^\infty(X;\mathbb{R})$ such that $\omega_\varphi > 0$ and $s(\omega_\varphi) = c$. In local coordinates, this means

$$-g_\varphi^{\bar{j}i}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\det(g_{p\bar{q}} + \varphi_{p\bar{q}}) = c,$$

which is a fourth-order nonlinear PDE.

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$$\Delta h = n - \frac{\lambda}{2\pi} s_g = n - \frac{\lambda}{2\pi} \frac{2n\pi\lambda^{n-1} c_1(X)^n}{\lambda^n c_1(X)^n} = 0.$$

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- h is harmonic and hence a constant, so $Ric(\omega_g) = \frac{2\pi}{\lambda}\omega_g$.

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- Another easy to check obstruction is the vanishing of Calabi-Futaki invariants: Let $\omega_g \in [\omega]$, then $\int_X (s_g - \underline{s}) dV_g = 0$, then we can find $h \in C^\infty(X; \mathbb{R})$ such that $s_g - \underline{s} = \Delta h$. The Calabi-Futaki invariant is defined to be the map $v \mapsto \int_X v(h) dV_g$, which is a character on the space of holomorphic vector fields.

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- In general, when $[\omega_g] = \sqrt{-1}\Theta(h)$ for some holomorphic line bundle (L, h) (we say "(X, ω_g) is a polarized manifold"), then it is conjectured that the existence of cscK metric in $[\omega_g]$ is equivalent to the K-stability of the polarized pair (X, L) (Yau-Tian-Donaldson conjecture). This is still open.

Variational approach

- The cscK metric is the critical point of a functional K on the space of Kähler potentials $\mathcal{H}(\omega) := \{\varphi \in C^\infty(X;\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$ called "K-energy" (introduced by Mabuchi).

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- We also introduce the following functionals on $\mathcal{H}(\omega)$:

$$I_\omega(\varphi) := \int_X \varphi(\frac{\omega^n}{n!} - \frac{\omega_\varphi^n}{n!}), \quad J_\omega(\varphi) := \int_0^1 \frac{I_\omega(t\varphi)}{t}dt = \int_0^1 \Big(\int_X \varphi(\frac{\omega^n}{n!} - \frac{\omega_{t\varphi}^n}{n!})\Big)dt.$$

Then it is easy to see that $I_\omega, J_\omega, I_\omega - J_\omega$ are all non-negative and equivalent to each other.

Tian's "properness condition"

- Motivated by "direct methods" in the calculus of variations, Tian introduced a condition similar to the "coercive condition": a functional h on $\mathcal{H}(\omega)$ is called proper, if $h \geq \epsilon(I_\omega - J_\omega) - C$ where $\epsilon > 0$ and C are constants.

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- Motivated by his joint work with Ding on Kähler-Einstein metrics, Tian conjectured that properness of K_ω should imply the existence of cscK metric in $[\omega]$, and in "nice" cases, they should be equivalent.
- This conjecture is confirmed by Chen-Cheng in 2018 (published in JAMS, 2021).

When can we get the properness?

- To prove properness of K_ω, a useful tool is the following formula due to Chen and Tian:

$$K_\omega(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega^n} \frac{\omega_\varphi^n}{n!} + J_{\omega,-Ric(\omega)}(\varphi),$$

where $J_{\omega,\chi}(\varphi)$ is a functional where χ is an auxiliary closed $(1,1)$-form, defined by

$$J_{\omega,\chi}(\varphi) = \int_0^1 \int_X \varphi\Big(\chi \wedge \frac{\omega_{t\varphi}^{n-1}}{(n-1)!} - c\frac{\omega_{t\varphi}^n}{n!}\Big)dt,$$

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where $c = \frac{n[\chi][\omega]^{n-1}}{[\omega]^n}$
- The critical point of $J_{\omega,\chi}$ (if exist) satisfies $tr_{\omega_\varphi}\chi = c$.

