§7. Bishop-Gromov relative volume comparison.

§7.1. Volume form.

- Let (M^n, g) be a Riem. mfd.
- An orientation is a family of coord. covering $\{(\varphi_\alpha, U_\alpha)\}$, s.t.

$$\mathrm{Jac}(\varphi_{\alpha\beta}) > 0, \quad \text{on } \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)$$

- Recall: $f = (f_1, \cdots, f_n): \mathbb{R}^n \to \mathbb{R}^n$, C^1 function.

$$\mathrm{Jac}(f) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

- We call M^n is orientable, if there is an orientation on M.
- Equivalently, there is an global n-form on M, that is non-vanishing everywhere.
- Let ω be a nowhere vanishing n-form, ω is called to be compatible with the orientation, if on each coord. chart $(\varphi_\alpha, U_\alpha, x_\alpha = (x_\alpha^1, \cdots, x_\alpha^n))$, ω can be written as:

$$\omega = f_\alpha(x_\alpha) \cdot dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n, \quad f_\alpha(x_\alpha) > 0 \text{ on } U_\alpha.$$

• Let $\{(\varphi_\alpha, U_\alpha)\}$ be an orientation on M. A coord. (φ, U) is called compatible with the orientation, if $Jac(\varphi_\alpha^{-1}\circ\varphi)>0$, $\forall\, \alpha$.

• **Def 1.** • Let $\{(\varphi_\alpha, U_\alpha)\}$ be an orientation and ω be a nowhere vanishing n-form, compatible with the orientation. In each orientation compatible coord. (φ, U), define:

$$\int_{\varphi(U)} \omega = \int_U f(x)\cdot dx^1\cdots dx^n,$$

where $\omega = f\cdot dx^1\wedge\cdots\wedge dx^n$ on U.

• Let β be another n-form, $\beta = h\cdot\omega$, $h\in C^\infty(M)$.

• Let $\{\eta_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$. Then, define integration on compact subset $K\subset\subset M$ by:

$$\int_K \beta = \sum_\alpha \int_{K\cap\varphi_\alpha(U_\alpha)} \eta_\alpha\cdot\beta = \sum_\alpha \int_{\varphi_\alpha^{-1}(K\cap\varphi_\alpha(U_\alpha))} (\eta_\alpha\circ\varphi_\alpha)\cdot f_\alpha\cdot dx_\alpha^1\cdots dx_\alpha^n.$$

where $x_\alpha = (x_\alpha^1,\dots,x_\alpha^n)$ is coord. on U_α, and $\omega = f_\alpha dx_\alpha^1\wedge\cdots\wedge dx_\alpha^n$ on $\varphi_\alpha(U_\alpha)$.

• The integration is independent of the choice of $\{\eta_\alpha\}$.

• Riem. volume form.

• (M^n, g) : oriented Riem. mfd.

• Define a canonical n-form:

$$\omega_g := \sqrt{\det(g_{ij})} \cdot dx^1 \wedge \cdots \wedge dx^n,$$

on each coord. chart $(\varphi, U, x=(x^1, \cdots, x^n))$, compatible with orientation.

• ω_g is independent of the coord.

• We can define the volume of compact subset $K \subset\subset M$:

$$\mathrm{Vol}(K) := \int_K \omega_g.$$

• For an open set $U \subset M$, with an exhaustion by compact subsets K_i, i.e., $\bigcup K_i = U$, $K_i \subset K_{i+1}$, define:

$$\mathrm{Vol}(U) = \lim_{i \to +\infty} \mathrm{Vol}(K_i).$$

• §7.2. Cut locus.

• Define the cut point in direction $v \in T_pM$, $|v|=1$, by:

$$t_v := \sup\{t_0 > 0 \mid \exp_p(t \cdot v) \text{ is minimal up to } t_0\}.$$

• By Gauss Lemma, $t_v > 0$, $\forall v \in T_pM$, $|v|=1$.

