Introduction to Complex Geometry

Selected Topics 2: L^2-theory for the $\bar{\partial}$-equation

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Outline

§1 L^2-theory of $\bar{\partial}$ in a pseudoconvex domain

Domain of holomorphy

Definition

Let Ω be a domain in $\mathbb{C}^n$. We say Ω is a "domain of holomorphy" if for any strictly larger domain $\Omega' \supsetneq \Omega$, the restriction map $\mathscr{O}(\Omega') \to \mathscr{O}(\Omega)$ is not surjective.

By a classical theorem of Cartan-Thullen, being a domain of holomorphy is equivalent to certain "holomorphic convexity". One equivalent version of the holomorphic convexity says that for any sequence $\{z_\nu\} \subset \Omega$ such that $z_\nu \to \partial\Omega$, we can find $f \in \mathscr{O}(\Omega)$ such that $\{|f(z_\nu)|\}$ is unbounded.

Pseudoconvexity

Motivated by Euclidean convexity, Levi introduced the following:

Definition

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2-boundary, i.e. $\forall z_0 \in \partial\Omega$, we can find an open neighborhood $U \ni z_0$ and $\varphi \in C^2(U;\mathbb{R})$ such that $\Omega \cap U = \{z \in U \big| \varphi(z) < 0\}$ and $d\varphi|_{\partial\Omega\cap U} \neq 0$. We say $\partial\Omega$ is pseudoconvex at z_0, if for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ satisfying $\sum_i \frac{\partial\varphi}{\partial z_i}(z_0)\xi_i = 0$ we have (called the "Levi form")

$$L_\varphi(z_0, \xi) := \sum_{i,j=1}^{n} \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i\bar{\xi}_j \geq 0.$$

If $L_\varphi(z_0, \xi) > 0$ when $\xi \neq 0$, we say $\partial\Omega$ is strongly pseudoconvex at z_0. If all the boundary points of $\partial\Omega$ are pseudoconvex (resp. strongly pseudoconvex), we say Ω is a pseudoconvex (resp. strongly pseudoconvex) domain.

Levi problem

- It is quite easy to see that a domain of holomorphy with C^2-boundary is necessarily pseudoconvex. Then it is natural to ask: are pseudoconvex domains holomorphically convex?

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- There are later proofs by H. Grauert in 1958 using sheaf theory, by J. Kohn in 1963 using $\bar{\partial}$-Neumann problem and by Hörmander in 1965 using L^2-theory in PDE.

- The main advantage of Hörmander's approach over Kohn's is his introduction of weight functions, bypassing the difficult problem of boundary regularity in Kohn's approach. Similar trick was introduced about at the same time by Andreotti-Vesentini in the vector bundle setting (Both works are motivated by Carleman type estimates in PDE). It relates geometry more closely and is very flexible to use.

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Let $\Omega \subset \mathbb{C}^n$ be a domain, $f = \sum_j f_j d\bar{z}_j$ be a smooth $(0,1)$-form, satisfying $\bar{\partial} f = 0$. The question is: can we find $u \in C^\infty(\Omega, \mathbb{C})$ such that $\bar{\partial} u = f$, i.e. $\frac{\partial u}{\partial \bar{z}_j} = f_j \; \forall j$? □

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- Existence of weak solutions;
- Regularity of weak solutions.

Introducing the weight function

The regularity problem here is not difficult, so we concentrate on the existence of weak solutions.

Let $\varphi \in C^\infty(\Omega)$ be a smooth $\mathbb{R}$-valued function to be determined later, we introduce:

$$L^2(\Omega, \varphi) := \left\{ f : \Omega \to \mathbb{C} \middle| \, \|f\|_\varphi^2 := \int_\Omega |f|^2 e^{-\varphi} < \infty \right\}$$

$$L^2_{(0,1)}(\Omega, \varphi) := \left\{ f = \sum_j f_j d\bar{z}_j \middle| \, \|f\|_\varphi^2 := \int_\Omega \sum_j |f_j|^2 e^{-\varphi} < \infty \right\}$$

$$L^2_{(0,2)}(\Omega, \varphi) := \left\{ f = \sum_{i<j} f_{\bar{i}\bar{j}} d\bar{z}_i \wedge d\bar{z}_j \middle| \, \|f\|_\varphi^2 := \int_\Omega \sum_{i<j} |f_{\bar{i}\bar{j}}|^2 e^{-\varphi} < \infty \right\}$$

$\bar{\partial}$ can be viewed as a densely defined closed operator between these Hilbert spaces.