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- The first term of Chen-Tian formula is always proper: by Tian's result, we can find $\alpha > 0$ such that $\int_X e^{-\alpha(\varphi - \sup\varphi)} \frac{\omega^n}{n!} \leq C$. From this we get

$$\begin{aligned} C &\geq \frac{1}{V}\int_X e^{-\alpha(\varphi-\sup\varphi)}\frac{\omega^n}{n!} = \frac{1}{V}\int_X e^{-\alpha(\varphi-\sup\varphi)-\log\frac{\omega_\varphi^n}{\omega^n}}\frac{\omega_\varphi^n}{n!} \\ &\geq \exp\Big(-\frac{\alpha}{V}\int_X(\varphi-\sup\varphi)\frac{\omega_\varphi^n}{n!} - \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{\omega^n}\frac{\omega_\varphi^n}{n!}\Big). \end{aligned}$$

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\begin{aligned}
\int_X \log\frac{\omega_\varphi^n}{\omega^n}\frac{\omega_\varphi^n}{n!} &\geq \alpha\int_X(\sup\varphi-\varphi)\frac{\omega_\varphi^n}{n!} - C = \alpha\int_X\Big(\sup\varphi\frac{\omega^n}{n!} - \varphi\frac{\omega_\varphi^n}{n!}\Big) - C \\
&\geq \alpha\int_X \varphi\Big(\frac{\omega^n}{n!} - \frac{\omega_\varphi^n}{n!}\Big) - C = \alpha I_\omega(\varphi) - C.
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- When $\chi > 0$, the problem of whether or not $tr_{\omega_\varphi}\chi = c$ is solvable is a pure PDE problem, and it is now completely understood, by Weinkove, Song-Weinkove, Lejmi-Szekelyhidi, Chen and Song.

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- In summary, if α is not too small, $c\omega - Ric(\omega) > 0$ for a not too big c and $J_{\omega,c\omega-Ric(\omega)}$ has a critical point, then K_ω is proper.

Calabi's extremal metric

- In 1970's, Calabi proposed a variational problem, motivated by Yang-Mills theory: consider the functional on $\mathcal{H}(\omega)$:

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- Ca's critical points are called "extremal Kähler metrics".
- CscK metrics are minimizers of Ca:

$$Ca(\varphi) = \int_X (s(\omega_\varphi) - \underline{s} + \underline{s})^2 \frac{\omega_\varphi^n}{n!} = \int_X (s(\omega_\varphi) - \underline{s})^2 \frac{\omega_\varphi^n}{n!} + C.$$

Existence of extremal Kähler metrics

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- There are also manifolds without any extremal metric in any Kähler class.
- There is a generalized version of Yau-Tian-Donaldson conjecture, which is also open at present.

§2 Bergman kernel and its asymptotic expansion

Setting

- Let (X, L) be a polarized pair, with $\omega_g = \frac{\sqrt{-1}}{2\pi}\Theta_{L,h}$, we can define a global inner product on $H^0(X, \mathscr{O}(L^{\otimes m}))$:

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- Let $s_0, \ldots, s_{N_m}$ be an orthonormal basis of $H^0(X, \mathscr{O}(L^{\otimes m}))$. We define a smooth function

$$\rho_m(z) := \sum_{j=0}^{N_m} |s_i(z)|^2_{h^m},$$

then it is smooth and in fact independent of the choice of the orthonormal basis.

Bergman kernel

Definition

The above defined function ρ_m is called the m-th Bergman kernel of (L, h).

Remark

Why the name "kernel"? If we write s_i^* the metric dual of s_i, which is a smooth section of $(L^{\otimes m})^*$, the "2-variable function" $K(z,w) := \sum_j s_j(z) \otimes s_j^*(w)$ is a "reproducing kernel" in the sense that for any section $s \in H^0(L^{\otimes m})$, we have

$$s(z) = \int_X \langle K(z,w), s(w) \rangle \frac{\omega^n}{n!}(w).$$

And $K(z,z) = \rho_m(z)$. So strictly speaking, ρ_m is "Bergman kernel restricted to the diagonal".

Basic properties of Bergman kernel

Lemma

For any $z \in X$*, we have*

$$\rho_m(z) = \sup\left\{|s(z)|^2_{h^m} \,\middle|\, \|s\|_{L^2} = 1\right\}.$$

Proof

Consider the evaluation map $ev_z : H^0(L^{\otimes m}) \to L_z^{\otimes m}$ *given by* $s \mapsto s(z)$*. Then the right hand side is precisely* $\|ev_z\|^2_{op}$*. On the other hand, we can find a* $h_z \in H^0(L^{\otimes m})$ *such that* $ev_z(s) = (s, h_z)$ *for any* s*, and* $\|ev_z\|_{op} = \|h_z\|_{L^2}$*. On the other hand, for the given orthonormal basis* $\{s_j\}$*, we have*

$$\|h_z\|^2_{L^2} = \sum_j |(s_j, h_z)|^2 = \sum_j |s_j(z)|^2_{h^m} = \rho_m(z).$$