- t_v may be infinity.
- Set $\mathcal{U}_p := \{ t\cdot v : v \in T_pM,\ |v|=1,\ 0 \le t < t_v \} \subseteq T_pM$.
- Define the cut point in the direction $v \in T_pM$, $|v|=1$, to be the unique point $\gamma_v(t_v \cdot v) = \exp_p(t_v \cdot v)$.
- $S_pM := \{ v \in T_pM : |v| = 1 \}$, then $t_v : S_pM \to \mathbb{R}$ is a continuous map.
- The cut point along a geodesic γ has a characterization:

  Either: ① a conjugate point of $\gamma(0)$, or

  ② $\exists$ at least two minimal geodesic from $\gamma(0)$ to this point.
- $\mathcal{U}_p$ is a star-shaped, open domain of T_pM.

  $\exp_p : \mathcal{U}_p \longrightarrow \exp_p(\mathcal{U}_p)$ is a diffeomorphism.
- Let $\mathcal{C}_p := \{ \gamma_v(t_v \cdot v) : v \in S_pM \}$ to be the set of cut-locus.

  Then $\mathrm{Vol}(\mathcal{C}_p) = 0$.
- For any compact/open subset $A \subseteq M$, the volume

$$\mathrm{Vol}(A) = \mathrm{Vol}(A \setminus \mathcal{C}_p).$$

- The volume can be calculated in the normal coord.

$\exp_p : \mathcal{U}_p \to \exp_p(\mathcal{U}_p)$.

- Let $\{e_i\}$ be an orthonormal frame at p, $e_i \in T_pM$, compatible with orientation.
- Let $x = x^i \cdot e_i$ be decomposition of $x \in T_pM$, then $(x^1, \cdots, x^n)$ is Descartes coord. on T_pM.
- $\forall\, v \in T_pM$, at $\exp_p(v)$, $\frac{\partial}{\partial x^i} := d(\exp_p)_v(e_i) \in T_{\exp_p(v)}M$.
- For any $x_0 \in T_pM$, let γ_{x_0} be the geodesic in the direction x_0.
- Define variations of geodesics:

$$\gamma_{i,u}(t) = \gamma_i(t,u) := \exp_p\left(t \cdot (x_0 + u e_i)\right), \quad i = 1, \cdots, n.$$

- The Jacobi field:

$$J_i(t,u) = \frac{\partial}{\partial u}\gamma_i = t \cdot d(\exp_p)_{t\cdot(x_0+ue_i)}(e_i) = t \cdot \frac{\partial}{\partial x^i}$$

$$\Rightarrow \frac{\partial}{\partial x^i} = t^{-1} \cdot J_i, \quad t \in (0, t_{x_0}).$$

- Along the geodesic $\gamma_{x_0} = \gamma_{i,0}$, for $t \in (0, t_{x_0})$,

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = t^{-2} \cdot g(J_i, J_j)$$

$\gamma_{x_0}(t)$

- After orthogonal transformation on T_pM, we may assume

$$x_0 = |x_0| \cdot e_n$$

- Then, by Gauss Lemma, along $\gamma_{x_0} = \gamma_{i,0}$,

$$g_{in} = 0 = g_{ni}, \quad i = 1, \cdots, n-1; \quad g_{nn} = 1.$$

Γ. $g_{in} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^n} \rangle = \langle d(\exp_p)_{tx_0}(e_i), d(\exp_p)_{tx_0}(e_n) \rangle$

$= \langle e_i, e_n \rangle = \delta_{in}.$ $\lrcorner$

- Along $\gamma_{x_0} = \gamma_{i,0}$, we have:

$$J_i(0) = 0, \quad J_i'(0) = e_i = \frac{\partial}{\partial x^i}, \quad J_i(t) = t \cdot \frac{\partial}{\partial x^i}.$$

- Since $d(\exp_p)_0 = id$, at $p = \gamma_{x_0}(0)$, $\frac{\partial}{\partial x^i} = d(\exp_p)_0(e_i) = e_i$,

$$g_{ij}(p) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = \langle e_i, e_j \rangle = \delta_{ij}$$

