# §5. Index of vector fields, Rauch comparison, energy functional.

## §5.1. Index of variation fields.

- Let $\gamma : [0, l] \to M$ be a unit speed geodesic.
- For vector field $X$ along $\gamma$, define the index:

$$I(X, X) = \int_0^l \left[ |\nabla_{\dot\gamma} X|^2 - Rm(\dot\gamma, X, \dot\gamma, X) \right] dt.$$

- $I$ defines a symmetric form on the space of vector fields on $\gamma$:

$$I(X, Y) = \int_0^l \left[ \langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} Y \rangle - Rm(\dot\gamma, X, \dot\gamma, Y) \right] dt.$$

**Lemma 1**. • Suppose $\gamma$ has no conjugate point on $(0, l]$ to $\gamma(0)$.

- Let $J \in \mathcal{J}_0^\perp$, i.e., $J(0) = 0$, $J \perp \dot\gamma$.
- Let $X$ be vector field along $\gamma$ satisfying:

$$X(0) = 0, \quad X \perp \dot\gamma, \quad X(l) = J(l).$$

- Then: $I(X, X) \geq I(J, J)$, with "=" hold iff $X \equiv J$.

**Proof**: • Since $\gamma$ has no conjugate point to $\gamma(0)$ on $(0, l]$ and $J(0) = 0$, we must have $J(l) \neq 0$.

- Since $\dim_{\mathbb{R}} \mathcal{J}_0^\perp = n-1$, we can take a basis $J_1, \cdots, J_{n-1} = J \in \mathcal{J}_0^\perp$.

For each $t \in [0,l]$, $\{J_i(t)\}_{i=1}^{n-1}$ is a basis of $\dot\gamma(t)^{\perp} \subset T_{\gamma(t)}M$.

Hence $\exists$ function $a^i(t)$ s.t.

$$X(t) = \sum_{i=1}^{n-1} a_i(t) \cdot J_i(t), \quad t \in [0, l].$$

From $X(l) = J(l) = J_{n-1}(l)$, we have:

$$a_i(l) = 0, \quad i = 1, \cdots, n-2; \quad a_{n-1}(l) = 1.$$

- By def.

$$I(J,J) = \int_0^l \left[ |\nabla_{\dot\gamma} J| - Rm(\dot\gamma, J, \dot\gamma, J) \right] dt.$$

where:

$$Rm(\dot\gamma, J, \dot\gamma, J) = -Rm(\dot\gamma, J, J, \dot\gamma) = -\langle R(\dot\gamma, J)\dot\gamma, J \rangle$$

$$= -\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J, J \rangle = -\dot\gamma \langle \nabla_{\dot\gamma} J, J \rangle + \langle \nabla_{\dot\gamma} J, \nabla_{\dot\gamma} J \rangle$$

hence:

(1). $$I(J,J) = \int_0^l \frac{d}{dt} \langle \nabla_{\dot\gamma} J, J \rangle dt = \langle \nabla_{\dot\gamma} J, J \rangle \Big|_0^l = \langle \nabla_{\dot\gamma} J, J \rangle (l).$$

- Next:

$$I(X,X) = \sum_{i,j} \int_0^l \left[ \langle \nabla_{\dot\gamma}(a_i J_i), \nabla_{\dot\gamma}(a_j J_j) \rangle - a_i a_j \cdot Rm(\dot\gamma, J_i, \dot\gamma, J_j) \right] dt$$

$$= \sum_{i,j} \int_0^l \left[ \langle a_i' J_i + a_i J_i', a_j' J_j + a_j J_j' \rangle + a_i a_j \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J_i, J_j \rangle \right] dt$$

• For the 2nd term,

$$\sum_{i,j}\int_0^\ell a_i a_j \cdot \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J_i, J_j \rangle \, dt$$

$$= \sum_{i,j}\int_0^\ell a_i a_j \cdot \left[\frac{d}{dt}\langle \nabla_{\dot\gamma} J_i, J_j\rangle - \langle \nabla_{\dot\gamma} J_i, \nabla_{\dot\gamma} J_j\rangle\right] dt$$

$$= \sum_{i,j}\left\{ a_i a_j \cdot \langle \nabla_{\dot\gamma} J_i, J_j\rangle \Big|_0^\ell - \int_0^\ell (a_i' a_j + a_i a_j') \cdot \langle \nabla_{\dot\gamma} J_i, J_j\rangle \, dt \right\}$$

$$- \int_0^\ell a_i a_j \cdot \langle \nabla_{\dot\gamma} J_i, \nabla_{\dot\gamma} J_j\rangle \, dt$$