Basic properties of Bergman kernel

Assume $L^{\otimes m}$ is very ample. Consider the Kodaira map $\iota_m : X \to \mathbb{C}P^{N_m}$

$$z \mapsto [s_0(z), \dots, s_{N_m}(z)].$$

Lemma

For the Fubini-Study metric $\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_{N_m}|^2)$ *on* $\mathbb{C}P^{N_m}$, *we have*

$$\iota_m^*\omega_{FS} = m\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\rho_m.$$

Proof

We have $\iota_m^*\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|s_0(z)|^2 + \cdots + |s_{N_m}(z)|^2)$ *using local trivialization. But this is* $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|s_0(z)|^2_{h^m} + \cdots + |s_{N_m}(z)|^2_{h^m}) - m\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ *which is precisely* $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\rho_m + m\omega$.

Tian's convergence theorem

Theorem (Tian, 1990)

We have

$$\|\frac{1}{m}\iota_m^*\omega_{FS} - \omega\|_{C^2} = O(\frac{1}{\sqrt{m}}).$$

In particular, $\frac{1}{m}\iota_m^*\omega_{FS} \to \omega$ *in* C^2*-topology.*

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Remark

- *Later, Wei-Dong Ruan (1998) proved that we in fact have* C^∞ *convergence.*
- *Tian's main tool is still Hörmander's* L^2*-theory of* $\bar{\partial}$*-equation.*
- *The idea is that around any given point* p*, one can construct a family of global "peak sections", essentially concentrated in coordinate balls of radius* $\frac{\log m}{\sqrt{m}}$*. We use these peak sections to compute* ρ_m *and its derivatives.*

Refinement: asymptotic expansion

Theorem (Catlin 1997, Zelditch 1998)

We have an asymptotic expansion $\rho_m(z) \sim \sum_{j=0}^{\infty} a_j(z) m^{n-j}$ *with* $a_0 = 1$, *in the sense that for any* k *and* $R > 0$, *we have*

$$\|\rho_m - \sum_{j<R} a_j m^{n-j}\|_{C^k} \leq C_{k,R} m^{n-R}.$$

Remark

- *There is a later proof via heat kernel by Dai-Liu-Ma (2004).*

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- *The first several coefficients are computed by Zhiqin Lu (2000). The first two are:*

$$a_1 = \frac{1}{2} s_g, \quad , a_2 = \frac{1}{3}\Delta s_g + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3 s_g^2).$$

Some corollaries

Corollary (Asymptotic Riemann-Roch)

As $m \to \infty$, we have

$$\dim H^0(X, L^{\otimes m}) = m^n \int_X \frac{\omega^n}{n!} + \frac{m^{n-1}}{2} \int_X s_g \frac{\omega^n}{n!} + O(m^{n-2})$$

Proof

We have

$$\dim H^0(X, L^{\otimes m}) \quad = \quad \int_X \rho_m \frac{\omega^n}{n!} = \int_X \Big(m^n + \frac{s_g}{2} m^{n-1} + O(m^{n-2}) \Big) \frac{\omega^n}{n!}.$$

Some corollaries

Corollary (C^∞ convergence)

We have

$$\frac{1}{m}\iota_m^*\omega_{FS} - \omega = O(m^{-2})$$

in C^∞ topology.

Proof

$$Left = \frac{\sqrt{-1}}{2m\pi}\partial\bar{\partial}\Big(\log \rho_m\Big) = \frac{\sqrt{-1}}{2m\pi}\partial\bar{\partial}\Big(\log m^n + \log(1 + O(\frac{1}{m}))\Big) = O(\frac{1}{m^2}).$$

Partial C^0-estimate

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- The reason for the name is that in the problem of existence of Kähler-Einstein metrics on Fano manifolds, Tian observed that if we have uniform lower bound of Bergman kernel of K_X^{-1} for certain m along Aubin's continuity path, then we will get C^0 estimate of φ outside the zero locus of certain holomorphic section of K_X^{-m}.

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- In good cases, if we have partial C^0-estimates, we can usually prove that the Gromov-Hausdorff limit is in fact projective algebraic.