- Along $\gamma_{x_0} = \gamma_{i,0}$, we compute:

$$\frac{d}{dt} \log \sqrt{\det(g_{ij})} = \frac{1}{2} \cdot g^{ij} \cdot \frac{\partial}{\partial t} g_{ij}$$

$$= \frac{1}{2} \cdot g^{ij} \cdot \left(t^{-2} \langle J_i', J_j \rangle + t^{-2} \langle J_i, J_j' \rangle - 2t^{-3} \langle \underset{t^2 \cdot g_{ij}.}{\underset{\parallel}{J_i, J_j}} \rangle \right)$$

$$= t^{-2} \cdot g^{ij} \cdot \langle J_i', J_j \rangle - n \cdot t^{-1}.$$

- By Gauss Lemma, we have $\langle J_n, J_n \rangle = t^2$, hence:

$$2t = \frac{\partial}{\partial t} t^2 = \frac{\partial}{\partial t}\langle J_n, J_n \rangle = 2 \cdot \langle J_n' \cdot J_n \rangle.$$

By $g_{in} = g_{ni} = \delta_{in}$, we have $g^{in} = g^{ni} = \delta_{in}$, hence:

$$g^{ij} \cdot \langle J_i', J_j \rangle = \sum_{i,j \le n-1} g^{ij} \cdot \langle J_i', J_j \rangle + \langle J_n', J_n \rangle$$

$$= \sum_{i,j \le n-1} g^{ij} \cdot \langle J_i' \cdot J_j \rangle + t.$$

Hence:

(1) $$\frac{d}{dt} \log \sqrt{\det(g_{ij})} = t^{-2} \cdot \sum_{i,j \le n-1} g^{ij} \cdot \langle J_i' \cdot J_j \rangle - (n-1) \cdot t^{-1}.$$

- At any $0 < t < t_{x_0}$, we rewrite $\sum_{i,j \le n-1} g^{ij} \cdot \langle J_i', J_j \rangle$ as follows.

Note that the formula $g^{ij} \cdot \langle J_i', J_j \rangle$ is indep. of linear transformation of the Jacobi fields $\{J_i\}$, we may assume that $\left\{ \frac{J_i(t_0)}{|J_i(t_0)|} \right\}$ is an orthonormal basis of $T_{\gamma_{x_0}(t_0)} M$.

and $\frac{J_n}{|J_n|}(t_0) = \frac{\dot{\gamma}_{x_0}(t_0)}{|\dot{\gamma}_{x_0}(t_0)|}$. Then, at $\gamma_{x_0}(t_0)$,

$$g_{ij}(t_0) = t_0^{-2} \cdot \langle J_i, J_j \rangle (t_0) = t_0^{-2} \cdot |J_i(t_0)| \cdot |J_j(t_0)| \cdot \delta_{ij},$$

hence we can rewrite:

$$\sum_{i,j\leq n-1} g^{ij}\cdot\langle J_i', J_j\rangle(t_0) = \sum_{i\leq n-1} t_0^2\,|J_i(t_0)|^{-2}\langle J_i', J_i\rangle(t_0)$$

$$= \sum_{i\leq n-1} t_0^2\cdot\langle \tilde{J}_i', \tilde{J}_i\rangle(t_0).$$

where $\tilde{J}_i = \dfrac{J_i}{|J_i(t_0)|}$ is a normalized Jacobi field.

- Recall that $J_i(t_0)\neq 0$, since $\gamma_{x_0}(t_0)$ is not conjugate to p.

Therefore by (1), along γ_{x_0}, at $t_0\in(0, t_{x_0})$,

$$\frac{d}{dt}\log\sqrt{\det(g_{ij})}\,(t_0) = \sum_{i\leq n-1}\langle \tilde{J}_i', \tilde{J}_i\rangle(t_0) - (n-1)\cdot t_0^{-1},$$

where $\{\tilde{J}_i\}_{i=1}^{n-1}\subset \mathcal{J}_0^{\perp}$ is a family of Jacobi fields,

s.t. $\{\tilde{J}_i(t_0)\}_{i=1}^{n-1}$ is an orthonormal basis of $\dot{\gamma}_{x_0}(t_0)^{\perp}\subset T_{\gamma_{x_0}(t_0)}M$.

§7.3. Volume comparison.