• Boundary term $= \sum_{i,j} a_i a_j \cdot \langle \nabla_{\dot\gamma} J_i, J_j \rangle \Big|_0^\ell$

$$= \langle \sum_i a_i \nabla_{\dot\gamma} J_i, X\rangle \Big|_0^\ell \overset{X(0)=0}{=} \langle \sum_i a_i \nabla_{\dot\gamma} J_i, X\rangle(\ell)$$

$a_1(\ell) = \cdots = a_{n-2}(\ell) = 0$, $a_{n-1}(\ell) = 1$, $J_{n-1} = J$

$$= \langle \nabla_{\dot\gamma} J, J\rangle(\ell) \overset{(1)}{=} I(J, J).$$

• Hence,

$$I(X,X) = I(J,J) + \sum_{i,j}\int_0^\ell \Big[ \langle a_i' J_i, a_j' J_j\rangle + a_i' a_j \cdot \langle J_i, J_j'\rangle$$

$$- a_i' a_j \langle J_i', J_j\rangle \Big] dt$$

• We claim that : $\langle J_i', J_j\rangle = \langle J_i, J_j'\rangle$ along $\gamma$, $\forall\, i, j$.

• Actually, $\langle J_i, J_j'\rangle(0) = \langle J_i', J_j\rangle(0)$, and:

$$\frac{d}{dt}\left(\langle J_i', J_j\rangle - \langle J_i, J_j'\rangle\right) = \langle J_i'', J_j\rangle - \langle J_i, J_j''\rangle$$

$$= \langle R(\dot\gamma, J_i)\dot\gamma, J_j\rangle - \langle J_i, R(\dot\gamma, J_j)\dot\gamma\rangle$$

$$= -Rm(\dot\gamma, J_i, \dot\gamma, J_j) + Rm(\dot\gamma, J_j, \dot\gamma, J_i) = 0.$$

- Therefore:

$$I(X,X) = I(J,J) + \sum_i \int_0^\ell |a_i' \cdot J_i|^2 dt \geq I(J,J)$$

with "=" only if $a_i' \equiv 0$. But $a_1(\ell) = \cdots = a_{n-2}(\ell) = 0$, $a_{n-1}(\ell) = 1$,

hence only if $X \equiv J$ along $\gamma$. $\blacksquare$

- **Theorem 2. (Rauch comparison).**

  - Let $M^n$, $\underline{M}^n$ be complete Riem. mfds.
  - Let $\gamma / \underline{\gamma} : [0, \ell] \to M / \underline{M}$ be unit speed geodesics.
  - Let $J / \underline{J}$ be Jacobi fields along $\gamma / \underline{\gamma}$.
  - **Assume**:

    ① $K(\dot\gamma, v) \geq \underline{K}(\dot{\underline{\gamma}}, \underline{v})$, for any $v \in T_{\gamma(t)}M$, $\underline{v} \in T_{\underline{\gamma}(t)}\underline{M}$,
    of unit length, $v \perp \dot\gamma$, $\underline{v} \perp \dot{\underline{\gamma}}$, $\forall t \in [0, \ell]$.

    ② $J(0) = 0 = \underline{J}(0)$, $|J'(0)| = |\underline{J}'(0)|$, $\langle J', \dot\gamma\rangle(0) = \langle \underline{J}', \dot{\underline{\gamma}}\rangle(0)$.

③ There is no conjugate point along $\gamma$ to $\gamma(0)$.

• **Then**: $\quad |J(t)| \leq |\underline{J}(t)|, \quad \forall t \in [0, \ell]$.

• **Proof**: • Recall that:

$$\langle J, \dot\gamma \rangle (t) = \langle J, \dot\gamma \rangle (0) + \langle J', \dot\gamma \rangle (0) \cdot t .$$

By condition ②, we have:

$$\langle J, \dot\gamma \rangle (t) = \langle \underline{J}, \dot{\underline{\gamma}} \rangle (t), \quad \forall t \in [0, \ell].$$

Hence $J''(t) = \langle J, \dot\gamma \rangle (t) \cdot \dot\gamma(t)$ and $\underline{J}''(t) = \langle \underline{J}, \dot{\underline{\gamma}} \rangle (t) \cdot \dot{\underline{\gamma}}(t)$ have equal norms. So we may assume that:

$$J \perp \dot\gamma, \quad \underline{J} \perp \dot{\underline{\gamma}}.$$

• If $J'(0) = 0$, then $J \equiv 0$, the result is trivial.

