§6. Manifolds of non-positive curvature.

§6.1. Fundamental group.

- M^n: connected diff. mfd. Define its fundamental group:

$$\pi_1(M,p) := \{\gamma : [0,1] \to M \text{, continuous , } \gamma(0) = \gamma(1) = p\} / \sim$$

where $\gamma_0 \sim \gamma_1$ iff $\exists$ homotopy equivalence based at p:

$$h : [0,1] \times [0,1] \longrightarrow M, \quad \text{continuous with:}$$

$$\gamma_0(\cdot) = h(0,\cdot), \quad \gamma_1(\cdot) = h(1,\cdot), \quad h(0,\cdot) \equiv p \equiv h(1,\cdot).$$

- Trivial element: $\gamma(t) \equiv p$, $t \in [0,1]$, denoted by $[pt]$.
- Group structure:
  - product: $(\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2}. \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$
  - inverse: $\gamma^{-1}(t) := \gamma(1-t)$.
- The manifold M is called simply connected. if $\pi_1(M,p) = \{[pt]\}$

Def 1. (Covering map).

- A covering map $\phi : M^n \longrightarrow N^n$, is a diff. map, s.t.

for any $p \in N$, $\exists$ nbhd $U \subset N$, s.t.

$$\phi^{-1}(U) = \bigcup_{\alpha \in I} U_\alpha \subset M$$

with each $\phi|_{U_\alpha} : U_\alpha \longrightarrow U$ is diffeomorphism, and

$U_\alpha \cap U_\beta = \phi$, $\forall\ \alpha \neq \beta$.

• The covering map has curve lifting property:

Any curve $\gamma : [0, \ell] \to N$, $\gamma(0) = p$, for any $\tilde{p} \in M$ with

$\phi(\tilde{p}) = \gamma(0) = p$, $\exists!$ curve $\tilde{\gamma} : [0, \ell] \to M$, s.t. $\tilde{\gamma}(0) = \tilde{p}$, and:

$$\phi \circ \tilde{\gamma} = \gamma .$$

$$\begin{tikzcd} & \tilde{M} \arrow[d, "\phi"] \\ {[0,\ell]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowleft"] & M \end{tikzcd}$$

• The covering map also has homotopy lifting property:

For any homotopy $h : [0,1] \times [0,1] \to M$ based at $p \in N$,

for any $\tilde{p} \in M$, with $\tilde{p} \in \phi^{-1}(p)$, $\exists!$ lifting homotopy:

$$\tilde{h} : [0,1] \times [0,1] \longrightarrow M,$$

s.t. $\tilde{h}(\cdot, 0) \equiv \tilde{p}$, $\phi \circ \tilde{h} = h$.

$$\begin{tikzcd} & M \arrow[d, "\phi"] \\ {[0,1]\times[0,1]} \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "\gamma"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowleft"] & N \end{tikzcd}$$

- For any connected mfd M, $\exists !$ universal covering space $\tilde{M}$, together with a covering map $\phi : \tilde{M} \to M$, s.t. $\tilde{M}$ is simply connected.

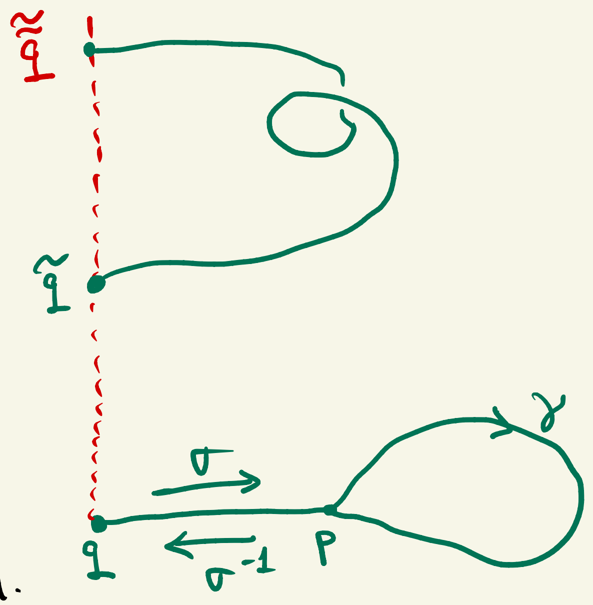
- Any continuous map $f : M \to N$ induce a morphism:

$$f_* : \pi_1(M, p) \longrightarrow \pi_1(N, f(p))$$
$$[\gamma] \longmapsto [f \circ \gamma].$$