- Polar coord.:

$$\Phi: \mathcal{U}_p \longrightarrow \exp_p(\mathcal{U}_p)\subset M, \quad (t,v)\longmapsto \exp_p(t\cdot v).$$

where $(t,v)\in\mathbb{R}_+\times S_pM$, $0\leq t< t_v$.

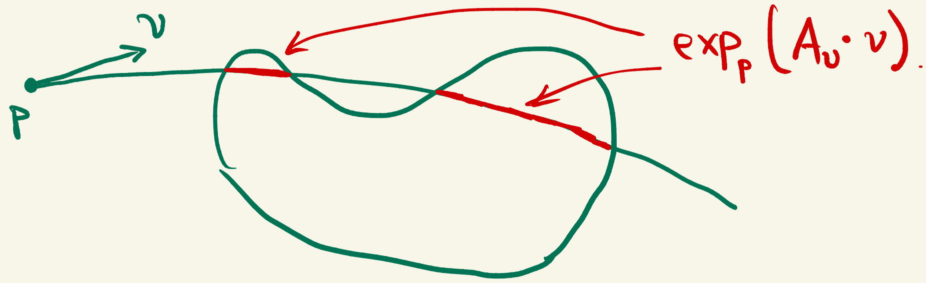
- Let $\omega_g =: \mathcal{A}(t,v)\,dt\wedge\omega_{S^{n-1}}$ be the Riem. volume form,

where $\omega_{S^{n-1}}$ is the volume form on unit sphere in $\mathbb{R}^n$.

• Then, for any subset $A \subset M$, by Fubini Thm,

$$\mathrm{Vol}(A) = \mathrm{Vol}(A \setminus \mathcal{C}_p) = \int_{S^{n-1}} \left(\int_{A_v} \mathcal{A}(t,v)\, dt \right) \omega_{S^{n-1}},$$

where $A_v := \{ 0 < t < t_v : t \cdot v \in A \}$, $\forall\, v \in S_pM$.



• <u>Theorem 2.</u> (Bishop–Gromov).

• (M^n, g): connected, complete Riem. mfd, $\mathrm{Ric} \geq (n-1) \cdot K_0$, $K_0 \in \mathbb{R}$.

• (i). Along any minimal geodesic $\gamma : [0, l] \to M$ of unit speed, suppose $\omega_g = \sqrt{\det(g_{ij})} \cdot dx^1 \wedge \cdots \wedge dx^n$ in normal coord.. <u>Then</u>:

$$\frac{d}{dt} \log \frac{\sqrt{\det(g_{ij})}}{\sqrt{\det(\underline{g}_{ij})}} \leq 0, \quad \text{along } \gamma$$

where $\underline{g}$ is the corresponding metric on space form $\underline{M}$ of constant sectional curvature K_0, computed along a geodesic $\underline{\gamma} : [0, l] \to \underline{M}$.

• (ii). Define annulus:

$$A_{s,r}(p) := \{\exp_p(t\cdot v) : s < t < \min\{r, t_v\},\ v \in S_pM\}.$$

Then:

$$\frac{\mathrm{Vol}(A_{r_3,r_4}(p))}{\underline{\mathrm{Vol}}(\underline{A}_{r_3,r_4}(\underline{p}))} \leq \frac{\mathrm{Vol}(A_{r_1,r_2}(p))}{\underline{\mathrm{Vol}}(\underline{A}_{r_1,r_2}(\underline{p}))},$$

for any $0 \leq r_1 \leq \min\{r_2, r_3\} \leq \max\{r_2, r_3\} \leq r_4$.

In particular, choose $r_1 = r_3 = 0$ above,

$$\frac{\mathrm{Vol}(B_{r_2}(p))}{\mathrm{Vol}(\underline{B}_{r_2}(\underline{p}))} \leq \frac{\mathrm{Vol}(B_{r_1}(p))}{\mathrm{Vol}(\underline{B}_{r_1}(\underline{p}))}, \quad \forall\ r_1 \leq r_2.$$

• Proof: • For any $v \in S_pM$, $0 < t_0 < t_v$, by Eqn. (2),

(3) $$\frac{d}{dt} \log \sqrt{\det(g_{ij})}\,(t_0) = \sum_{i \leq n-1} \langle \tilde{J}_i', \tilde{J}_i \rangle (t_0) - (n-1)t_0^{-1},$$

where $\{\tilde{J}_i\}_{i=1}^{n-1} \subset \mathcal{J}_0^{\perp}$ satisfies that, $\{\tilde{J}_i(t_0)\}_{i=1}^{n-1}$ is an orthonormal basis of $\dot{\gamma}_v(t_0)^{\perp} \subset T_{\gamma_v(t_0)}M$.