Various results on partial C^0-estimates

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- In higher dimensions, Tian proved the KE case, and Donaldson-Sun proved the bounded Ricci case.
- Tian's partial C^0-conjecture is also confirmed by Chen-Wang using their solution of Hamilton-Tian conjecture for Kähler-Ricci flow. (Previously W. Jiang proved it for $n \leq 3$, using Tian-Zhang's proof of Hamilton-Tian conjecture in dim$\leq$ 3).

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- There is a recent work of G. Liu-Szekelyhidi, removed the Ricci upper bound in Donaldson-Sun's result.

§3 cscK metrics and Chow stability

Looking back at the asymptotic expansion

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- Can we really make it a constant by changing the bundle metric?
- What is the geometric implication for $\rho_m \equiv const$?

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- We call the matrix

$$\left(\int_{\iota_m(X)} \frac{Z_i \bar{Z}_j}{\sum_k |Z_k|^2} \omega_{FS}^n\right)$$

the center of mass of $\iota_m(X)$. If it is a multiple of I, we say ι_m is a “balanced embedding”.

Balance condition and algebraic geometry

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- One can easily prove that being balance is equivalent to $\rho_m \equiv const$.
- The polarized pair $(X, L^{\otimes m})$ can be balanced iff $(X, L^{\otimes m})$ is "Chow stable" in the sense of geometric invariant theory. This is first observed by S.W. Zhang (1996) and later reproved by H.Z. Luo, Phong-Sturm and S. Paul.

Donaldson's theorems

Theorem (Donaldson, 2001)

Suppose that $Aut(X, L)$ is discrete and (X, L^k) is balanced for all sufficiently large k. Suppose that the metrics ω_k converge in C^∞ to some limit ω_∞ as $k \to \infty$. Then ω_∞ has constant scalar curvature.

Question: can we always find a converging subsequence of $\{\omega_k\}$? This is still open to my knowledge.

Proof

This is just an application of the asymptotic expansion formula:

$$\|\rho_k(\omega_k) - k^n - \frac{s(\omega_k)}{2}k^{n-1}\|_{C^0} \leq ck^{n-2}.$$

Now $\rho_k(\omega_k) = \frac{\dim H^0(L^k)}{V} = k^n + \frac{k^{n-1}}{2}\underline{s} + O(k^{n-2})$. *Together with the previous inequality, we get*

$$\|s(\omega_k) - \underline{s}\|_{C^0} = O(\frac{1}{k}).$$

Let $k \to \infty$, *we get at once* $s(\omega_\infty) \equiv \underline{s}$.

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Theorem (Donaldson 2001)

Suppose that $Aut(X, L)$ *is discrete and* ω_∞ *is a Kähler metric in the class* $2\pi c_1(L)$ *with constant scalar curvature. Then* (X, L^k) *is balanced for large enough* k *and the sequence of metrics* ω_k *converge in* C^∞ *to* ω_∞ *as* $k \to \infty$.

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- Roughly speaking, this means "no holomorphic vector fields"+"cscK" implies "asymptotically Chow stable".
- When there are non-trivial holomorphic vector fields, there are counterexamples (7-dim toric variety, due to Ono-Sano-Yotsutani, 2012).
- In the general case, Mabuchi (2005) proved that when there are non-trivial holomorphic vector fields, one need extra conditions to get asymptotically Chow stability.

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- The rough outline of the proof is as follows: first using the asymptotic expansion to get a family of "nearly balanced embedding", this is done by Donaldson.
- Restricted to the finite dimensional spaces of pulling back metrics, balnaced embeddings are critical points of a functional I.
- Donaldson's estimates essentially tell us that at nearly balanced point, the first order derivative of I is very small and the second order derivative of I is quite large. Then elementary arguments show that there is a critical point not far away from the nearly balanced point.

Reference

- About cscK and extremal metrics, one can learn the basics from Tian and Szekelyhidi's books.
- About Bergman kernel, a good book is Ma-Marinescu "Holomorphic Morse Inequalities and Bergman kernels". Their tool of study is mainly the heat kernel. For the "peak section" method, one can start with Zhiqin Lu's Amer. J. Math. paper.
- For applications of Bergman kernels in cscK problems, besides Szekelyhidi's book, one can also start with Chi Li's master thesis at PKU.
- For K-stability, there is a quite detailed survey paper by Chenyang Xu, with title "K-stability of Fano varieties: an algebro-geometric approach".