The abstract setting

Given three (complex) Hilbert spaces H_1, H_2, H_3 and two densely defined closed operators $T : H_1 \to H_2$ and $S : H_2 \to H_3$ satisfying $ST = 0$. Our question is that given $f \in Ker\ S$, can we find $u \in Dom(T)$ such that $Tu = f$?

Theorem (Hörmander)

If

$$||g||^2_{H_2} \leq C(||T^* g||^2_{H_1} + ||Sg||^2_{H_3}), \quad \forall g \in Dom(T^*) \cap Dom(S),$$

then $\forall f \in Ker\ S$, $\exists u \in H_1$ *such that* $Tu = f$, $u \in (Ker\ T)^\perp$ *and*

$$||u||_{H_1} \leq \sqrt{C}||f||_{H_2}.$$

Translate into our language

- Let Ω be a pseudoconvex domain and $\varphi_1, \varphi_2, \varphi_3$ smooth functions to be determined later. Let

$$H_1 := L^2(\Omega, \varphi_1), \quad H_2 := L^2_{(0,1)}(\Omega, \varphi_2), \quad H_3 := L^2_{(0,2)}(\Omega, \varphi_3).$$

  S and T are both the (closed extension of the usual) $\bar{\partial}$-operators.

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- We need to verify that

$$\|f\|^2_{\varphi_2} \leq C^2\Big(\|T^* f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3}\Big), \quad \forall f \in Dom(T^*) \cap Dom(S).$$

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- It suffices to check this for $f \in \mathcal{D}_{(0,1)}(\Omega)$, smooth $(0, 1)$-forms with compact supports.

Formula for T^*

For $f \in \mathcal{D}_{(0,1)}(\Omega)$ and $u \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}
(Tu, f)_{\varphi_2} &= (u, T^* f)_{\varphi_1} = \int_\Omega u \overline{T^* f} e^{-\varphi_1} \\
&= \int_\Omega \sum_j \frac{\partial u}{\partial \bar{z}_j} \bar{f}_j e^{-\varphi_2} = -\int_\Omega u \sum_j \frac{\partial}{\partial \bar{z}_j} \left(\bar{f}_j e^{-\varphi_2} \right) \\
&= \int_\Omega u \overline{\left(-e^{\varphi_1} \sum_j \frac{\partial}{\partial z_j} (f_j e^{-\varphi_2}) \right)} e^{-\varphi_1},
\end{aligned}$$

So we get

$$T^* f = -e^{\varphi_1} \sum_j \frac{\partial}{\partial z_j} (f_j e^{-\varphi_2})$$

Special choice of weights

- We set $\varphi_3 = \varphi$, $\varphi_2 := \varphi - \psi$ and $\varphi_1 := \varphi - 2\psi$, with φ, ψ to be determined. Then we have

$$
\begin{aligned}
T^* f &= -e^{\varphi - 2\psi} \sum_j \frac{\partial}{\partial z_j}(f_j e^{-\varphi+\psi}) \\
&= -e^{\varphi - 2\psi} \sum_j \Big[\frac{\partial}{\partial z_j}(f_j e^{-\varphi}) e^{\psi} + f_j e^{-\varphi+\psi} \partial_j \psi \Big]
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- Write $\delta_j := e^{\varphi} \frac{\partial}{\partial z_j}(\cdot e^{-\varphi})$ (this is the formal adjoint of $-\frac{\partial}{\partial \bar{z}_j}$ with respect to $e^{-\varphi} dV$), then $T^* f = -e^{-\psi} \sum_j \delta_j f_j - e^{-\psi} \sum_j \partial_j \psi \cdot f_j$.

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- From which we get

$$
\int_\Omega |\sum_j \delta_j f_j|^2 e^{-\varphi} \leq 2 ||T^* f||^2_{\varphi_1} + 2 \int_\Omega |f|^2 |\partial \psi|^2 e^{-\varphi}
$$

Using commutator to help

- It is easy to check that $[\delta_i, \frac{\partial}{\partial \bar{z}_j}] = \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}$. Also, direct computation gives us

  $|Sf|^2 = \frac{1}{2}\sum_{i,j} |\frac{\partial f_{\bar{j}}}{\partial \bar{z}_i} - \frac{\partial f_{\bar{i}}}{\partial \bar{z}_j}|^2 = \sum_{i,j} |\frac{\partial f_{\bar{j}}}{\partial \bar{z}_i}|^2 - \sum_{i,j} \frac{\partial f_{\bar{j}}}{\partial \bar{z}_i} \overline{\left(\frac{\partial f_{\bar{i}}}{\partial \bar{z}_j}\right)}$.