• For each $1 \leq i \leq n-1$, since $\tilde{J}_i(0) = 0$, using Jacobi eqn.

$$\tilde{J}_i'' = R(\dot{\gamma}_v, \tilde{J}_i)\dot{\gamma}_v$$

we have:

$$\langle \tilde{J}_i', \tilde{J}_i \rangle(t_0) = \int_0^{t_0} \left(|\tilde{J}_i'|^2 + \langle \tilde{J}_i'', \tilde{J}_i \rangle \right) dt$$

$$= \int_0^{t_0} \left(|\tilde{J}_i'|^2 - Rm(\dot{\gamma}_v, \tilde{J}_i, \dot{\gamma}_v, \tilde{J}_i) \right) dt$$

$$= I(\tilde{J}_i, \tilde{J}_i).$$

- Similar computation on $\underline{M}$ gives:

$$\langle \underline{\tilde{J}}_i', \underline{\tilde{J}}_i \rangle(t_0) = I(\underline{\tilde{J}}_i, \underline{\tilde{J}}_i).$$

where $\{\underline{\tilde{J}}_i(t_0)\}_{i=1}^n$ is orthonormal at $\underline{\gamma}(t_0)$, satisfying

$$\underline{\tilde{J}}_i(0) = 0, \quad \{\underline{\tilde{J}}_i\}_{i=1}^{n-1} \subset \underline{\mathcal{J}}_0^\perp,$$

- Let $\{\underline{e}_i\}$ be parallel fields along $\underline{\gamma}$ with $\underline{e}_i(t_0) = \underline{\tilde{J}}_i(t_0)$. Then:

$$\underline{\tilde{J}}_i(t) = f(t) \cdot \underline{e}_i(t), \quad i = 1, \cdots, n-1,$$

for a common function f with $f(0) = 0$, since we assume $\tilde{J}_i(0) = 0$.

- Define comparison fields:

$$X_i(t) := f(t) \cdot e_i(t),$$

where $\{e_i\}$ are parallel fields along γ with $e_i(t_0) = \tilde{J}_i(t_0)$.

- Then, $X_i(0) = 0$, $X_i(t_0) = \tilde{J}_i(t_0)$, $X_i \perp \dot{\gamma}$, by §5, Lemma 1, we have for $i=1,\cdots,n-1$,

$$I(X_i, X_i) \geq I(\tilde{J}_i, \tilde{J}_i).$$

- Next,

$$\sum_{i\leq n-1} I(\underline{\tilde{J}_i}, \underline{\tilde{J}_i}) = \sum_{i\leq n-1} \int_0^{t_0} \left[|f'|^2 - f^2 \cdot \mathrm{Rm}(\underline{\dot{\gamma}}, \underline{e_i}, \underline{\dot{\gamma}}, \underline{e_i}) \right] dt$$

$$= (n-1) \cdot \int_0^{t_0} |f'|^2 dt - \int_0^{t_0} f^2 \cdot \underline{\mathrm{Ric}}(\underline{\dot{\gamma}}, \underline{\dot{\gamma}})\, dt$$

$\mathrm{Ric} \geq \underline{\mathrm{Ric}}$.

$$\geq (n-1) \cdot \int_0^{t_0} |f'|^2 dt - \int_0^{t_0} f^2 \cdot \mathrm{Ric}(\dot{\gamma}, \dot{\gamma})\, dt$$

$$= \sum_{i\leq n-1} I(X_i, X_i)$$

$$\geq \sum_{i\leq n-1} I(\tilde{J}_i, \tilde{J}_i).$$

Using Eqn. (3), we conclude:

$$\frac{d}{dt} \log \sqrt{\det(\underline{g}_{ij})}\,(t_0) \geq \frac{d}{dt} \log \sqrt{\det(g_{ij})}\,(t_0).$$

