Introduction to Complex Geometry

Chapter 4 Hodge theorem and Kodaira vanishing theorem

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Outline

§4.1 Hodge Theorem

Motivation

Let (M^m, g) be a compact oriented Riemannian manifold. Then we can define inner product on the space of real differential forms: for $\omega, \eta \in A^p(M)$

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle_g dV_g.$$

The idea of Hodge theorem is to represent a de Rham cohomology class by a "best"closed form. Since we can define norm of a differential form, a natural idea is to find a closed form of minimal norm within its cohomology class. To be precise, start with a closed p-form $\eta \in A^p(M)$, we want to minimize the functional:

$$\Phi(\xi) := \|\eta + d\xi\|^2, \quad \xi \in A^{p-1}(M).$$

We can solve this variational problem by considering the corresponding Euler-Lagrange equation, which is an elliptic system.

Harmonic forms

Suppose $\eta_0 = \eta + d\xi_0$ achieves the minimum of $||\eta + d\xi||^2$, then for any $\xi \in A^{p-1}(M)$,

$$||\eta_0 + td\xi||^2 = (\eta_0 + td\xi, \eta_0 + td\xi) = ||\eta_0||^2 + 2t(\eta_0, d\xi) + t^2||d\xi||^2$$

achieves its minimum at $t = 0$. This happens if and only if $(\eta_0, d\xi) = 0$ for any $\xi \in A^{p-1}(M)$. We can define an operator d^*, the "formal adjoint" of d, such that $(\alpha, d\beta) = (d^*\alpha, \beta)$ for any $\alpha \in A^p(M)$ and $\beta \in A^{p-1}(M)$. Then $(\eta_0, d\xi) = 0$ for any $\xi \in A^{p-1}(M)$ if and only if $(d^*\eta_0, \xi) = 0$ for any $\xi \in A^{p-1}(M)$, which implies $d^*\eta_0 = 0$.

Definition

Let (M^m, g) be a compact oriented Riemannian manifold. A smooth differential form $\omega \in A^p(M)$ is called a "harmonic p-form" if $d\omega = 0, d^*\omega = 0$.

Laplacian and Hodge's $\star$-operator

If we define the Laplacian operator to be $\Delta_d : A^p(M) \to A^p(M)$, $\Delta_d := dd^* + d^* d$, then for any smooth p-form $\omega \in A^p(M)$, we have

$$(\omega, \Delta_d \omega) = (\omega, dd^* \omega) + (\omega, d^* d\omega) = \|d^* \omega\|^2 + \|d\omega\|^2.$$

So we conclude that $\omega \in A^p(M)$ is harmonic if and only if $\Delta_d \omega = 0$.
To write down a precise formula for d^*, we introduce Hodge's "star"-operator: $* : A^p(M) \to A^{m-p}(M)$. If $\omega_1, \ldots, \omega_m$ is an orthonormal basis of $T_x^* M$, such that $\omega_1 \wedge \cdots \wedge \omega_m = dV_g$ gives the positive orientation, then we define

$$*\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = \delta_{1,2,\ldots,m}^{i_1,\ldots,i_p,j_1,\ldots,j_{m-p}} \omega_{j_1} \wedge \cdots \wedge \omega_{j_{m-p}}.$$

(Note that this implies $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge *(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) = \omega_1 \wedge \cdots \wedge \omega_m$.) Then we extend $*$ linearly. It is direct to check that this is well-defined.

Basic properties of $\star$

Moreover, if $\alpha = \sum_{i_1<\cdots<i_p} a_{i_1,\ldots,i_p}\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}$, $\beta = \sum_{i_1<\cdots<i_p} b_{i_1,\ldots,i_p}\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}$, we have

$$\begin{aligned}
\alpha\wedge *\beta &= \sum_{k_1<\cdots<k_p}\sum_{i_1<\cdots<i_p} a_{k_1,\ldots,k_p} b_{i_1,\ldots,i_p}\omega_{k_1}\wedge\cdots\wedge\omega_{k_p}\wedge *(\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}) \\
&= \sum_{i_1<\cdots<i_p} a_{i_1,\ldots,i_p} b_{i_1,\ldots,i_p}\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}\wedge *(\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}) \\
&= \langle\alpha,\beta\rangle_g dV_g = \beta\wedge *\alpha.
\end{aligned}$$

From the definition, it is easy to check that $** = (-1)^{p(m-p)} = (-1)^{pm+p}$ on $A^p(M)$. Also, we have

$$\langle *\alpha, *\beta\rangle_g dV_g = *\alpha\wedge **\beta = (-1)^{p(m-p)} *\alpha\wedge\beta = \beta\wedge *\alpha = \langle\beta,\alpha\rangle_g dV_g = \langle\alpha,\beta\rangle_g dV_g.$$

So $*$ is a point-wise isometry.

Laplacian and ⋆

Using $*$, we can also express d^* as:

Lemma

We have $d^* = (-1)^{mp+m+1} * d *$ *on* $A^p(M)$.

From this, it is easy to check:

Lemma

We always have $*\Delta_d = \Delta_d *$ *and* $\Delta_d d = d\Delta_d, \Delta_d d^* = d^*\Delta_d$.