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- So we get

$$\begin{aligned} \|Sf\|_\varphi^2 &= \int_\Omega \sum_{i,j} |\frac{\partial f_{\bar{j}}}{\partial \bar{z}_i}|^2 e^{-\varphi} + \int_\Omega \sum_{i,j} \delta_i(\partial_{\bar{j}} f_{\bar{i}})\overline{f_{\bar{j}}} e^{-\varphi} \\ &= \int_\Omega \sum_{i,j} |\frac{\partial f_{\bar{j}}}{\partial \bar{z}_i}|^2 e^{-\varphi} - \int_\Omega |\sum_j \delta_j f_{\bar{j}}|^2 e^{-\varphi} + \int_\Omega \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} f_{\bar{i}} \overline{f_{\bar{j}}} e^{-\varphi}. \end{aligned}$$

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&= \int_\Omega \sum_{i,j} |\frac{\partial f_{\bar{j}}}{\partial \bar{z}_i}|^2 e^{-\varphi} - \int_\Omega |\sum_j \delta_j f_{\bar{j}}|^2 e^{-\varphi} + \int_\Omega \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} f_{\bar{i}} \overline{f_{\bar{j}}} e^{-\varphi}.
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- We get

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\int_\Omega \sum_{i,j} |\frac{\partial f_{\bar{j}}}{\partial \bar{z}_i}|^2 e^{-\varphi} + \int_\Omega \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} f_{\bar{i}} \overline{f_{\bar{j}}} e^{-\varphi} \le \|Sf\|_\varphi^2 + 2\|T^* f\|_{\varphi_1}^2 + 2\int_\Omega |f|^2 |\partial \psi|^2 e^{-\varphi}.
$$

Condition for φ and ψ

- If we have $\sum_{i,j} \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\xi_i\bar{\xi}_j \geq 2(|\partial\psi|^2 + e^{\psi})|\xi|^2$ on Ω for any $\xi \in \mathbb{C}^n$, then we will get

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$$\|f\|_{\varphi_2}^2 \leq \|T^* f\|_{\varphi_1}^2 + \|Sf\|_{\varphi_3}^2, \quad \forall f \in Dom(T^*) \cap Dom(S).$$

- A C^2-function u with $(u_{i\bar{j}}) := (\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}) \geq 0$ everywhere is called a plurisubharmonic function ("PSH"-for short). If $(u_{i\bar{j}})$ is positive definite everywhere, we say u is strictly plurisubharmonic. Generally, a PSH function is defined to be an upper semicontinuous function with values in $[-\infty, \infty)$, such that its restriction to any complex line is subharmonic.

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- When can we find such φ and ψ? It is known that a pseudoconvex domain always admits a strictly PSH function $p \in C^\infty(\Omega)$ with $\{z \in \Omega \mid p(z) \leq c\}$ is compact for any $c \in \mathbb{R}$. Then one can prove that for a suitable choice of convex increasing function $\chi \in C^\infty(\mathbb{R})$ (depending on ψ), $\varphi := \chi \circ p$.

Hörmander's L^2-existence theorem

Theorem

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\varphi \in C^2(\Omega)$ satisfies

$$\sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \bar{\xi}_j \geq c(z)|\xi|^2, \forall z \in \Omega, \forall \xi \in \mathbb{C}^n,$$

where $c(z)$ is a positive continuous function on Ω. If $f \in L^2_{(0,1)}(\Omega, \varphi)$ s.t. $\bar{\partial} f = 0$, then there exists $u \in L^2(\Omega, \varphi)$ such that $\bar{\partial} u = f$, (when f is smooth, so is u) and

$$\int_\Omega |u|^2 e^{-\varphi} \leq 2 \int_\Omega \frac{|f|^2}{c} e^{-\varphi}.$$

A cleaner form

Theorem (Hörmander)

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$$\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-2} \leq \int_\Omega |f|^2 e^{-\varphi}.$$

Idea of proof:

- May first assume φ is C^2, and apply the previous theorem to $\tilde{\varphi} := \varphi + 2\log(1+|z|^2)$, where we take $c(z) := \frac{2}{(1+|z|^2)^2}$.

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Idea of proof:

- May first assume φ is C^2, and apply the previous theorem to $\tilde{\varphi} := \varphi + 2\log(1 + |z|^2)$, where we take $c(z) := \frac{2}{(1+|z|^2)^2}$.
- For general $\varphi \in PSH(\Omega)$, we use smooth PSH functions φ_ϵ (defined on larger and larger subdomains) decreasing to φ. Solving the equation on smaller domains with respect to φ_ϵ and take limit $\epsilon \to 0$.

Application: The Levi Problem

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\{z_\nu\} \subset \Omega$ satisfies $z_\nu \to \partial\Omega$. Also given an arbitrary family of $c_\nu \in \mathbb{C}$, we need to construct $f \in \mathscr{O}(\Omega)$ such that $f(z_\nu) = c_\nu$.