- $\pi_1(M, p)$ acts on $\tilde{M}$ via homotopy lifting.
  - $\forall \tilde{q} \in \tilde{M}$, $q = \phi(\tilde{q})$.
  - $\forall [\gamma] \in \pi_1(M, p)$, take a curve σ from q to p. Then take the lifting of $\sigma^{-1} * \gamma * \sigma$, starting at $\tilde{q}$. The endpoint $\tilde{\tilde{q}} \in \phi^{-1}(q)$ is defined to be the action:

$$\tilde{\tilde{q}} = [\gamma](\tilde{q}).$$

$$\pi_1(M, p) \times \tilde{M} \longrightarrow \tilde{M}$$
$$(\gamma, \tilde{q}) \longmapsto \tilde{\tilde{q}} = [\gamma](\tilde{q}).$$



- Homotopy lifting property

$\Rightarrow$ the action $[\gamma]$ is well-defined.

• §6.2 Cartan–Hadamard Theorem.

• Def.2 (Local isometry). A map $f: M \to N$ is a local isometry, if:

① f is a local diffeomorphism, i.e., $\forall p \in M$, $\exists$ nbhd $U \subset M$. s.t. $f|_U : U \to f(U) \subset N$ is a diffeomorphism.

② f is isometry on each T_pM, i.e.,

$$\langle df_p(u), df_p(v) \rangle = \langle u, v \rangle, \quad \forall u, v \in T_pM. \quad \#$$

• The local isometry induces isomorphism of Levi-Civita connection and curvatures:

• $df(\nabla_X Y) = \nabla_{df(X)} df(Y), \quad \forall X, Y \in \Gamma(TM).$

• $df(R(X,Y)Z) = R(df(X), df(Y))\, df(Z), \ \forall X, Y, Z \in \Gamma(TM).$

• Local isometry maps geodesics to geodesics, with same speed:

if γ is a geodesic on M, then:

$$\nabla_{(f\circ\gamma)'} (f\circ\gamma)' = \nabla_{df(\dot\gamma)} df(\dot\gamma) = df(\nabla_{\dot\gamma}\dot\gamma) = 0.$$

hence $f\circ\gamma$ is a geodesic on N.

• **Lemma 3** (Gauss Lemma). • (M^n, g): Riem. mfd.

① At any $v \in T_pM$ where $\exp_p$ is defined, $v \neq 0$,

$$\langle d(\exp_p)_v w, d(\exp_p)_v v \rangle_{g_{\exp_p(v)}} = \langle w, v \rangle_p, \quad \forall\, w \in T_pM.$$

② $\forall\, p \in M$, $0 < r < \mathrm{inj}_p$, $\gamma_v(t) := \exp_p(t \cdot v)$, $0 < t \leq 1$, is well-defined, is the unique minimal geodesic joining p and $\exp_p(v)$, whenever $|v| < r$.

In particular, $d(p, \exp_p(v)) = |v|$, $\forall\, |v| < r$.

• **Proof**:

• **①**: Define the variation of geodesics:

$$\gamma(t,u) := \exp_p\big(t \cdot (v + u \cdot w)\big), \quad (t,u) \in [0,1] \times [-1,1].$$

The Jacobi field $J(t) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma(t,u)$ satisfies: (see §4 ⑥).

• $J(0) = 0$, $J'(0) = w$, $J(1) = d(\exp_p)_v w$.

• $\langle J, \dot\gamma \rangle(0) = 0$, $\langle J', \dot\gamma \rangle(0) = \langle w, v \rangle$, here $\gamma = \gamma_0$

• $\langle J'', \dot\gamma \rangle(t) = \langle R(\dot\gamma, J)\dot\gamma, \dot\gamma \rangle(t) \equiv 0$.

by antisymmetry of the Rm curvature.

• Hence:

$$\langle J', \dot\gamma\rangle(t) - \langle J', \dot\gamma\rangle(0) = \int_0^t \frac{d}{ds}\langle J', \dot\gamma\rangle(s)\, ds$$

$$= \int_0^t \langle J'', \dot\gamma\rangle(s) + \langle J', \nabla_{\dot\gamma}\dot\gamma\rangle(s)\, ds = 0.$$