Exercise: check that in case of $\mathbb{R}^n$, we have

$$\Delta_d\Big(\sum_{1\le i_1<\cdots<i_p\le m} f_{i_1\dots i_p} dx_{i_1}\wedge\cdots\wedge dx_{i_p}\Big) = -\sum_{1\le i_1<\cdots<i_p\le m}\Big(\sum_i \frac{\partial^2 f_{i_1\dots i_p}}{\partial x_i^2}\Big) dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

Theorem (Hodge)

Let (M^m, g) be a compact oriented Riemannian manifold. Then each de Rham cohomology class has a unique harmonic representative, so we have a linear isomorphism

$$\mathcal{H}^p(M) := \{\omega \in A^p(M) \mid \Delta_d \omega = 0\} \cong H^p_{dR}(M; \mathbb{R}), \quad p = 0, \ldots, m.$$

Moreover, $\mathcal{H}^p(M)$ is always a finite dimensional vector space, and we have a linear operator $G : A^p(M) \to A^p(M)$ such that for any $\omega \in A^p(M)$, if we denote its orthogonal projection to $\mathcal{H}^p(M)$ by ω_h, then we have the decomposition:

$$\omega = \omega_h + \Delta_d G\omega = \omega_h + d(d^* G\omega) + d^*(dG\omega).$$

In fact, we have a orthogonal direct sum decomposition

$$A^p(M) = \mathcal{H}^p(M) \oplus Im\, d \oplus Im\, d^*.$$

Remark

- G *is usually called the "Green operator". It is constructed in the following way: suppose the eigenvalues of* Δ_d *on* $A^p(M)$ *are* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$. *The corresponding eigenspaces are* $\mathcal{H}^p(M)$ *and* $E_1, E_2, \ldots$. *Then we define* $G|_{\mathcal{H}^p(M)} \equiv 0$ *and* $G|_{E_i} := \frac{1}{\lambda_i} id_{E_i}$.

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- *$\mathcal{H}^p(M)$, $\mathrm{Im}\, d$, $\mathrm{Im}\, d^*$ are indeed orthogonal to each other: Let $\omega_h \in \mathcal{H}^p(M)$, $\xi \in A^{p+1}(M), \eta \in A^{p-1}(M)$, then*

$$(\omega_h, d^*\xi) = (d\omega_h, \xi) = 0, \quad (\omega_h, d\eta) = (d^*\omega_h, \eta) = 0, \quad (d^*\xi, d\eta) = (\xi, dd\eta) = 0.$$

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- $\mathcal{H}^p(M)$, $Im\ d$, $Im\ d^*$ *are indeed orthogonal to each other: Let* $\omega_h \in \mathcal{H}^p(M)$, $\xi \in A^{p+1}(M), \eta \in A^{p-1}(M)$, *then*

$$(\omega_h, d^*\xi) = (d\omega_h, \xi) = 0, \quad (\omega_h, d\eta) = (d^*\omega_h, \eta) = 0, \quad (d^*\xi, d\eta) = (\xi, dd\eta) = 0.$$

- *Hodge decomposition implies Hodge isomorphism: the map* $\mathcal{H}^p(M) \to H^p_{dR}(M)$, $\eta \mapsto [\eta]$ *is injective: if* η_1, η_2 *are both harmonic and* $\eta_2 = \eta_1 + d\xi$, *then* $(d\xi, d\xi) = (\eta_2 - \eta_1, d\xi) = (d^*\eta_2 - d^*\eta_1, \xi) = 0$, *and hence* $\eta_1 = \eta_2$. *For surjectivity, if* $\omega = \omega_h + d\xi + d^*\eta$ *and* $d\omega = 0$, *then* $(d^*\eta, d^*\eta) = (\omega, d^*\eta) = (d\omega, \eta) = 0$. *So we get* $\omega = \omega_h + d\xi$, $[\omega] = [\omega_h]$.

Outline of existence of decomposition

Δ_d is a 2nd order elliptic operator, and we have a "basic estimate" :

$$\|\omega\|^2_{W^{1,2}} \leq C\Big(\Delta_d\omega + \omega, \omega\Big) = C\Big(\|\omega\|^2 + \|d\omega\|^2 + \|d^*\omega\|^2\Big).$$

We consider the quadratic form on $W^{1,2}(M, \Lambda^p T^* M)$:

$$\mathcal{D}(\xi, \eta) := (\xi, \eta) + (d\xi, d\eta) + (d^*\xi, d^*\eta).$$

Basic inequality implies that $\mathcal{D}(\omega)$ is an equivalent norm on $W^{1,2}(M, \Lambda^p T^* M)$. Given $\eta \in L^2(M, \Lambda^p T^* M)$, $\xi \mapsto (\xi, \eta)$ is a bounded linear functional on $A^p(M) \subset W^{1,2}(M, \Lambda^p T^* M)$: $|(\xi, \eta)| \leq \|\xi\| \cdot \|\eta\| \leq \|\eta\| \cdot \|\xi\|_{W^{1,2}} \leq C\sqrt{\mathcal{D}(\xi, \xi)}$. This extends to a bounded linear functional on $W^{1,2}(M, \Lambda^p T^* M)$, and we can use Riesz representation theorem to get a unique $\varphi \in W^{1,2}(M, \Lambda^p T^* M)$ such that for all $\xi \in A^p(M)$: $(\xi, \eta) = \mathcal{D}(\xi, \varphi)$.

Outline of existence of decomposition(continued)

Using this to define a linear map $T(\eta) := \varphi$. It is a bounded linear operator from $L^2(M, \Lambda^p T^* M)$ to $W^{1,2}(M, \Lambda^p T^* M)$. Its composition with the compact embedding $W^{1,2} \to L^2$ (also denoted by T) gives us a compact self-adjoint operator on $L^2(M, \Lambda^p T^* M)$. Intuitively, $T = (id + \Delta_d)^{-1}$.

By spectrum theorem and elliptic regularity, we have a Hilbert space direct sum decomposition $L^2(M, \Lambda^p T^* M) = \oplus_{m=0}^{\infty} E_m$, where each E_m is a finite dimensional space of smooth p-forms, satisfying $T\varphi = \rho_m \varphi, \forall \varphi \in E_m$, with $\rho_0 = 1 > \rho_1 > \rho_2 \dots$ and $\rho_m \to 0$. Then $E_0 = \mathcal{H}^p(M)$ and for $\varphi \in E_m$, we have $\Delta_d \varphi = \left(\frac{1}{\rho_m} - 1\right)\varphi =: \lambda_m \varphi$, $\lambda_m \nearrow \infty$.

Hermitian case

Now let X^n be a n-dimensional compact complex manifold, with almost complex structure J and Hermitian metric g. As before, we define $\omega_g := \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. It is a real $(1,1)$-form. A direct computation shows that we always have $dV_g = \frac{\omega_g^n}{n!}$.