- Around each z_ν, choose cut-off functions ρ_ν such that their supports do not intersect each other. Then construct $h(z)$ such that near z_ν, $h(z) = c_\nu \rho_\nu$.

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- Need to solve the equation $\bar{\partial} u = \bar{\partial} h$ with $u(z_\nu) = 0$, then $f := h - u$ is what we need.

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- Need to solve the equation $\bar\partial u = \bar\partial h$ with $u(z_\nu) = 0$, then $f := h - u$ is what we need.
- The vanishing of u at z_ν is guaranteed by using weight of the form $\phi + \psi$ where $\phi(z) = n \sum_\nu \rho_\nu(z) \log |z - z_\nu|^2$ and $\psi \in C^\infty(\Omega) \cap PSH(\Omega)$ such that $\phi + \psi \in PSH(\Omega)$. Note that $|\bar\partial h|^2 e^{-\phi-\psi} \in L^1_{loc}$, so $|u|^2 e^{-\phi-\psi} \in L^1_{loc}$ will imply $u(z_\nu) = 0$ for all z_ν.

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- Using special property of pseudoconvex domains, we can find such a ψ.

Application: Bombieri's theorem

Theorem (Bombieri)

Let $\varphi \in PSH(\Omega)$, then the set $E := \{z \in \Omega \mid e^{-\varphi} \notin L^1(B(z,r)), \forall r > 0\}$ is an analytic subset of Ω.

Proof

- *Consider the space $A^2(\Omega,\varphi)$ consists of holomorphic functions on Ω with $\int_\Omega |f|^2 e^{-\varphi} < \infty$, we claim that $E = \cap_{f \in A^2(\Omega,\varphi)} f^{-1}(0)$. This immediately implies that E is analytic.*

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- *On one hand, if $\int_\Omega |f|^2 e^{-\varphi} < +\infty$, we necessarily have $f(z) = 0$ for any $z \in E$. This implies $E \subset \cap_{f \in A^2(\Omega,\varphi)} f^{-1}(0)$.*

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- *On one hand, if $\int_\Omega |f|^2 e^{-\varphi} < +\infty$, we necessarily have $f(z) = 0$ for any $z \in E$. This implies $E \subset \cap_{f \in A^2(\Omega,\varphi)} f^{-1}(0)$.*
- *On the other hand, if $z_0 \in \Omega \setminus E$, then we can find a neighborhood of z_0 on which $e^{-\varphi}$ is integrable. We start with a cut-off function ρ and solve the equation $\bar{\partial} u = \bar{\partial} \rho$ using weight $\varphi - n \log |z - z_0|^2$ to get $f = \rho - u \in A^2$ with $f(z_0) = 1$.*

§2 Global version

Global version

Theorem

Let (X, ω) *be a compact Kähler manifold.* (F, h) *be a holomorphic line bundle with Hermitian metric* h*. Assume* $\varphi \in L^1(X)$ *satisfies that there is a continuous function* $c > 0$ *on* X *such that*

$$\sqrt{-1}\Theta(h) + \sqrt{-1}\partial\bar{\partial}\varphi \geq c\omega.$$

Then for any $g \in L^2(X, \Lambda^{n,1}T^*X \otimes F)$ *such that* $D''g = 0$*, there exists* $f \in L^2(X, \Lambda^{n,0}T^*X \otimes F)$ *such that* $D''f = g$*, (when* g *is smooth, so is* f*) and*

$$\int_X |f|^2_{h,g} e^{-\varphi} dV_g \leq \int_X \frac{|g|^2_{h,g}}{c} e^{-\varphi} dV_g.$$

A convenient version

Usually we want to construct holomorphic section of a holomorphic line bundle L with bundle metric h, then we can take $F := K_X^{-1} \otimes L$ with bundle metric $\det(g) \otimes h$, then the curvature condition is $\sqrt{-1}\Theta(h) + Ric(\omega_g) + \sqrt{-1}\partial\bar{\partial}\varphi \geq c\omega_g$.