• Then:

$$\langle J, \dot\gamma\rangle(1) - \langle J, \dot\gamma\rangle(0) = \int_0^1 \frac{d}{dt}\langle J, \dot\gamma\rangle(t)\, dt$$

$$= \int_0^1 \langle J', \dot\gamma\rangle(t) + \langle J, \nabla_{\dot\gamma}\dot\gamma\rangle(t)\, dt$$

$$= \int_0^1 \langle J', \dot\gamma\rangle(0)\, dt = \langle v, w\rangle.$$

Combining with $\dot\gamma(1) = d(\exp_p)_v v$, this is exactly:

$$\langle d(\exp_p)_v w,\ d(\exp_p)_v v\rangle_{g_{\exp_p v}} = \langle w, v\rangle_{g_p}.$$

• ②: Let $0 < r < \mathrm{inj}_p$, where:

$$\mathrm{inj}_p := \sup\left\{ r > 0 : \exp_p : \underset{\subset T_pM}{B(0,r)} \to M \text{ is a diffeomorphism onto its image} \right\}.$$

• **Claim**: For any curve on tangent space:

$$\sigma : [0, \ell] \to B(0,r) \subset T_pM, \quad \text{with } \sigma(0) = 0,$$

the curve $\gamma(t) := \exp_p(\sigma(t))$ has length:

$$L(\gamma) \geq |\sigma(\ell)|,$$

with "=" iff $\sigma(t) = f(t)\cdot\sigma(\ell)$, for some monotone increasing function f.

Proof: • We may assume $\sigma(t) \neq 0$, $\forall t \in [0, \ell]$. Then write:

$$\sigma'(t) = r(t)\cdot\sigma(t) + w(t), \quad \text{with} \quad w(t) \perp \sigma(t).$$

Then:

$$L(\gamma) = \int_0^\ell \left| d(\exp_p)_{\sigma(t)}(\sigma'(t)) \right| dt$$

$$= \int_0^\ell \left[r^2(t)\cdot\left| d(\exp_p)_{\sigma(t)}(\sigma(t))\right|^2 + \left| d(\exp_p)_{\sigma(t)}(w(t))\right|^2 \right]^{\frac{1}{2}} dt$$

$$\geq \int_0^\ell |r(t)|\cdot\left| d(\exp_p)_{\sigma(t)}(\sigma(t))\right| dt$$

By (i), we have:

$$\left| d(\exp_p)_{\sigma(t)}(\sigma(t))\right|^2 = |\sigma(t)|^2,$$

hence:

$$L(\gamma) \geq \int_0^\ell |r(t)\sigma(t)|\, dt$$

• Note that:

$$|\sigma|' = \frac{\langle \sigma', \sigma\rangle}{|\sigma|} = \frac{\langle r\cdot\sigma + w, \sigma\rangle}{|\sigma|} = r\cdot|\sigma|.$$

hence:

$$L(\gamma) \geq \int_0^{\ell} |r(t)\cdot\sigma(t)|\,dt \geq \int_0^{\ell} r(t)\cdot|\sigma(t)|\,dt$$

$$= \int_0^{\ell} |\sigma|'(t)\,dt = |\sigma|(\ell) - |\sigma|(0) = |\sigma|(\ell)$$

• "=" holds iff $r(t) \geq 0$, $w(t) \equiv 0$, $\forall\, t \in (0, \ell]$, i.e.,

$$\sigma'(t) = r(t)\cdot\sigma(t), \quad r(t) \geq 0, \quad \forall\, t \in (0, \ell].$$

$\Rightarrow \sigma(t) = f(t)\cdot\sigma(\ell)$, for some continuous function with $f' \geq 0$.

• Now, for fixed $v \in B(0,r)$, $q := \exp_p(v)$, set:

$$\ell := |v|, \quad \gamma_0(t) := \exp_p\left(\frac{t}{\ell}\cdot v\right), \quad t \in [0, \ell].$$

is a geodesic from p to q, of speed: $\forall\, t \in (0, \ell]$,

$$|\dot{\gamma}_0(t)|^2 = \left|d(\exp_p)_{\frac{t}{\ell}\cdot v}\left(\frac{v}{\ell}\right)\right|^2 = \frac{1}{t^2}\left|d(\exp_p)_{\frac{t}{\ell}\cdot v}\left(\frac{t}{\ell}\cdot v\right)\right|^2$$

$$\overset{(i)}{=} \frac{1}{t^2}\cdot\left|\frac{t}{\ell}\cdot v\right|^2 = 1.$$

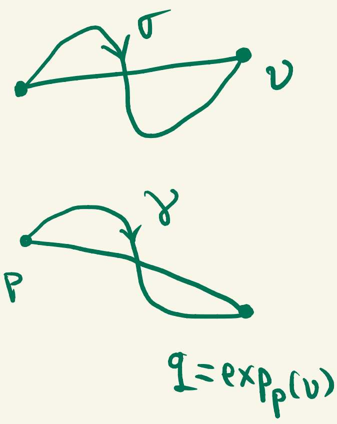
Hence $L(\gamma_0) = |v|$.