- (ii). This is equivalent to:

$$\frac{\mathrm{Vol}(A_{s,r}(p))}{\underline{\mathrm{Vol}}(\underline{A}_{s,r}(\underline{p}))} \text{ is decreasing in both } s \text{ and } r.$$

• Rewrite this as:

$$\frac{Vol(A_{s,r}(p))}{Vol(\underline{A}_{s,r}(\underline{p}))} = \frac{\int_{S^{n-1}}\left(\int_s^r \mathcal{A}(t,v)\,dt\right)\omega_{S^{n-1}}}{\int_{S^{n-1}}\left(\int_s^r \underline{\mathcal{A}}(t,\underline{v})\,dt\right)\omega_{S^{n-1}}}$$

$$= \frac{1}{Vol(S^{n-1})}\cdot\int_{S^{n-1}} \frac{\int_s^r \mathcal{A}(t,v)\,dt}{\int_s^r \underline{\mathcal{A}}(t,\underline{v})\,dt}\,\omega_{S^{n-1}}$$

since $\underline{\mathcal{A}}(t,\underline{v})$ is indep. of $\underline{v}\in S_{\underline{p}}^{n-1}\underline{M}$. Hence it suffices to show that, for each $v\in S_pM$,

$$\frac{\int_s^r \mathcal{A}(t,v)\,dt}{\int_s^r \underline{\mathcal{A}}(t,\underline{v})\,dt} \text{ is decreasing in both } s \text{ and } r.$$

• Suppose $s<r_1<r_2$, then by putting $\eta(t)=\dfrac{\mathcal{A}(t,v)}{\underline{\mathcal{A}}(t,\underline{v})}$. From (i),

$$\frac{d}{dt}\log\eta(t)\le 0 \implies \frac{d}{dt}\eta(t)\le 0.$$

Hence $\eta(t)$ is decreasing in t. Hence:

$$\int_s^{r_1}\mathcal{A}(t,v)\,dt\cdot\int_{r_1}^{r_2}\underline{\mathcal{A}}(t,\underline{v})\,dt$$

$$= \int_s^{r_1} \eta(t) \cdot \underline{\mathcal{A}}(t, \underline{v})\, dt \cdot \int_{r_1}^{r_2} \underline{\mathcal{A}}(t, \underline{v})\, dt$$

(4). $$\geq \int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt \cdot \eta(r_1) \cdot \int_{r_1}^{r_2} \underline{\mathcal{A}}(t, \underline{v})\, dt$$

$$\geq \int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt \cdot \int_{r_1}^{r_2} \eta(t) \cdot \underline{\mathcal{A}}(t, \underline{v})\, dt$$

$$= \int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt \cdot \int_{r_1}^{r_2} \mathcal{A}(t, v)\, dt.$$

- Hence:

$$\int_s^{r_1} \mathcal{A}(t, v)\, dt \cdot \int_s^{r_2} \underline{\mathcal{A}}(t, \underline{v})\, dt$$

$$= \int_s^{r_1} \mathcal{A}(t, v)\, dt \cdot \left(\int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt + \int_{r_1}^{r_2} \underline{\mathcal{A}}(t, \underline{v})\, dt \right)$$

(4).

$$\geq \int_s^{r_1} \mathcal{A}(t, v)\, dt \int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt + \int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt \int_{r_1}^{r_2} \mathcal{A}(t, v)\, dt$$

$$\geq \int_s^{r_1} \underline{\mathcal{A}}(t, \underline{v})\, dt \cdot \int_s^{r_2} \mathcal{A}(t, v)\, dt$$

Hence we conclude:

$$\frac{\int_s^{r_1} \mathcal{A}(t,v)\,dt}{\int_s^{r_1} \underline{\mathcal{A}}(t,\underline{v})\,dt} \geq \frac{\int_s^{r_2} \mathcal{A}(t,v)\,dt}{\int_s^{r_2} \underline{\mathcal{A}}(t,\underline{v})\,dt}.$$

- Similarly, we can get monotonicity in s. $\blacksquare$