Introduction to Complex Geometry

Chapter 3 supplementary notes

SHI, Yalong
(Nanjing University)

BICMR Summer School 2022

§3.S More about the bundle $\mathit{End}\ E$ and $\mathit{End}\ E$-valued forms

The bundle *End E*

- Let E be a complex vector bundle of rank r over X, the bundle $End\ E$ is defined to be (as a set) $End\ E := \coprod_{p\in X} End_{\mathbb{C}}(E_p) = \coprod_{p\in X} Hom_{\mathbb{C}}(E_p, E_p)$, with the natural projection map $\tilde{\pi} : End\ E \to X$.

The bundle *End E*

- Let E be a complex vector bundle of rank r over X, the bundle $End\ E$ is defined to be (as a set) $End\ E := \coprod_{p\in X} End_{\mathbb{C}}(E_p) = \coprod_{p\in X} Hom_{\mathbb{C}}(E_p, E_p)$, with the natural projection map $\tilde{\pi} : End\ E \to X$.
- If we have a local trivialization of E, $\pi^{-1}(U) \to U \times \mathbb{C}^r$, equivalently, we choose a local frame $\{e_\alpha\}_{\alpha=1}^r$. Now we have an induced local frame for $End\ E$: $E_{\alpha\beta} \in C^\infty(U, End\ E)$, defined by $E_{\alpha\beta}(e_\beta) = e_\alpha$ and $E_{\alpha\beta}(e_\gamma) = 0$ when $\gamma \neq \beta$. The notation indicates that its matrix representation is precisely the elementary matrix $E_{\alpha\beta}$ (whose (α, β) entry is 1 and others are 0).

The bundle $End\ E$

- Let E be a complex vector bundle of rank r over X, the bundle $End\ E$ is defined to be (as a set) $End\ E := \coprod_{p\in X} End_{\mathbb{C}}(E_p) = \coprod_{p\in X} Hom_{\mathbb{C}}(E_p, E_p)$, with the natural projection map $\tilde{\pi} : End\ E \to X$.
- If we have a local trivialization of E, $\pi^{-1}(U) \to U \times \mathbb{C}^r$, equivalently, we choose a local frame $\{e_\alpha\}_{\alpha=1}^r$. Now we have an induced local frame for $End\ E$: $E_{\alpha\beta} \in C^\infty(U, End\ E)$, defined by $E_{\alpha\beta}(e_\beta) = e_\alpha$ and $E_{\alpha\beta}(e_\gamma) = 0$ when $\gamma \neq \beta$. The notation indicates that its matrix representation is precisely the elementary matrix $E_{\alpha\beta}$ (whose (α, β) entry is 1 and others are 0).
- Under this frame, we get a trivialization map: $\tilde{\pi}^{-1}(U) \to U \times M_r(\mathbb{C})$, where $M_r(\mathbb{C})$ is the (linear) space of $r \times r$ complex matrices. The trivialization map is given by

$$(p, \sum_{\alpha,\beta} a_{\alpha\beta} E_{\alpha\beta}(p)) \mapsto (p, (a_{\alpha\beta})).$$

Local section of *End E* acts on local sections of *E*

A local section $\sigma = \sum_{\alpha,\beta} a_{\alpha\beta} E_{\alpha\beta} \in C^\infty(U, End\ E)$ can be identified with a $M_r(\mathbb{C})$ valued smooth function $A = (a_{\alpha\beta}) : U \to M_r(\mathbb{C})$. For a section $s = \sum_\alpha f^\alpha e_\alpha$, we have

$$\sigma s := \sigma(s) = \sum_{\alpha,\beta} a_{\alpha\beta} E_{\alpha\beta} (\sum_\gamma f^\gamma e_\gamma) = \sum_{\alpha,\beta} a_{\alpha\beta} f^\beta e_\alpha.$$

So under the trivialization, the action of σ on s is just the matrix $(a_{\alpha\beta})$ times the column vector (f^α).

Effect of changing frames

If we change the local frame $\{e_\alpha\}$ to $\tilde{e}_\alpha = a_\alpha^\beta e_\beta$, then we have a corresponding induced frame $\tilde{E}_{\alpha\beta}$. Then for a local section $\sigma \in C^\infty(U, End\ E)$(set $(b_\alpha^\beta) = (a_\alpha^\beta)^{-1}$), if $\sigma = \sum_{\alpha,\beta} c_{\alpha\beta} E_{\alpha\beta} = \sum_{\alpha,\beta} \tilde{c}_{\alpha\beta} \tilde{E}_{\alpha\beta}$, then

$$\sigma(e_\beta) = \sum_\alpha c_{\alpha\beta} e_\alpha, \quad \sigma(\tilde{e}_\beta) = \sum_\alpha \tilde{c}_{\alpha\beta} \tilde{e}_\alpha$$

so we get

$$\sum_\alpha \tilde{c}_{\alpha\beta} \tilde{e}_\alpha = \sigma(\sum_\gamma a_\beta^\gamma e_\gamma) = \sum_\gamma a_\beta^\gamma \sigma(e_\gamma) = \sum_{\gamma,\mu} a_\beta^\gamma c_{\mu\gamma} e_\mu = \sum_{\gamma,\mu,\alpha} a_\beta^\gamma c_{\mu\gamma} b_\mu^\alpha \tilde{e}_\alpha$$

and hence $\tilde{c} = a^{-1} c a$.