In this case, we also extend Hodge's star operator complex linearly to complex differential forms. Then we also have $** = (-1)^{p(2n-p)} = (-1)^p$ on $A^p(X)$ and

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_{\mathbb{C}} dV_g.$$

On the space of smooth complex differential forms, the correct Hermitian inner product should be

$$(\alpha, \beta) := \int_X \alpha \wedge *\bar{\beta}.$$

One can check that the $*$ operator maps $A^{p,q}(X)$ to $A^{n-q,n-p}(X)$.

Formal adjoint of $\bar\partial$

As in the real case, we consider the Hermitian inner product on $A^{p,q}(X)$, and define an operator $\bar\partial^*$ by

$$(\xi, \bar\partial\eta) = (\bar\partial^*\xi, \eta), \quad \forall \xi \in A^{p,q}(X), \eta \in A^{p,q-1}(X).$$

Then we get

$$\begin{aligned}
(\bar\partial^*\xi, \eta) &= \int_X \bar\partial^*\xi \wedge *\bar\eta \\
&= (\xi, \bar\partial\eta) = \overline{(\bar\partial\eta, \xi)} = \overline{\int_X \bar\partial\eta \wedge *\bar\xi} = \int_X \partial\bar\eta \wedge *\xi \\
&= \int_X \partial\Big(\bar\eta \wedge *\xi\Big) - (-1)^{p+q-1}\bar\eta \wedge \partial(*\xi) = (-1)^{p+q} \int_X \bar\eta \wedge \partial(*\xi) \\
&= -\int_X \partial(*\xi) \wedge \bar\eta = -\int_X *\partial(*\xi) \wedge *\bar\eta.
\end{aligned}$$

So we get that on $A^{p,q}(X)$, we always have $\bar\partial^* = - * \partial *$.

The $\bar{\partial}$-laplacian and $\bar{\partial}$-harmonic forms

We define the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} : A^{p,q}(X) \to A^{p,q}(X)$ by

$$\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

We look for $\bar{\partial}$-closed form of minimal norm within a given Dolbeault cohomology class. Suppose $\xi \in A^{p,q}(X)$ is such a $\bar{\partial}$-closed form, then for any $\eta \in A^{p,q-1}(X)$, the quadratic function of $t \in \mathbb{R}$:

$$\|\xi + t\bar{\partial}\eta\|^2 = (\xi + t\bar{\partial}\eta, \xi + t\bar{\partial}\eta) = \|\xi\|^2 + 2tRe(\xi, \bar{\partial}\eta) + t^2\|\bar{\partial}\eta\|^2$$

takes its minimum at $t = 0$. We get $Re(\xi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. Using $\|\xi + t\sqrt{-1}\bar{\partial}\eta\|^2$ instead, we get $Im(\xi, \bar{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. So we get $(\xi, \bar{\partial}\eta) = (\bar{\partial}^*\xi, \eta) = 0$ for all $\eta \in A^{p,q-1}(X)$. This implies $\bar{\partial}^*\xi = 0$.

Definition

If $\omega \in A^{p,q}(X)$ satisfies $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$ (equivalently, $\Delta_{\bar{\partial}}\omega = 0$), then ω is called a "$\bar{\partial}$-harmonic (p, q)-form".

Hodge theorem for compact complex manifolds

Theorem (Hodge)

Let (X^n, J, g) be a compact Hermitian manifold. Then each Dolbeault cohomology class has a unique $\bar\partial$-harmonic representative, so we have a complex linear isomorphism

$$\mathcal{H}^{p,q}(X) := \{\omega \in A^{p,q}(X) \mid \Delta_{\bar\partial}\omega = 0\} \cong H^{p,q}_{\bar\partial}(X), \quad p, q = 0, \dots, n.$$

Moreover, $\mathcal{H}^{p,q}(X)$ is always a finite dimensional complex vector space, and we have a complex linear operator $G : A^{p,q}(X) \to A^{p,q}(X)$ such that for any $\omega \in A^{p,q}(X)$, if we denote its orthogonal projection to $\mathcal{H}^{p,q}(X)$ by ω_h, then we have the decomposition:

$$\omega = \omega_h + \Delta_{\bar\partial} G\omega = \omega_h + \bar\partial(\bar\partial^* G\omega) + \bar\partial^*(\bar\partial G\omega).$$

This is an orthogonal direct sum decomposition: $A^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus Im\, \bar\partial \oplus Im\, \bar\partial^*$.

Twisting with a bundle

Assume also that we have a holomorphic vector bundle $E \to X$ of rank r, with Hermitian metric h. X is compact. We define an Hermitian inner product on $C^\infty(X, \Lambda^{p,q}(X) \otimes E)$ by

$$(s,t) := \int_X \langle s,t\rangle_{g,h} dV_g,$$

where the pointwise Hermitian inner product $\langle , \rangle_{g,h}$ is induced from the Hermitian metric g on X and bundle metric h on E. We can define a $\bar{\partial}$-operator on $A^{p,q}(X,E)$, which we shall write $\bar{\partial}_E : A^{p,q}(X,E) \to A^{p,q+1}(X,E)$. We can also define a formal adjoint operator $\bar{\partial}_E^* : A^{p,q}(X,E) \to A^{p,q-1}(X,E)$ by requiring that

$$(s, \bar{\partial}_E t) = (\bar{\partial}_E^* s, t), \quad \forall s \in A^{p,q}(X,E), t \in A^{p,q-1}(X,E).$$

Then we define $\Delta_{\bar{\partial}_E} := \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^* : A^{p,q}(X,E) \to A^{p,q}(X,E)$, and $\mathcal{H}^{p,q}(X,E) := Ker\left(\Delta_{\bar{\partial}_E}|_{A^{p,q}(X,E)}\right)$. The elements of $\mathcal{H}^{p,q}(X,E)$ are called "E-valued harmonic (p,q)-forms".