• If $J'(0) \neq 0$, then condition ③ implies that $J(t) \neq 0$, $\forall t \in (0, \ell]$.

• Put $f(t) = |\underline{J}(t)|^2 / |J(t)|^2$, $t \in (0, \ell]$, then:

$$\lim_{t\to 0} f(t) = \lim_{t \to 0} \frac{\langle \underline{J}, \underline{J}' \rangle (t)}{\langle J, J' \rangle (t)} = \lim_{t\to 0} \frac{|\underline{J}'|^2 (t)}{|J'|^2 (t)} \overset{②}{=} 1 .$$

• Use L'Hospital here, and:

$$\frac{d}{dt} \langle J, J' \rangle = \underbrace{|J'|^2(t)}_{\to |J'|(0) \neq 0} - \overbrace{Rm(\dot\gamma(t), J(t), \dot\gamma(t), J(t))}^{\to 0 \text{ as } t \to 0, \text{ since } J(0)=0.}$$

• So it suffices to prove that $f'(t_0) \geq 0$, $\forall\, t_0 \in (0, l]$.

Equivalently, $\dfrac{\langle \underline{J}, \underline{J}' \rangle}{|\underline{J}|^2}(t_0) \geq \dfrac{\langle J, J' \rangle}{|J|^2}(t_0)$.

• Define Jacobi fields:

$$\underline{X}(t) = \frac{\underline{J}(t)}{|\underline{J}|(t_0)}, \quad X(t) = \frac{J(t)}{|J|(t_0)}, \quad t \in (0, t_0].$$

then it suffices to prove that:

(2). $$\langle \underline{X}, \underline{X}' \rangle(t_0) \geq \langle X, X' \rangle(t_0).$$

• Note that:

(3)
$$\begin{cases} \langle \underline{X}, \underline{X}' \rangle(t_0) = \displaystyle\int_0^{t_0} \frac{d}{dt} \langle \underline{X}, \underline{X}' \rangle \, dt = \int_0^{t_0} \left[ |\underline{X}'|^2 + \langle \underline{X}, \underline{X}'' \rangle \right] dt \\ \qquad = \displaystyle\int_0^{t_0} \left[ |\underline{X}'|^2 - \underline{Rm}(\dot{\underline{\gamma}}, \underline{X}, \dot{\underline{\gamma}}, \underline{X}) \right] dt = \underline{I}(\underline{X}, \underline{X}). \\ \langle X, X' \rangle(t_0) = I(X, X). \end{cases}$$

• We are going to construct a comparison vector field along $\gamma$, and apply comparison of index. Lemma 1.

• Let $\{e_i(t)\}_{i=1}^n$ be orthonormal parallel vector fields along $\gamma$,

$\{\underline{e}_i(t)\}_{i=1}^{n}$ be orthonormal parallel vector fields along $\underline{\gamma}$.

- Assume that $e_n(t) = \dot{\gamma}(t)$, $\underline{e_n}(t) = \underline{\dot{\gamma}}(t)$, and:

$$\underline{e}_{n-1}(t_0) = \underline{X}(t_0), \qquad e_{n-1}(t_0) = X(t_0).$$

- Write: $\underline{X}(t) = \sum_i \eta_i(t)\cdot \underline{e}_i(t)$, $t \in [0, t_0]$.
  - $\underline{X}(0) = 0 \Rightarrow \eta_1(0) = \cdots = \eta_n(0) = 0$.
  - $\underline{X}(t_0) = e_{n-1}(t_0) \Rightarrow \eta_{n-1}(t_0) = 1$, $\eta_i(t_0) = 0$, $\forall i \neq n-1$.
  - $\underline{X} \perp \underline{\dot{\gamma}} \Rightarrow \eta_n(t) \equiv 0$, $\forall t \in [0, t_0]$.