End E-valued differential forms

So a smooth section of *End E* is given by a family of locally defined matrix-valued smooth functions $c_i : U_i \to M_r(\mathbb{C})$, and when $U_i \cap U_j \neq \emptyset$, we have $c_i = \psi_{ij}^{-1} c_j \psi_{ij}$.

Similarly, *End E*-valued differential forms are locally given by $\eta = \sum_{i=1}^{N} \omega_i \otimes c_i$, where c_i is a matrix-valued smooth function and ω_i is a smooth $k - form$ on a trivialization neighborhood U. To make it well-defined, we require that when we change the local frame by $\tilde{e}_\alpha = a_\alpha^\beta e_\beta$, we have $\tilde{\eta} = a^{-1} \eta a = \sum_i \omega_i \otimes (a^{-1} c_i a)$.

“Matrix-valued form” as “matrix of differential forms”

A $M_r(\mathbb{C})$-valued differential form can always be written as a $r \times r$ matrix of differential forms: let $\eta = \sum_{i=1}^N \omega_i \otimes c_i$. Suppose $c_i = (c^i_{\alpha\beta})$ with $c_{\alpha\beta}$ are smooth functions, then we have $c_i = \sum_{\alpha,\beta} c^i_{\alpha\beta} E_{\alpha\beta}$ and hence

$$\begin{aligned}\eta &= \sum_i \omega_i \otimes c_i = \sum_i \sum_{\alpha,\beta} \omega_i \otimes (c^i_{\alpha\beta} E_{\alpha\beta}) \\ &= \sum_{\alpha,\beta} (\sum_i \omega_i c^i_{\alpha\beta}) \otimes E_{\alpha\beta} \\ &=: \sum_{\alpha\beta} \eta_{\alpha\beta} \otimes E_{\alpha\beta}\end{aligned}$$

This means that we can view η as a matrix whose (α, β)-entry is precisely the differential form $\eta_{\alpha\beta} = \sum_i c^i_{\alpha\beta} \omega_i$.

Taking trace

When take trace for an $End\ E$-valued differential form, these two ways of representation are equivalent:

$$\begin{aligned}\sum_\alpha \eta_{\alpha\alpha} &= \sum_\alpha \sum_i c^i_{\alpha\alpha}\omega_i \\ &= \sum_i (\sum_\alpha c^i_{\alpha\alpha})\omega_i = \sum_i (tr\ c_i)\omega_i.\end{aligned}$$

(super)-commutator

We also compute the super-commutator of two $End\ E$-valued forms with respect to these two ways of representations: by linearity, we only need to compute $[\omega \otimes A, \eta \otimes B]$ with $A = (A_{\alpha\beta}), B = (B_{\alpha\beta})$.

On one hand, we have by definition $[\omega \otimes A, \eta \otimes B] = (\omega \wedge \eta) \otimes [A, B]$.

On the other hand, as matrix of forms, they are $\omega A := (A_{\alpha\beta}\omega)$ and $\eta B := (B_{\alpha\beta}\eta)$, so the (α, β)*entry* of $\omega A \wedge \eta B - (-1)^{deg(\omega)deg(\eta)} \eta B \wedge \omega A$ is

$$\begin{aligned} & \sum_{\gamma} (\omega A)_{\alpha\gamma} \wedge (\eta B)_{\gamma\beta} - (-1)^{deg(\omega)deg(\eta)} \sum_{\gamma} (\eta B)_{\alpha\gamma} \wedge (\omega A)_{\gamma\beta} \\ = \ & [\sum_{\gamma} (A_{\alpha\gamma} B_{\gamma\beta} - B_{\alpha\gamma} A_{\gamma\beta})] \omega \wedge \eta = [A, B]_{\alpha\beta} \omega \wedge \eta. \end{aligned}$$

This is precisely $(\omega \wedge \eta) \otimes [A, B]$ viewed as a matrix of differential forms.

About the Bianchi identity

I shall prove that $[D,\Theta]=0$ is your familiar 2nd Bianchi identity.
In fact, let $s=f^\alpha e_\alpha$ be a local section of E, then

$$\begin{aligned} 0 &= [D,\Theta]s = D(\Theta_\alpha^\beta f^\alpha e_\beta) - \Theta_\alpha^\beta \wedge (df^\alpha + \theta_\gamma^\alpha f^\gamma)e_\beta \\ &= [df^\alpha \wedge \Theta_\alpha^\beta + f^\alpha d\Theta_\alpha^\beta + \Theta_\alpha^\gamma f^\alpha \wedge \theta_\gamma^\beta - \Theta_\alpha^\beta \wedge (df^\alpha + \theta_\gamma^\alpha f^\gamma)]e_\beta \\ &= f^\alpha[d\Theta_\alpha^\beta + \Theta_\alpha^\gamma \wedge \theta_\gamma^\beta - \Theta_\gamma^\beta \wedge \theta_\alpha^\gamma]e_\beta \end{aligned}$$

That is, $d\Theta+\theta\wedge\Theta-\Theta\wedge\theta=0$. In the RIemannian case, $\Theta_j^i=\frac{1}{2}R^i_{jpq}dx^p\wedge dx^q$ and $\theta_j^i=\Gamma^j_{ki}dx^k$, the equation $d\Theta+\theta\wedge\Theta-\Theta\wedge\theta=0$ gives us (using the fact that Levi-Civita connection is torsion-free)

$$\sum_{k,p,q}\nabla_k R^i_{jpq}dx^k\wedge dx^p\wedge dx^q=0,$$

this is nothing but the more familiar formula $\nabla_k R^i_{jpq}+\nabla_p R^i_{jqk}+\nabla_q R^i_{jkp}=0$.