Generalized version

Theorem

Let (X^n, J, g) *be a compact Hermitian manifold.* $E \to X$ *be a holomorphic vector bundle of rank* r*, with Hermitian metric* h*. Then each cohomology class in* $H^{p,q}_{\bar\partial}(X, E)$ *has a unique harmonic representative, so we have a complex linear isomorphism*

$$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}_{\bar\partial}(X, E), \quad p, q = 0, \ldots, n.$$

Moreover, $\mathcal{H}^{p,q}(X, E)$ *is always a finite dimensional complex vector space, and we have a complex linear operator* $G_E : A^{p,q}(X, E) \to A^{p,q}(X, E)$ *such that for any* $\omega \in A^{p,q}(X, E)$*, if we denote its orthogonal projection to* $\mathcal{H}^{p,q}(X, E)$ *by* ω_h*, then we have an orthogonal direct sum decomposition* $A^{p,q}(X, E) = \mathcal{H}^{p,q}(X, E) \oplus Im\bar\partial_E \oplus Im\bar\partial_E^*$*, given by*

$$\omega = \omega_h + \Delta_{\bar\partial_E} G_E \omega = \omega_h + \bar\partial_E(\bar\partial_E^* G_E \omega) + \bar\partial_E^*(\bar\partial_E G_E \omega).$$

§4.2 Applications

Poincaré duality for de Rham cohomology

Theorem (Poincaré duality for de Rham cohomology)

Let M^m be a compact oriented differentiable manifold. Then

$$H^p_{dR}(M,\mathbb{R}) \cong H^{m-p}_{dR}(M,\mathbb{R}).$$

In particular, $b_p(M) = b_{m-p}(M)$.

Proof

Since $*$ commutes with Δ_d, and $** = \pm 1$, we conclude that $*$ induces a linear isomorphism between $\mathcal{H}^p(M)$ and $\mathcal{H}^{m-p}(M)$. Then the result follows from Hodge theorem.

Kodaira-Serre duality theorem

Theorem (Kodaira-Serre duality)

Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X of complex dimension n. Then we have a conjugate-linear isomorphism

$$\sigma : H^r(X, \Omega^p(E)) \xrightarrow{\cong} H^{n-r}(X, \Omega^{n-p}(E^*)).$$

Proof

We introduce a conjugate-linear operator $\bar{*}_E$, constructing from $* : A^{p,q} \to A^{n-q,n-p}$ and the conjugate-linear isomorphism $\tau : E \to E^*$ via bundle metric h. To make everything conjugate-linear, we also define $\bar{*} : A^{p,q}(X) \to A^{n-p,n-q}(X)$ by $\bar{*}(\eta) := *\bar{\eta}$. Then $\bar{*}_E : A^{p,q}(X, E) \to A^{n-p,n-q}(X, E^*)$ is defined by $\bar{*}_E(\eta \otimes s) := \bar{*}(\eta) \otimes \tau(s)$. Then we have $\bar{\partial}_E^* = -\bar{*}_{E^*} \circ \bar{\partial}_{E^*} \circ \bar{*}_E$ and hence $\bar{*}_E \Delta_{\bar{\partial}_E} = \Delta_{\bar{\partial}_{E^*}} \bar{*}_E$.

Proof (continued)

By Dolbeault isomorphism theorem and the generalized version of Hodge theorem, we have

$$H^r(X, \Omega^p(E)) \cong H^{p,r}_{\bar{\partial}}(X, E) \cong \mathcal{H}^{p,r}(X, E),$$

$$H^{n-r}(X, \Omega^{n-p}(E^*)) \cong H^{n-p,n-r}_{\bar{\partial}}(X, E^*) \cong \mathcal{H}^{n-p,n-r}(X, E^*).$$

Then $\bar{*}_E$ *induces a conjugate-linear map* $\sigma : H^r(X, \Omega^p(E)) \to H^{n-r}(X, \Omega^{n-p}(E^*))$*, and the Kodaira-Serre duality follows from the fact* $\bar{*}_E \circ \bar{*}_{E^*} = \pm 1$.

The Kähler case

Now we assume (X^n, J, g) is a compact Kähler manifold. Then we will have a better understanding of harmonic forms and Dolbeault cohomology. We shall begin by exploring the relation between Δ_d and $\Delta_{\bar{\partial}}$.
We introduce some operators that will be useful in our discussion:

$$d^c := \sqrt{-1}(\bar{\partial} - \partial).$$

Here my notation is the same as Wells, but differs from Griffiths-Harris by a factor 4π. Then $dd^c = \sqrt{-1}(\partial + \bar{\partial})(\bar{\partial} - \partial) = 2\sqrt{-1}\partial\bar{\partial}$. We define the "Lefschetz operator" $L : A^{p,q}(X) \to A^{p+1,q+1}(X)$ by:

$$L(\eta) := \omega_g \wedge \eta =: L\eta.$$

Its adjoint will be denoted by $\Lambda : A^{p+1,q+1}(X) \to A^{p,q}(X)$. We have

$$(\xi, L\eta) = (\Lambda\xi, \eta), \quad \forall \xi \in A^{p+1,q+1}(X), \eta \in A^{p,q}(X).$$

Kähler identity

The basic equality in the Kähler case is:

Lemma

On $A^{p,q}(X)$, we have $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$.

Given this, since L is a real operator, so is Λ, and we have

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*.$$

Combining these two identities, we further get

$$[\Lambda, d] = -d^{c*}, \quad [\Lambda, d^c] = d^*.$$

For the proof, one can use the Kähler normal coordinates to reduce it to the $\mathbb{C}^n$ case (since only first order derivatives are involved).