- Define vector field $Y(t) = \sum_i \eta_i(t)\cdot e_i(t)$ along $\gamma$, then:

$$Y(0) = 0, \quad Y(t_0) = e_{n-1}(t_0) = X(t_0), \qquad Y \perp \dot{\gamma} \text{ along } \gamma.$$

- Then:

$$I(\underline{X}, \underline{X}) = \int_0^{t_0} \left[ |\underline{X}'|^2 - \underline{Rm}(\underline{\dot{\gamma}}, \underline{X}, \underline{\dot{\gamma}}, \underline{X}) \right] dt$$

$$= \int_0^{t_0} \left[ \sum_i (\eta_i')^2 - |\underline{X}|^2 \cdot \underline{K}(\underline{\dot{\gamma}}, \underline{X}) \right] dt$$

① $\quad \bullet\ |\underline{X}|^2 = \sum_i \eta_i^2 = |Y|^2$

$$\geq \int_0^{t_0} \left[ \sum_i (\eta_i')^2 - |Y|^2 \cdot K(\dot{\gamma}, Y) \right] dt$$

$$= I(Y, Y) \overset{\text{Lemma 1.}}{\geq} I(X, X)$$

where we have used Lemma 1 in the last inequality.

- Combining this with (3), we obtain (2). $\blacksquare$

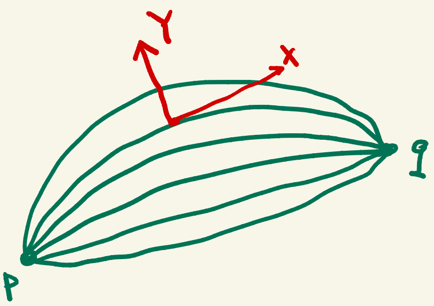
## §5.2 Energy functional.

- $(M, g)$: complete Riem. mfd.
- $\gamma: [0,1] \to M$, smooth curve. Define:

$$E(\gamma) := \frac{1}{2} \cdot \int_0^1 \langle \dot\gamma(t), \dot\gamma(t) \rangle \, dt.$$

- $\gamma(t, u): [0,1] \times (-\varepsilon, \varepsilon) \to M$, with:

$$\gamma(0,u) \equiv p, \quad \gamma(1,u) \equiv q.$$




- $\gamma_u(\cdot) = \gamma(\cdot, u)$.
- $X := \frac{\partial}{\partial t}\gamma, \quad Y = \frac{\partial}{\partial u}\gamma$.
- Then we compute:

$$\frac{\partial}{\partial u}\Big|_{u=0} E(\gamma_u) = \left( \int_0^1 \langle \nabla_Y X, X \rangle \, dt \right)\Big|_{u=0}$$

$$= \left( \int_0^1 \langle \nabla_X Y, X \rangle dt \right)\Big|_{u=0} = - \left( \int_0^1 \langle Y, \nabla_X X \rangle dt \right)\Big|_{u=0}$$

$\Rightarrow$ $\gamma_0$ is a critical point of $E$ on $\{\gamma: [0,1] \to M;\ \gamma(0) = p, \gamma(1) = q\}$

iff $\gamma_0$ is a geodesic.

- If $\gamma : [0,1] \to M$ is a geodesic, then:

$$E(\gamma) = \frac{1}{2} \cdot L(\gamma)^2.$$

- For any $C^1$-curve $\mu : [0,1] \to M$, we have:

$$E(\mu) = \frac{1}{2}\int_0^1 |\dot{\mu}|^2 dt \geq \frac{1}{2}\left(\int_0^1 |\dot{\mu}| dt\right)^2 = \frac{1}{2} \cdot L(\gamma)^2,$$

with equality iff $|\dot{\mu}| = \text{const}$.

- Jacobi fields are determined by the energy functional.

  - $\tilde{\gamma}(t,u) : [0,1] \times (-\varepsilon, \varepsilon) \to M$, $\gamma_u(t) := \tilde{\gamma}(t,u)$.

  - Fix $p \in M$, assume $\tilde{\gamma}(0,u) \equiv p$.

  - Assume $\gamma_0$ is a geodesic.

  - $\tilde{X} = \frac{\partial}{\partial t}\tilde{\gamma}$, $\tilde{Y} = \frac{\partial}{\partial u}\tilde{\gamma}$, $X = \tilde{X}|_{\gamma_0}$, $Y = \tilde{Y}|_{\gamma_0}$.