Theorem

Let (X, ω) be a compact Kähler manifold. (L, h) be a holomorphic line bundle with Hermitian metric h. Assume $\varphi \in L^1(X)$ satisfies that

$$\sqrt{-1}\Theta(h) + Ric(\omega) + \sqrt{-1}\partial\bar{\partial}\varphi \geq \epsilon\omega,$$

with $\epsilon > 0$. Then for any $g \in L^2(X, \Lambda^{0,1}T^*X \otimes L)$ such that $D''g = 0$, there exists $f \in L^2(X, L)$ such that $\bar{\partial}f = g$, (when g is smooth, so is f) and

$$\int_X |f|_h^2 e^{-\varphi} dV_g \leq \frac{1}{\epsilon} \int_X |g|_{h,g}^2 e^{-\varphi} dV_g.$$

Rough outline of proof

- We have the basic equality of Bochner-Kodaira-Nakano

$$\Delta_{D''} - \Delta_{D'} = [\sqrt{-1}\Theta_{F,h}, \Lambda]$$

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- Assume at a given point $g_{i\bar{j}} = \delta_{ij}$ and $\sqrt{-1}\Theta_{F,h} = \sqrt{-1}\sum_j \lambda_j dz_j \wedge d\bar{z}_j$, then for a section u of $\Lambda^{n,1}T^*X \otimes F$, write $u = \sum_j u_{\bar{j}} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_j \otimes e$ we have

$$\langle [\sqrt{-1}\Theta_{F,h}, \Lambda]u, u\rangle = \sum_j \lambda_j |u_{\bar{j}}|^2 h(e, e).$$

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$$\langle[\sqrt{-1}\Theta_{F,h}, \Lambda]u, u\rangle = \sum_j \lambda_j |u_{\bar{j}}|^2 h(e, e).$$

- If $\lambda_j \geq \epsilon > 0$ for all j, we have

$$\epsilon ||u||^2 \leq ||D''u||^2 + ||D''^* u||^2.$$

This is precisely the condition we need in Hörmander's abstract lemma.

Application: Kodaira's embedding theorem

- Suppose $L \to X$ is a holomorphic line bundle over a compact complex manifold. h is an Hermitian metric on L such that $\sqrt{-1}\Theta(h) > 0$. We can take $\omega := \sqrt{-1}\Theta(h)$ as a Kähler metric on X.

Application: Kodaira's embedding theorem

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- We want to prove that there is a $m > 0$ such that global sections of $L^{\otimes}$ will embeds X into $\mathbb{C}P^N$ with $N = \dim H^0(X, \mathscr{O}(L^{\otimes m})) - 1$.

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- Need to prove: for any $p \in X$, we can find global holomorphic sections $s_0, s_1, \ldots, s_n$ of $L^{\otimes m}$, such that $s_0(p) \neq 0$, $s_i(p) = 0, i = 1, \ldots, n$ and $s_1/s_0, \ldots, s_n/s_0$ can be viewed as a coordinate system near p. (Same argument shows that for any two different points p, q, we can find global section s of $L^{\otimes m}$ such that $s(p) \neq 0$ but $s(q) = 0$. We omit this.)

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- Using compactness, we can find m such that any basis of $H^0(X, \mathscr{O}(L^{\otimes m}))$ will give rise to an embedding of X.

- Fix $p \in X$ and choose holomorphic coordinates $(z_1, \ldots, z_n)$ near p such that $z_i(p) = 0, i = 1, \ldots, n$. Let ρ be a cut-off function near p. Also assume that e is local holomorphic frame of L over U.

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- We define $f_0 := \rho e^{\otimes m}$ and $f_i := \rho z_i e^{\otimes m}$, they are smooth sections of $L^{\otimes m}$ and are holomorphic in a smaller neighborhood of p. Then $g_i := \bar{\partial} f_i$ are smooth $L^{\otimes m}$-valued (0,1)-forms and $D'' g_i = 0$ for $i = 0, \ldots, n$.

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- We need to find smooth sections u_i satisfying $\bar{\partial} u_i = g_i$ and $u_i = O(|z|^2)$, so we need singular weight φ, which is smooth away from the chart and is of the form $\varphi = (n+1)\log|z|^2$ near p.

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- We need to find smooth sections u_i satisfying $\bar{\partial} u_i = g_i$ and $u_i = O(|z|^2)$, so we need singular weight φ, which is smooth away from the chart and is of the form $\varphi = (n+1)\log|z|^2$ near p.
- Note that $|g_i|^2_{h,g} e^{-\varphi}$ is always integrable on X, so the condition we need to solve $\bar{\partial} u_i = g_i$ with estimates is

$$m\sqrt{-1}\Theta(h) + Ric(\omega) + \sqrt{-1}\partial\bar{\partial}\varphi > \epsilon\omega.$$

  Since $\sqrt{-1}\Theta(h) > 0$ and $\sqrt{-1}\partial\bar{\partial}\varphi$ is non-negative near p, this is easily achieved by enlarging m.