Consequences of the Kähler identity

A direct consequence of Kähler identities is that Δ_d commutes with both L and Λ: Since ω_g is closed, we have $dL(\eta) = d(\omega_g \wedge \eta) = \omega_g \wedge d\eta$, so $[L, d] = 0$. Taking adjoints, we get $[\Lambda, d^*] = 0$. So using $[\Lambda, d] = -d^{c*}$, we get

$$\begin{aligned} \Lambda\Delta_d &= \Lambda(dd^* + d^*d) = [\Lambda, d]d^* + d\Lambda d^* + d^*\Lambda d \\ &= -d^{c*}d^* + dd^*\Lambda + d^*[\Lambda, d] + d^*d\Lambda \\ &= -d^{c*}d^* - d^*d^{c*} + \Delta_d\Lambda = \Delta_d\Lambda. \end{aligned}$$

Taking adjoints, we also get $[L, \Delta_d] = 0$.

Consequences of the Kähler identity

Besides Δ_d and $\Delta_{\bar\partial}$, we can similarly define Δ_∂. For compact Kähler manifolds, we have the following:

Proposition

In the Kähler case, we always have $\Delta_{\bar\partial} = \Delta_\partial = \frac{1}{2}\Delta_d$.

Proof

Use $d = \partial + \bar\partial$ *and* $d^* = \partial^* + \bar\partial^*$ *to compute:*

$$\begin{aligned}\Delta_d &= dd^* + d^*d = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= (\partial\partial^* + \partial^*\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) + \partial\bar\partial^* + \bar\partial\partial^* + \partial^*\bar\partial + \bar\partial^*\partial \\ &= \Delta_\partial + \Delta_{\bar\partial} + (\partial\bar\partial^* + \bar\partial^*\partial) + (\bar\partial\partial^* + \partial^*\bar\partial).\end{aligned}$$

We need to prove: $\partial\bar\partial^* + \bar\partial^*\partial = 0$, $\bar\partial\partial^* + \partial^*\bar\partial = 0$ *and* $\Delta_\partial = \Delta_{\bar\partial}$.

Proof (continued)

To prove $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$*, we use the identity* $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$*:*

$$\sqrt{-1}(\partial\bar{\partial}^* + \bar{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial = 0.$$

Now we compute Δ_∂ *and* $\Delta_{\bar{\partial}}$ *separately, both using Kähler identities:*

$$-\sqrt{-1}\Delta_\partial = \partial[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\partial = \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial.$$

$$\begin{aligned} \sqrt{-1}\Delta_{\bar{\partial}} &= \bar{\partial}[\Lambda, \partial] + [\Lambda, \partial]\bar{\partial} = \bar{\partial}\Lambda\partial - \bar{\partial}\partial\Lambda + \Lambda\partial\bar{\partial} - \partial\Lambda\bar{\partial} \\ &= \bar{\partial}\Lambda\partial + \partial\bar{\partial}\Lambda - \Lambda\bar{\partial}\partial - \partial\Lambda\bar{\partial} = \sqrt{-1}\Delta_\partial. \end{aligned}$$

From the above computations, we conclude that $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}} = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.

Hodge decomposition for compact Kähler manifolds

From this we conclude that Δ_d preserves $A^{p,q}(X)$, and $\mathcal{H}_d^{p+q}(X,\mathbb{C}) \cap A^{p,q}(X)$ equals $\mathcal{H}_{\bar\partial}^{p,q}(X)$. Since $\mathcal{H}_d^r(X,\mathbb{C}) = \oplus_{p+q=r}\left(\mathcal{H}_d^{p+q}(X,\mathbb{C}) \cap A^{p,q}(X)\right) = \oplus_{p+q=r}\mathcal{H}_{\bar\partial}^{p,q}(X)$, and $\overline{\mathcal{H}_{\bar\partial}^{p,q}(X)} = \mathcal{H}_{\bar\partial}^{q,p}(X)$, applying Hodge theorem, we get:

Theorem (Hodge decomposition for compact Kähler manifolds)

Let (X^n, J, g) be a compact Kähler manifold, then we have isomorphisms

$$H_{dR}^r(X,\mathbb{C}) \cong \oplus_{p+q=r} H_{\bar\partial}^{p,q}(X) \cong \oplus_{p+q=r} H^q(X,\Omega^p), \quad r = 0, 1, \ldots, 2n,$$

and

$$\overline{H_{\bar\partial}^{p,q}(X)} \cong H_{\bar\partial}^{q,p}(X).$$

In particular, we have

$$b_r = \sum_{p+q=r} h^{p,q}, \quad h^{p,q} = h^{q,p}.$$

Direct application

For example, we always have

$$\mathcal{H}_{\bar{\partial}}^{p,0}(X) = H^0(X, \Omega^p),$$

since any $(p,0)$-form is $\bar{\partial}^*$-closed and it is $\bar{\partial}$-closed if and only if it is holomorphic. Then we conclude that *any holomorphic p-form on a compact Kähler manifold is also d-closed and even d-harmonic.*

Corollary

The odd Betti number b_{2k+1} of a compact Kähler manifold X^n is always even.

Proof

We have

$$b_{2k+1} = \sum_{0 \le p,q \le n, p+q=2k+1} h^{p,q} = 2 \sum_{p<q, p+q=2k+1} h^{p,q} \equiv 0 \mod 2.$$

Example (Computations for $\mathbb{C}P^n$)

The topological structure of $\mathbb{C}P^n$ is rather simple: we have $\mathbb{C}P^n = U_0 \cup \{z_0 = 0\}$, with $U_0 \cong \mathbb{C}^n$ and $\{z_0 = 0\} \cong \mathbb{C}P^{n-1}$. So we can construct $\mathbb{C}P^n$ in the following way: start with a point (a "0-cell"), glue a $\mathbb{C}^1$ (a "2-cell") to get $\mathbb{C}P^1$, then glue a $\mathbb{C}^2$ (a "4-cell") to get $\mathbb{C}P^2$, ... So the cellular cohomologies of $\mathbb{C}P^n$ are:

$$H^{2k+1}(\mathbb{C}P^n, \mathbb{Z}) = 0, \quad H^{2k}(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}, k = 0, \dots, n.$$

Now ω_{FS} is a Kähler forms on $\mathbb{C}P^n$. Since $\omega_{FS}^k = L^k 1$ and $\Delta_d L = L\Delta_d$, each ω_{FS}^k is a harmonic (k, k)-form. So we conclude that $h^{p,p} \geq 1, p = 0, \dots, n$. On the other hand, $1 = b_{2p} \geq h^{p,p}$, we must have $b_{2p} = h^{p,p}$. Also, $h^{p,q} = 0$ when $p + q$ is odd. So the only non-zero Dolbeault cohomologies of $\mathbb{C}P^n$ are $H^{p,p}_{\bar{\partial}}(X) \cong \mathbb{C}, p = 0, \dots, n$. In particular, there are no non-zero holomorphic forms on $\mathbb{C}P^n$.