  - $$\frac{d}{du}E(\gamma_u) = \int_0^1 \langle \nabla_{\tilde{X}}\tilde{Y}, \tilde{X} \rangle dt.$$

hence:

$$\frac{d^2}{du^2}E(\gamma_u) = \int_0^1 \langle \nabla_{\tilde{Y}}\overset{①}{\overset{\shortparallel}{\nabla_{\tilde{X}}\tilde{Y}}}, \tilde{X} \rangle + \langle \overset{②}{\overset{\shortparallel}{\nabla_{\tilde{X}}\tilde{Y}}}, \nabla_{\tilde{Y}}\tilde{X} \rangle dt.$$

- $① = \int_0^1 \langle \nabla_{\tilde{X}} \nabla_{\tilde{Y}} \tilde{Y}, \tilde{X} \rangle + \langle R(\tilde{Y}, \tilde{X})\tilde{Y}, \tilde{X} \rangle \, dt$

$$= \int_0^1 \frac{d}{dt} \langle \nabla_{\tilde{Y}} \tilde{Y}, \tilde{X} \rangle - \langle \nabla_{\tilde{Y}} \tilde{Y}, \nabla_{\tilde{X}} \tilde{X} \rangle \, dt - \int_0^1 Rm(\tilde{X}, \tilde{Y}, \tilde{X}, \tilde{Y}) \, dt$$

$\tilde{Y}(0,u) \equiv 0$

$$= \langle \nabla_{\tilde{Y}} \tilde{Y}, \tilde{X} \rangle(1) - \int_0^1 \langle \nabla_{\tilde{Y}} \tilde{Y}, \nabla_{\tilde{X}} \tilde{X} \rangle \, dt + \int_0^1 \langle R(\tilde{X}, \tilde{Y})\tilde{X}, \tilde{Y} \rangle \, dt.$$

- $② = \int_0^1 \langle \nabla_{\tilde{X}} \tilde{Y}, \nabla_{\tilde{X}} \tilde{Y} \rangle \, dt$

$$= \int_0^1 \tilde{X} \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Y} \rangle \, dt - \int_0^1 \langle \nabla_{\tilde{X}} \nabla_{\tilde{X}} \tilde{Y}, \tilde{Y} \rangle \, dt$$

$$= \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Y} \rangle(1) - \int_0^1 \langle \nabla_{\tilde{X}} \nabla_{\tilde{X}} \tilde{Y}, \tilde{Y} \rangle \, dt.$$

- Hence, restricting to $\gamma_0$, since $\gamma_0$ is geodesic, $\nabla_X X = 0$, recall:

$$\mathrm{Jac}(Y) := \nabla_X \nabla_X Y - R(X, Y)X$$

hence:

$$\frac{d^2}{du^2}\Big|_{u=0} E(\gamma_u) = \langle \nabla_X Y, Y \rangle(1) - \langle X(1), \nabla_Y \tilde{Y}(1,0) \rangle - \int_0^1 \langle \mathrm{Jac}(Y), Y \rangle \, dt.$$

- Similarly, for a two-parameter families:

$$\tilde{\gamma}(t,u_1,u_2): [0,1]\times(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon) \longrightarrow M.$$

with $\gamma(t)=\tilde{\gamma}(t,0,0)$ being geodesic, $\tilde{\gamma}(0,u_1,u_2)\equiv p$, set:

$$\tilde{Y}_1 := \frac{\partial\tilde{\gamma}}{\partial u_1},\quad \tilde{Y}_2 := \frac{\partial\tilde{\gamma}}{\partial u_2},\quad Y_1 := \tilde{Y}_1\big|_{\gamma},\quad Y_2 := \tilde{Y}_2\big|_{\gamma}.$$

then:

$$\frac{\partial^2 E(\tilde{\gamma})}{\partial u_1 \partial u_2}\bigg|_{u_1=u_2=0}$$

$$= \langle \nabla_X Y_1, Y_2\rangle(1) + \langle X(1), \nabla_{Y_1}\tilde{Y}_2(1,0)\rangle - \int_0^1 \langle \mathrm{Jac}(Y_1), Y_2\rangle\, dt.$$

- If $Y_1(1)=Y_2(1)=0$, i.e., $\tilde{\gamma}(1,u_1,u_2)\equiv q$, then:

$$\frac{\partial^2 E(\tilde{\gamma})}{\partial u_1 \partial u_2}\bigg|_{u_1=u_2=0} = -\int_0^1 \langle \mathrm{Jac}(Y_1), Y_2\rangle\, dt =: B_\gamma(Y_1,Y_2).$$

But $\frac{\partial^2}{\partial u_1\partial u_2} = \frac{\partial^2}{\partial u_2 \partial u_1}$, hence:

$$B_\gamma(Y_1,Y_2) = B_\gamma(Y_2,Y_1).$$

• **Proposition 3**: • Suppose $\gamma:[0,1]\to M$ is a minimal geodesic.