• Next, let $\gamma(t): [0, \ell] \to M$ be minimal geodesic joining p to $q := \exp_p(v)$ of unit speed, here $\ell = |v|$.

• If $\gamma([0,\ell]) \subset \exp_p(B(0,r))$, then $\exists\ \sigma: [0,\ell] \to B(0,r)$, with

$\sigma(\ell)=v$, s.t.

$$\gamma(t)=\exp_p(\sigma(t)), \quad \forall t\in[0,\ell].$$



Hence from Claim above,

$$L(\gamma)\geq|\sigma(\ell)|=|v|=L(\gamma_0)\geq L(\gamma).$$

and $\sigma(t)=f(t)\cdot v$, f is a monotone increasing function.

- From (i), we have:

$$1=|\dot\gamma(t)|^2=\left|d(\exp_p)_{\sigma(t)}(\dot\sigma(t))\right|^2=\left|d(\exp_p)_{f(t)\cdot v}(f'(t)\cdot v)\right|^2$$

$$=|f'(t)\cdot v|^2=f'(t)^2\cdot\ell^2,$$

but $f(0)=0$, $f'\geq 0$, hence $f(t)=\frac{t}{\ell}$, then $\sigma(t)=\frac{t}{\ell}\cdot v$, then

$$\gamma(t)=\exp_p(\sigma(t))=\exp_p\left(\frac{t}{\ell}\cdot v\right).$$

- If $\gamma([0,\ell])\not\subset\exp_p(B(0,r))$, then $\exists\, t_0\in(0,\ell]$, s.t.

$$\gamma(t_0)=\exp_p(v_0), \quad r<|v_0|<\mathrm{inj}_p$$

$$\gamma([0,t_0])\subset\exp_p\left(\overline{B(0,|v_0|)}\right).$$

Apply our previous case, we have:

$$L(\gamma)\geq L(\gamma|_{[0,t_0]})\geq|v_0|>r>|v|=L(\gamma_0).$$

contradicts to γ is a minimizing geodesic from p to q. $\boxed{\#}$

• **Lemma 4.** • Let $\phi : M^m \to N^n$ be a surjective, local isometry of Riem. mfds.

• If M is complete, then ϕ is a covering map.

Proof: • For any $p \in N$, let $0 < r < \frac{1}{2}\,\mathrm{inj}_p$. We show that $B(p,r) = \exp_p(B(0,r))$ satisfies the nbhd property in covering map.

• Let $\phi^{-1}(p) = \{p_\alpha\}_{\alpha \in I}$. Consider the diagram:

$$\begin{tikzcd} T_{p_\alpha}M \arrow[r, "d\phi"] \arrow[d, "\exp_{p_\alpha}"'] \arrow[dr, phantom, "\circlearrowright"] & T_pN \arrow[d, "\exp_p"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

• Commutable: $\forall\, v \in T_{p_\alpha}M$,

$$\phi(\exp_{p_\alpha}(v)) = \phi(\gamma_v(1)) = \gamma_{d\phi(v)}(1) = \exp_p(d\phi(v)).$$

• Restriction on $B(0,r) \subset T_pM$: since $d\phi : T_{p_\alpha}M \to T_pN$ is isometry,

$$\begin{tikzcd} B(0,r) \arrow[r, "d\phi"] \arrow[d, "\exp_{p_\alpha}"'] \arrow[dr, phantom, "\circlearrowright"] & B(0,r) \arrow[d, "\exp_p"] \\ B(p_\alpha, r) \arrow[r, "\phi"'] & B(p,r) \end{tikzcd}$$

which is also commutable. Hence $\exp_{p_\alpha}: B(0,r) \longrightarrow B(p_\alpha, r)$

is also a diffeomorphism map. Hence:

$$\bigcup_{\alpha \in I} B(p_\alpha, r) \subset \phi^{-1}(B(p,r)).$$

- Next, we show: $\phi^{-1}(B(p,r)) \subset \bigcup_{\alpha\in I} B(p_\alpha, r)$.

  - curve lifting: $\forall\, \tilde{q} \in \phi^{-1}(B(p,r))$, let $q = \phi(\tilde{q}) \in B(p,r)$.