Example (Computations for $\mathbb{C}P^n$)

Now we can prove that $Pic(\mathbb{C}P^n) \cong \mathbb{Z}$:

Recall that from the short exact sequence $0 \to \underline{\mathbb{Z}} \to \mathscr{O} \to \mathscr{O}^* \to 1$, we get a long exact sequence of cohomologies:

$$\cdots \to H^1(\mathbb{C}P^n, \mathscr{O}) \to H^1(\mathbb{C}P^n, \mathscr{O}^*) \to H^2(\mathbb{C}P^n, \mathbb{Z}) \to H^2(\mathbb{C}P^n, \mathscr{O}) \to \dots$$

Now we have $H^p(\mathbb{C}P^n, \mathscr{O}) \cong H^{0,p}_{\bar{\partial}}(\mathbb{C}P^n) = 0$ when $p > 0$. This implies that

$$Pic(\mathbb{C}P^n) \cong H^1(\mathbb{C}P^n, \mathscr{O}^*) \cong H^2(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}.$$

The “$\partial\bar{\partial}$-lemma”

Lemma

If η is any d-closed (p, q)-form on a compact Kähler manifold X^n, and η is d- or ∂- or $\bar{\partial}$-exact, then

$$\eta = \partial\bar{\partial}\gamma$$

for some $(p-1, q-1)$-form γ. When $p = q$ and η is real, then we can take $\gamma = \sqrt{-1}\xi$ for a real $(p-1, q-1)$-form ξ.

Proof

Recall that in the Kähler case we have $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$*, they share the same kernel: harmonic forms. Since* η *is* d*- or* ∂*- or* $\bar\partial$*-exact, its harmonic projection must be zero. So we have*

$$\eta = \Delta_{\bar\partial} G_{\bar\partial}\eta = \bar\partial\bar\partial^* G_{\bar\partial}\eta.$$

Here we use the fact that $\bar\partial$ *commutes with* $G_{\bar\partial}$ *and that* $d\eta = 0 \Rightarrow \bar\partial\eta = 0$.
Now we look at the form $\bar\partial^* G_{\bar\partial}\eta$*, it is also orthogonal to harmonic forms. Also since* $G_\partial = G_{\bar\partial}$*, we have* $\partial\bar\partial^* G_{\bar\partial}\eta = -\bar\partial^*\partial G_\partial\eta = -\bar\partial^* G_\partial\partial\eta = 0$*. Then we can use Hodge decomposition for* Δ_∂*:*

$$\bar\partial^* G_{\bar\partial}\eta = \Delta_\partial G_\partial\bar\partial^* G_{\bar\partial}\eta = \partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta.$$

So we get

$$\eta = \bar\partial\partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta = \partial\bar\partial\Big(-\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta\Big) = \partial\bar\partial\Big(-\partial^*\bar\partial^* G_{\bar\partial}^2\eta\Big).$$

Importance of the $\partial\bar\partial$-Lemma

The most often used case is about $(1,1)$-class. Let ω and $\tilde\omega$ be two Kähler forms on X such that $[\omega] = [\tilde\omega] \in H^2_{dR}(X)$. Then $\tilde\omega - \omega$ is a d–exact form, so by the $\partial\bar\partial$-lemma, we can find a smooth function $\varphi \in C^\infty(X;\mathbb{R})$ such that

$$\tilde\omega = \omega + \sqrt{-1}\partial\bar\partial\varphi.$$

φ is unique up to a constant. On the other hand, if $\varphi \in C^\infty(X;\mathbb{R})$ satisfies $\omega + \sqrt{-1}\partial\bar\partial\varphi > 0$, then it defines a Kähler metric with the same Kähler class. So we conclude that the space of Kähler metrics within the same cohomology class $[\omega]$ is isomorphic to

$$\{\varphi \in C^\infty(X;\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\}\Big/\mathbb{R}.$$

One of the most important problem in Kähler geometry is the existence of canonical metrics in a given Kähler class. Through the $\partial\bar\partial$-lemma, we can reduce the problem to a (usually non-linear) partial differential equation for φ. This is the starting point of using non-linear PDEs to solve problems in Kähler geometry.

Lefschetz decomposition

Remark

If we further introduce the operator $h : A^*(X) \to A^*(X)$ *by* $h = \sum_{p=0}^{2n}(n-p)\Pi_p$ *, then we will have*

$$[\Lambda, L] = h, \quad [h, \Lambda] = 2\Lambda, \quad [h, L] = -2L.$$

Recall the 3-dimensional complex Lie algebra sl_2*, generated by*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They satisfy

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So $H \mapsto h, X \mapsto \Lambda, Y \mapsto L$ *gives a representation of* sl_2 *on* $\mathcal{H}^*(X, \mathbb{C})$*. Using elementary representation theory, we can get a finer decomposition result, due to S. Lefschetz.*

Hodge conjecture

- Let $Z \subset X$ be a analytic subvariety of codimension k in a Kähler manifold X^n. The Poincaré dual of its fundamental class is denoted by $\eta_Z := PD(i_*[Z]) \in H^{2k}(X, \mathbb{Z})$. Its image in $H^{2k}_{dR}(X, \mathbb{C})$ must belong to $H^{k,k}_{\bar{\partial}}(X)$.