• Then: there is no conjugate point to $\gamma(0)$ on $\gamma\big|_{(0,1)}$.

i.e., $\forall\, t_0 \in (0,1)$, there is no non-zero Jacobi field along $\gamma|_{[0,t_0]}$, which vanishing at $\gamma(0)$ & $\gamma(t_0)$.

• Proof: First we prove:

• Claim: • $\gamma : [0,1] \to M$, minimal geodesic.

• $Y$: field along $\gamma$ with $Y(0) = 0 = Y(1)$.

• $\tilde{\gamma}(t,u) : [0,1] \times (-\epsilon, \epsilon) \to M$, with $\gamma_u(t) = \tilde{\gamma}(t,u)$, $\gamma_0 = \gamma$,

$\gamma(0,u) \equiv p \in M$.

• $\tilde{Y} = \frac{\partial \tilde{\gamma}}{\partial u}$, $Y = \tilde{Y}|_{\gamma_0}$.

• Then: $\frac{d^2}{du^2}\Big|_{u=0} E(\gamma_u) = 0$ iff $Y$ is a Jacobi field.

• Proof: • Since $\gamma$ is a minimal geodesic, it minimizes the energy functional $E$ among all possible variations. Hence, for any field $W$ along $\gamma$ with $W(0) = W(1) = 0$,

$$B_\gamma(W,W) = -\int_0^1 \langle \mathrm{Jac}(W), W \rangle \, dt = \frac{d^2}{du^2}\Big|_{u=0} E(\hat{\gamma}_u) \geq 0$$

where $\hat{\gamma}$ is variation curves of $W$, i.e. $\frac{\partial}{\partial u}\Big|_{u=0} \hat{\gamma} = W$.

- By our condition:

$$B_\gamma(Y, Y) = -\int_0^1 \langle Jac(Y), Y \rangle dt = \frac{d^2}{du^2}\Big|_{u=0} E(\gamma_u) = 0.$$

Hence, apply (4) to $W \leftarrow Y + tW$, $t \in \mathbb{R}$, we obtain:

$$0 \leq B_\gamma(tY+W, tY+W)$$

$$= t^2 \cdot B_\gamma(Y, Y) + t \cdot B_\gamma(Y, W) + t \cdot B_\gamma(W, Y) + B_\gamma(W, W).$$

$$= 2t \cdot B_\gamma(Y, W) + B_\gamma(W, W).$$

If $B_\gamma(Y, W) \neq 0$, then this is a contradiction by letting $t \to +\infty$ or $t \to -\infty$.

- Hence

$$B_\gamma(Y, \cdot) = -\int_0^1 \langle Jac(Y), \cdot \rangle dt = 0$$

as a linear functional on the space of vector fields along $\gamma$ vanishing at $\gamma(0)$ & $\gamma(1)$.

$\Rightarrow Jac(Y) \equiv 0.$

$\boxed{\text{VII}}$

- If for some $t_0 \in (0,1)$, $\gamma(t_0)$ is a conjugate point along $\gamma$ to $\gamma(0)$.

- Then $\exists$ a nontrivial Jacobi field $Y(t)$ along $\gamma|_{[0,t_0]}$ with

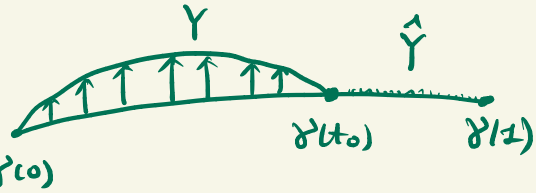
$$Y(0) = 0 = Y(t_0).$$

Claim $\Longrightarrow$ $$\left.\frac{d^2}{du^2}\right|_{u=0} E\left(\gamma_u|_{[0,t_0]}\right) = 0. \qquad (4).$$

- Since $Y \not\equiv 0$, $\nabla_X Y(t_0) \neq 0$.

- Extend $Y$ to a vector field $\hat{Y}$ along all $\gamma|_{[0,1]}$ by:

$$\hat{Y}(t) \equiv 0, \quad t \in (t_0, 1].$$




(4) $\Longrightarrow$ $$\left.\frac{d^2}{du^2}\right|_{u=0} E\left(\gamma_u|_{[0,1]}\right) = 0.$$

Claim $\Longrightarrow$ $\hat{Y}$ is a Jacobi field along $\gamma|_{[0,1]}$, which is impossible, since $\hat{Y}$ is not smooth at $\gamma(t_0)$,

$$\left(\nabla_X Y(t_0) \neq 0\right).$$