§3 Variations of the theme: L^2-extension theorems

L^2-extension theorem

In 1987, Ohsawa-Takegoshi proved the following extension theorem with estimates:

Theorem (Ohsawa-Takegoshi)

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, and H is a complex hyperplane in $\mathbb{C}^n$ with $sup_{z\in\Omega} d(z,H) < +\infty$. Then for any $\varphi \in PSH(\Omega)$ and $f \in \mathscr{O}(\Omega \cap H)$ satisfying $\int_{\Omega\cap H} |f|^2 e^{-\varphi} < \infty$, we can find $F \in \mathscr{O}(\Omega)$ such that $F|_{\Omega\cap H} = f$ and

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq C \int_{\Omega\cap H} |f|^2 e^{-\varphi}$$

where C is a constant, depending only on $sup_{z\in\Omega} d(z,H) < +\infty$.

L^2-extension theorem

By induction, we get

Theorem (Ohsawa-Takegoshi)

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and H is a complex affine subspace of $\mathbb{C}^n$. Then for any $\varphi \in PSH(\Omega)$ and $f \in \mathcal{O}(\Omega \cap H)$ satisfying $\int_{\Omega \cap H} |f|^2 e^{-\varphi} < \infty$, we can find $F \in \mathcal{O}(\Omega)$ such that $F|_{\Omega \cap H} = f$ and

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq C \int_{\Omega \cap H} |f|^2 e^{-\varphi}$$

where C is a constant, depending only on $n, diam\Omega$.

This theorem is very powerful, even the simplest case $H = point$.

Application: Demailly's approximation theorem

Theorem (Demailly, 1992)

Let $\varphi \in PSH(\Omega)$, where Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$. Denote by $A^2(\Omega, m\varphi)$ the Hilbert space of holomorphic functions f on Ω with $\int_\Omega |f|^2 e^{-m\varphi} < \infty$. And let $\varphi_m := \frac{1}{m}\log(\sum_l |\sigma_l|^2)$ where (σ_l) is an orthonormal basis of $A^2(\Omega, m\varphi)$. Then there are constants C_1, C_2 independent of m such that

1. $\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\zeta - z|<r} \varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^{2n}}$, *for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to \infty$.*

Application: Demailly's approximation theorem

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Let $\varphi \in PSH(\Omega)$*, where* Ω *is a bounded pseudoconvex domain in* $\mathbb{C}^n$*. Denote by* $A^2(\Omega, m\varphi)$ *the Hilbert space of holomorphic functions* f *on* Ω *with* $\int_\Omega |f|^2 e^{-m\varphi} < \infty$*. And let* $\varphi_m := \frac{1}{m}\log(\sum_l |\sigma_l|^2)$ *where* (σ_l) *is an orthonormal basis of* $A^2(\Omega, m\varphi)$*. Then there are constants* C_1, C_2 *independent of* m *such that*

1. $\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\zeta - z|<r} \varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^{2n}}$*, for every* $z \in \Omega$ *and* $r < d(z, \partial\Omega)$*. In particular,* φ_m *converges to* φ *pointwise and in* L^1_{loc} *topology on* Ω *when* $m \to \infty$*.*
2. $\nu(\varphi, z) - \frac{2n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z)$*, for every* $z \in \Omega$*. Here* $\nu(\varphi, z)$ *is the Lelong-number.*

Lelong number

For $\varphi \in PSH(\Omega)$ and $x_0 \in \Omega$, we define the Lelong number of φ at x_0 to be

$$\nu(\varphi, x_0) := \liminf_{z \to x_0} \frac{\varphi(z)}{\log|z - x_0|} = \lim_{r \to 0+} \frac{\sup_{B(x_0, r)} \varphi}{\log r}.$$

If $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0) = ord_{x_0} f$.

Proof of Demailly's theorem

- φ_m is well-defined: consider the evaluation functional $ev_z : A^2(\Omega, m\varphi) \to \mathbb{C}$, $f \mapsto f(z)$. It is bounded since the norm here dominates the L^∞ norm by Cauchy formula. By Riesz theorem, there is a $h_z \in A^2(\Omega, m\varphi)$ such that for any $f \in A^2(\Omega, m\varphi)$, we have $f(z) = \int_\Omega f\bar{h}_z e^{-m\varphi}$. Also, we have $h_z = \sum_l (h_z, \sigma_l)\sigma_l$ and so
$$\sup_{f\in B(1)} |f(z)|^2 = \|h_z\|^2 = \sum_l |(h_z, \sigma_l)|^2 = \sum_l |\sigma_l(z)|^2$$
So we get $\varphi_m = \frac{1}{m}\log\|h_z\|^2 = \sup_{f\in B(1)} \frac{1}{m}\log|f(z)|^2$. Since L^2-topology here is stronger than the uniform convergence topology, we know that $\sum_l |\sigma_l|^2$ converges uniformly on compact sets and is real analytic.