Conjecture (Hodge conjecture)

Let X be a projective algebraic manifold, then any Hodge class $\alpha \in H^{k,k}_{\bar{\partial}}(X) \cap H^{2k}(X, \mathbb{Q})$ can be represented by a rational algebraic cycle $\sum_i c_i Z_i$ of codimension k.

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- It is known for $k = 1, n-1$ (Lefschetz).
- It is not true for general Kähler manifolds.

§4.3 Kodaira Vanishing Theorem and Embedding Theorem

Kodaira vanishing theorem

We say a real $(1,1)$-form ω is "positive" if locally it can be written as $\omega = \sqrt{-1}\sum_{i,j} a_{i\bar{j}} dz_i \wedge d\bar{z}_j$ where $(a_{i\bar{j}})$ is positive definite everywhere. A line bundle L is called "positive" if there exists an Hermitian metric h on L such that $\sqrt{-1}\Theta(h)$ is positive.

Theorem (Kodaira-Nakano)

If $L \to X$ is a positive holomorphic line bundle on a compact Kähler manifold,[1] *then we have*

$$H^q(X, \Omega^p(L)) = 0, \quad \textit{for } p + q > n.$$

In particular, $H^q(X, \mathscr{O}(K_X \otimes L)) = 0$ *for* $q > 0$.

[1] We can just assume X is compact complex manifold. Then if $\sqrt{-1}\Theta(h) > 0$, then it is a Kähler form on X and so X is in fact Kähler. Later, by Kodaira's embedding theorem, X is in fact projective algebraic.

Preparation of the proof

We use $\omega := \sqrt{-1}\Theta(h)$ as our reference Kähler metric. The Hodge theorem ensures that $H^q(X, \Omega^p(L)) \cong \mathcal{H}^{p,q}(X, L)$. So we need to show that when $p + q > n$ each L-valued harmonic (p, q)-form must be zero.

We need the following lemma, whose proof is almost identical to the "un-twisted case" we proved before:

Lemma

Let E be a holomorphic vector bundle over a compact Kähler manifold (X, ω) with Hermitian metric h. Introduce the operator $L : A^{p,q}(X, E) \to A^{p+1,q+1}(X, E)$ as before and define $\Lambda := L^*$. If we denote the $(1, 0)$ and $(0, 1)$ components of the Chern connection D by D' and $D''(= \bar{\partial})$, then we have

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}D'^*, \quad [\Lambda, D'] = \sqrt{-1}\bar{\partial}^*.$$

Bochner's formula

The proof of Kodaira vanishing theorem essentially follows from the comparison of two "Laplacians", the so called "Bochner's technique":

$$\Delta_{\bar{\partial},E} - \Delta_{D',E} = [\sqrt{-1}\Theta(h), \Lambda],$$

where $\Delta_{D',E} := D'D'^* + D'^*D'$. The reason for this equality is:

$$-\sqrt{-1}\Delta_{D',E} = D'[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]D' = D'\Lambda\bar{\partial} - D'\bar{\partial}\Lambda + \Lambda\bar{\partial}D' - \bar{\partial}\Lambda D',$$

while $\sqrt{-1}\Delta_{\bar{\partial},E} = \bar{\partial}[\Lambda, D'] + [\Lambda, D']\bar{\partial} = \bar{\partial}\Lambda D' - \bar{\partial}D'\Lambda + \Lambda D'\bar{\partial} - D'\Lambda\bar{\partial}$. So we get

$$\sqrt{-1}\Delta_{\bar{\partial},E} - \sqrt{-1}\Delta_{D',E} = \Lambda(\bar{\partial}D' + D'\bar{\partial}) - (\bar{\partial}D' + D'\bar{\partial})\Lambda.$$

Note that $\Theta(h)$ is of type $(1,1)$, we get $D'D' = 0$, $\bar{\partial}\bar{\partial} = 0$, so

$$\Theta(h) = D^2 = (D' + \bar{\partial})(D' + \bar{\partial}) = D'\bar{\partial} + \bar{\partial}D'.$$

So we get

$$\Delta_{\bar{\partial},E} - \Delta_{D',E} = -\sqrt{-1}[\Lambda, \Theta(h)] = [\sqrt{-1}\Theta(h), \Lambda].$$

Proof of the theorem

Now back to the proof of Kodaira's vanishing theorem. We have $\sqrt{-1}\Theta(h) = \omega$, so the above Bochner formula reduces to

$$\Delta_{\bar\partial} - \Delta_{D'} = [L, \Lambda] = (p + q - n)id.$$

So if $s \in \mathcal{H}^{p,q}(X, L)$ is not identically zero, then we have

$$(\Delta_{\bar\partial} s - \Delta_{D'} s, s) = (p + q - n)\|s\|^2 > 0.$$

On the other hand,

$$(\Delta_{\bar\partial} s - \Delta_{D'} s, s) = -(\Delta_{D'} s, s) = -\|D' s\|^2 - \|D'^* s\|^2 \le 0.$$

This is a contradiction.

Application: the embedding theorem

One important application of the Kodaira vanishing theorem is the following embedding theorem of Kodaira:

Theorem

If a compact complex manifold X has a positive line bundle, then it is projective algebraic.

The basic construction we shall use is the following: Let $L \to X$ be a holomorphic line bundle, such that $H^0(X, \mathscr{O}(L)) \neq 0$. Then we can take a basis of $H^0(X, \mathscr{O}(L))$, $s_0, \ldots, s_N$, and define a "map" from X to $\mathbb{C}P^N$:

$$x \mapsto [s_0(x), \ldots, s_N(x)].$$

This is defined using a local trivialization, so that we can identify each s_i as a locally defined holomorphic function. This map is independent of the trivialization we choose, but it is un-defined on the set of common zeroes of $s_0, \ldots, s_N$.