    Let $\gamma : [0, l] \to N$ be a minimal geodesic joining q to p,

    then $\phi^{-1} \circ \gamma$ is a geodesic in M joining $\tilde{q}$ to some p_α, $\alpha \in I$,

    with $L(\phi^{-1} \circ \gamma) = L(\gamma) < r$, hence $\tilde{q} \in B(p_\alpha, r)$.

- Hence we conclude:

$$\phi^{-1}(B(p,r)) = \bigcup_{\alpha\in I} B(p_\alpha, r).$$

- Next, we show: $d(p_\alpha, p_\beta) > 2r$, $\forall\, \alpha \neq \beta$, $\alpha, \beta \in I$.

  - Suppose $d(p_\alpha, p_\beta) \leq 2r$, then choose a minimal geodesic $\tilde{\gamma}$

    connecting p_α to p_β, hence $L(\tilde{\gamma}) \leq 2r$.

  - Then $\gamma := \phi \circ \tilde{\gamma}$ is a closed geodesic based at p.

  - By Gauss Lemma, $\exists\ 0 \neq v \in T_pM$, $|v| < \frac{1}{2} \mathrm{inj}_p$, s.t. $\gamma(t) = \exp_p(t \cdot v)$, $t \in [0,1]$.

- Then $p = \exp_p(0) = \exp_p(v)$, contradicts to $r < \frac{1}{2} \mathrm{inj}_p$.

- Now, we have $B(p_\alpha, r) \cap B(p_\beta, r) = \phi$, $\forall \alpha \neq \beta$:
  - $\forall x \in B(p_\alpha, r)$, $y \in B(p_\beta, r)$,

    $d(x,y) \geq d(p_\alpha, p_\beta) - d(p_\alpha, x) - d(p_\beta, x) > 2r - r - r = 0$.

    hence $x \neq y$.

- In conclusion, ϕ is a covering map. $\blacksquare$

- **Theorem 5 (Cartan–Hadamard).**
  - Suppose (M^n, g) is complete and connected, with sectional curvature $K \leq 0$.
  - Then, $\forall p \in M$, $\exp_p : T_pM \to M$ is a covering map.
  - In particular, if M is simply connected, then $\exp_p$ is a diffeomorphism.

- **Proof**: - Let $\gamma : [0, \ell] \to M$ be any unit-speed geodesic.

- Let $J \in \mathcal{J}_0^\perp$ be any Jacobi field, i.e.,

$J(0)=0$, $J \perp \dot\gamma$ $\Big(\Leftrightarrow \langle J', \dot\gamma\rangle(0)=0$, since

$\langle J, \dot\gamma\rangle(t) = \langle J,\dot\gamma\rangle(0) + \langle J', \dot\gamma\rangle(0)\cdot t \Big)$

- Denote $a = |J'(0)| > 0$.
- Comparison space: $\mathbb{R}^n$. flat Euclidean space $\mathbb{R}^n$,

  comparison geodesic: $\underline{\gamma}$: unit speed geodesic on $\mathbb{R}^n$.

  comparison Jacobi field:

  $$\underline{J}(t) = at\cdot e, \quad \text{for some unit vector } e \perp \underline{\dot\gamma}(0).$$

- Hence by the Rauch comparison. [§5, Thm 2]. we have:

  $$at = |\underline{J}(t)| \leq |J(t)|, \quad \forall t \in [0, \ell].$$

- Hence we conclude:

  Any geodesic on M is conjugate point free.

- Hence, $\phi := \exp_p : T_pM \to M$ is non-degenerate everywhere,

  i.e., $\forall v \in T_pM$, $d\phi_v = d(\exp_p)_v : T_v(T_pM)^{\cong T_pM} \longrightarrow T_pM$ is

  an isomorphism.

- Then, $\tilde g := \phi^* g$ is a Riem. metric on T_pM, with:

$$\tilde{g}(u,v) = g\big(d\phi(u), d\phi(v)\big), \quad \forall u,v \in T_w(T_pM) \cong T_pM.$$

Then $\phi : (T_pM, \tilde{g}) \to (M, g)$ is a local isometry.

- $(T_pM, \tilde{g})$ is complete:

  - $\forall v \in T_pM$, $|v| = 1$, $\sigma(t) := t \cdot v$ is a geodesic on $(T_pM, \tilde{g})$, since $\gamma(t) := \exp_p(\sigma(t)) = \phi \circ \sigma(t)$ is a geodesic on M.

  - Hence by Hopf-Rinow thm, $(T_pM, \tilde{g})$ is complete.

- Now we can apply Lemma 4 to obtain that ϕ is a covering map. $\blacksquare$