- For any $r < d(z_0, \partial\Omega)$, we have

$$
\begin{aligned}
|f(z_0)|^2 &\leq \frac{1}{\omega_{2n} r^{2n}} \int_{|z-z_0|<r} |f(z)|^2 dV(z) \\
&\leq \frac{1}{\omega_{2n} r^{2n}} \exp\left(m \sup_{|z-z_0|<r} \varphi(z)\right) \int_\Omega |f|^2 e^{-m\varphi}
\end{aligned}
$$

- For any $r < d(z_0, \partial\Omega)$, we have

$$\begin{aligned} |f(z_0)|^2 &\leq \frac{1}{\omega_{2n} r^{2n}} \int_{|z-z_0|<r} |f(z)|^2 dV(z) \\ &\leq \frac{1}{\omega_{2n} r^{2n}} \exp\left(m \sup_{|z-z_0|<r} \varphi(z)\right) \int_\Omega |f|^2 e^{-m\varphi} \end{aligned}$$

- So we get

$$\varphi_m(z_0) \leq \frac{1}{m}\left(\log \frac{1}{\omega_{2n} r^{2n}} + m \sup_{|z-z_0|<r} \varphi(z)\right) = \sup_{|z-z_0|<r} \varphi(z) + \frac{1}{m} \log \frac{C_2}{r^{2n}}.$$

- Now we apply the L^2-extension theorem: start with the point z_0 and the function $a \in \mathbb{C}$, we can find f holomorphic on Ω and $f(z_0) = a$ with $\int_\Omega |f|^2 e^{-m\varphi} \leq C_3 |a|^2 e^{-m\varphi(z_0)}$ where C_3 depends only on n and $diam\Omega$ (may assume $\varphi(z_0) \neq -\infty$).

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- Choose a such that $C_3|a|^2 e^{-m\varphi(z_0)} = 1$, then we get

$$\varphi_m(z_0) \geq \frac{1}{m} \log |f(z_0)|^2 = \frac{1}{m} \log |a|^2 = \varphi(z_0) - \frac{C_1}{m}.$$

The conclusion for Lelong numbers follows easily form the first conclusion:

$$\sup_{|z-z_0|<r} \varphi(z) - \frac{C_1}{m} \leq \sup_{|z-z_0|<r} \varphi_m(z) \leq \sup_{|z-z_0|<2r} \varphi(z) + \frac{1}{m} \log \frac{C_2}{r^{2n}},$$

and hence

$$\frac{\sup_{|z-z_0|<2r} \varphi(z) + \frac{1}{m} \log \frac{C_2}{r^{2n}}}{\log r} \leq \frac{\sup_{|z-z_0|<r} \varphi_m(z)}{\log r} \leq \frac{\sup_{|z-z_0|<r} \varphi(z) - \frac{C_1}{m}}{\log r}.$$

Taking limit $r \to 0+$, we get

$$\nu(\varphi, z_0) - \frac{2n}{m} \leq \nu(\varphi_m, z_0) \leq \nu(\varphi, z_0).$$

§4 Recent progresses: multiplier ideal sheaf, openness conjecture and strong openness conjecture

Multiplier ideal sheaves

Definition

Let $\varphi \in PSH(\Omega)$, we define the multiplier ideal sheaf of φ to be the sheaf of germs $f \in \mathscr{O}_z$ such that $|f|^2 e^{-\varphi}$ is integrable near z. We denote it by $\mathcal{I}(\varphi)$. It is a subsheaf of $\mathscr{O}_\Omega$. Explicitly,

$$\mathcal{I}(\varphi)(U) = \{f \in \mathscr{O}(U) \mid |f|^2 e^{-\varphi} \in L^1_{loc}\}.$$

This is a useful tool, connecting complex analysis and potential theory to algebraic geometry.

Nadel's vanishing theorem

Theorem

Let (X, ω) be a compact Kähler manifold, F a holomorphic line bundle with a possibly singular Hermitian metric, locally of the form $e^{-\varphi}$. If $\sqrt{-1}\Theta(h) = \sqrt{-1}\partial\bar{\partial}\varphi \geq \epsilon\omega$ for some continuous positive function ϵ, then we have

$$H^q(X, \mathscr{O}(K_X \otimes F) \otimes \mathcal{I}(\varphi)) = 0, \forall q \geq 1.$$

It contains several useful generalizations of Kodaira vanishing theorem, e.g. the Kawamata-Viehweg vanishing theorem.

Demailly-Kollár's conjectures

Conjecture (Openness conjecture)

For a plurisubharmonic function φ defined in a neighbourhood of $z_0 \in \mathbb{C}^n$, the set of those $p \in \mathbb{R}$ such that $e^{-p\varphi}$ is integrable near z_0 is an open interval of the form $(-\infty, p_0)$.