Strategy of Kodaira's proof

What Kodaira actually proved is the following: If $L \to X$ is a positive line bundle on a compact complex manifold, then *we can find a large integer* $m_0 > 0$ *such that for all* $m > m_0$*:*

1. $L^{\otimes m}$ *is "base point free", i.e. for any point* $p \in X$*, there is a global section* $s \in \Gamma(X, L^{\otimes m})$ *such that* $s(p) \neq 0$*;*
2. *Choose a basis of* $H^0(X, \mathscr{O}(L^{\otimes m}))$*,* $s_0, \ldots, s_{N_m}$*, then the "Kodaira map"* $\iota_{L^m} : X \to \mathbb{C}P^{N_m}$ *defined by*

$$x \mapsto [s_0(x), \ldots, s_{N_m}(x)]$$

is a holomorphic embedding.

Some concepts from algebraic geometry

Definition

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold.

- If there is an integer $m_0 > 0$ such that for all $m > m_0$, $L^{\otimes m}$ is base point free, then we say L is *semi-ample*;
- If L is base point free and the Kodaira map ι_L is a holomorphic embedding, then we say L is *very ample*;
- If there is an integer $m_0 > 0$ such that for all $m > m_0$, $L^{\otimes m}$ is very ample, then we say L is *ample*.

Ample=positive

A corollary of Kodaira's theorem is that on a compact complex manifold, a holomorphic line bundle is ample if and only if it is positive:

In fact, if L is positive, then it is ample by Kodaira's theorem. On the other hand, if L is ample, we can find $m \in \mathbb{N}$ such that ι_{L^m} is a holomorphic embedding. Then the pulling back of the hyperplane bundle is isomorphic to $L^{\otimes m}$, and the induced metric has positive curvature. The corresponding metric on L also has positive curvature.

Proof (Outline of the proof of Kodaira embedding theorem:)

For simplicity, we only prove that there is a sufficiently large m *such that* ι_{L^m} *is an embedding. We need to prove the following 3 properties:*

1. *Prove that* $L^{\otimes m}$ *is base point free when* m *large enough. We only need to show that for any point* $p \in X$*, we can find a* $m_p \in \mathbb{N}$ *such that for all* $m \geq m_p$*, we can find a* $s \in H^0(X, \mathscr{O}(L^{\otimes m}))$ *such that* $s(p) \neq 0$*. That is, the linear map* $r_p : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m}$ *is surjective.*
2. *Prove that for* m *large, global sections of* $L^{\otimes m}$ *separate points. For this, we need to prove that for any two points* $p \neq q$ *in* X*, the linear map* $r_{p,q} : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m} \oplus L_q^{\otimes m}$ *is surjective for* m *sufficiently large.*
3. *Prove that for* m *large,* ι_{L^m} *is an immersion. That is, for any point* $p \in X$*, global sections of* $L^{\otimes m}$ *separate tangent directions at* p*. We only need to show the linear map* $r_{p,p} : H^0(X, \mathscr{O}(L^{\otimes m})) \to L_p^{\otimes m} \otimes \left(\mathscr{O}_p/\mathfrak{m}_p^2\right)$ *is surjective for* m *sufficiently large.*

Proof (continued)

Note that property 2 is stronger than property 1. So we only need to prove 2 and 3. Also note that if we denote by $\mathfrak{m}_p$ *the ideal sheaf of holomorphic germs vanishing at* p *and* $\mathfrak{m}_{p,q}$ *the ideal sheaf of holomorphic germs vanishing at* p *and* q*, then what we need prove is that*

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2)$$

are both surjective when m *is large enough. For this, we use short exact sequences of sheaves:*

$$0 \to \mathfrak{m}_{p,q} \to \mathscr{O} \to \mathscr{O}/\mathfrak{m}_{p,q} \to 0, \qquad 0 \to \mathfrak{m}_p^2 \to \mathscr{O} \to \mathscr{O}/\mathfrak{m}_p^2 \to 0.$$

Proof (continued)

Tensor with the locally free sheaf $\mathscr{O}(L^{\otimes m})$, *we get exact sequences*

$$0 \to \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q} \to \mathscr{O}(L^{\otimes m}) \to \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q} \to 0$$

and

$$0 \to \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2 \to \mathscr{O}(L^{\otimes m}) \to \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2 \to 0.$$

The induced long exact sequences give us:

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_{p,q}) \to H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathscr{O}(L^{\otimes m})) \to H^0(X, \mathscr{O}(L^{\otimes m}) \otimes \mathscr{O}/\mathfrak{m}_p^2) \to H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2).$$

We need to prove the vanishing of $H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q})$ *and* $H^1(X, \mathscr{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2)$.

Proof (continued)

Comparing with Kodaira's vanishing theorem, we found that the main problem is that $\mathfrak{m}_{p,q}$ *and* $\mathfrak{m}_p^2$ *are not sheaves of germs of holomorphic line bundles. They are examples of "coherent analytic sheaves". This "generalized Kodaira vanishing theorem" for coherent analytic sheaves is indeed true, but harder to prove. Kodaira's method (as appeared in Griffiths-Harris and Wells) is to replace* X *by its blown-up* $\tilde{X}$ *at* p *and* q*. Pulling everything back to* $\tilde{X}$ *we can work purely with line bundles, and then Kodaira's vanishing theorem works. Then one need to show that vanishing upstairs implies vanishing downstairs.*

Finally, since both property 2 and 3 are "open" properties, we can use a "finite covering trick" to find a uniform m*, independent of* $p, q \in X$.

Concluding remark

In short, the proof says that positivity of a line bundle L implies $L^{\otimes m}$ has so many global sections that they can separate points and tangent directions. Here we use Kodaira's cohomology vanishing to prove the existence of global sections satisfying special properties. This is typical when applying vanishing theorems.

Also, to prove the existence of global sections separating points and tangent directions, one can directly construct sections by solving $\bar{\partial}$-equations using Hörmander's L^2-method. It turns out that we also need a certain type of Bochner type identity, and the positivity of the line bundle is also crucial. We will give an overview of L^2-theory for the $\bar{\partial}$-equation later.