Conjecture (Strong openness conjecture)

For a plurisubharmonic function φ defined in the unit polydisc Δ_1 and a holomorphic function F satisfies $\int_{\Delta_1} |F|^2 e^{-\varphi} < \infty$, there is a $r \in (0, 1)$ and a $p > 1$ such that

$$\int_{\Delta_r} |F|^2 e^{-p\varphi} < \infty$$

- The openness conjecture was proved by Berndtsson in 2013.

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- The strong openness conjecture was proved by Guan-Zhou later in the same year. (Published in 2015).
- There is a shorter proof of openness conjecture by Berndtsson, using methods of Guan-Zhou.

Proof of Openness Conjecture

- We may assume that $z_0 = 0$, φ is defined in a neighborhood of $\bar{\Delta}^n$ and $\varphi \leq 0$. We shall prove by induction on n. The $n = 1$ case is easy and known before. So we omit this.

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- We may assume that $z_0 = 0$, φ is defined in a neighborhood of $\bar{\Delta}^n$ and $\varphi \leq 0$. We shall prove by induction on n. The $n = 1$ case is easy and known before. So we omit this.
- We first claim that if $e^{-\varphi}$ is not locally integrable near the origin then

$$\int_{\Delta^{n-1}} e^{-\varphi(\cdot, z_n)} dV(z') \geq \frac{c_n}{|z_n|^2}, \quad |z_n| \leq \frac{1}{2}.$$

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- Assuming the claim now, we can easily finish the proof:

- Assume the result is proven for $n-1$. Now suppose

$$\int_{\Delta^n} e^{-p_0\varphi} < \infty.$$

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- By definition of p_0, if $p > p_0$, $e^{-p\varphi}$ is not integrable near the origin. So we can use the claim to conclude

$$\int_{\Delta^{n-1}} e^{-p\varphi(\cdot,z_n)} dV(z') \geq \frac{c_n}{|z_n|^2}, \quad |z_n| \leq \frac{1}{2}.$$

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- Now for a.e. z_n, we have $\int_{\Delta^{n-1}} e^{-p_0\varphi(\cdot,z_n)} dV(z') < \infty$, so we can use induction assumption to get that for any such z_n and a $p > p_0$,

$$\int_{\Delta^{n-1}} e^{-p\varphi(\cdot,z_n)} dV(z') < \infty$$

- Now apply Lebesgue's dominated convergence theorem, we get for previous z_n, we have

$$\int_{\Delta^{n-1}} e^{-p_0\varphi(\cdot,z_n)}dV(z') = \lim_{p\to p_0+}\int_{\Delta^{n-1}} e^{-p\varphi(\cdot,z_n)}dV(z') \geq \frac{c_n}{|z_n|^2}.$$

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- But this implies $\int_{\Delta^n} e^{-p_0\varphi} = \infty$.

Proof of the claim

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- Fix z_n and assume $\int_{\Delta^{n-1}} e^{-\varphi(\cdot,z_n)} dV(z') < +\infty$. Apply Ohsawa-Takegoshi to $f \equiv 1$ on $\Delta^{n-1} \times z_n$, we get F holomorphic on Δ^n with

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- Using mean value inequality, we get

$$|F(0,\zeta)|^2 \leq C_2 \int_{\Delta^n} |F|^2 \leq C_2 \int_{\Delta^n} |F|^2 e^{-\varphi}, \quad |\zeta| \leq \frac{1}{2}$$

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$$|F(0,\zeta)|^2 \leq C_2 \int_{\Delta^n} |F|^2 \leq C_2 \int_{\Delta^n} |F|^2 e^{-\varphi}, \quad |\zeta| \leq \frac{1}{2}$$

- Now $F(0,0) = 0$, we can apply the one-variable Schwarz lemma to get

$$|F(0,\zeta)|^2 \leq C_3 |\zeta|^2 \int_{\Delta^n} |F|^2 e^{-\varphi}, \quad |\zeta| \leq \frac{1}{2}.$$

  Now take $\zeta = z_n$ and use the fact $F(0,z_n) = 1$.

References

We list some papers and books suitable for beginners.

- Hörmander: his book and paper "L^2 estimates and existence theorems for the $\bar{\partial}$-operator".
- Demailly's books and lecture notes.
- The book "Analytic and Algebraic Geometry: Common Problems Different Methods"(AMS) contains several nice lecture notes on L^2-method, written by Berndtsson, Demailly, Varolin...
- Blocki: "Cauchy-Riemann meet Monge-Ampère".
- Chen Boyong: "L^2 theory of the Cauchy-Riemann equation", Science Press,2022.