§4. 2nd variation formula, Jacobi fields. Space forms.

§4.1. 2nd variation formula.

- Let $\gamma : [0,l] \times (-\varepsilon, \varepsilon) \to M$ be a variation.
- Denote $\gamma_u(t) := \gamma(t,u)$.
- Assume γ_0 is a unit speed geodesic.
- $X = \frac{\partial}{\partial t}\gamma$, $Y = \frac{\partial}{\partial u}\gamma$.

Prop. 1. We have:

$$\frac{d^2}{du^2}\Big|_{u=0} L(\gamma_u) = \langle \nabla_Y Y, X\rangle\Big|_0^l + \int_0^l \left[|(\nabla_X Y)^\perp|^2 - Rm(X,Y,X,Y)\right] dt$$

where $(\nabla_X Y)^\perp = \nabla_X Y - \langle \nabla_X Y, X\rangle \cdot X$.

Proof: • Recall: $\nabla_X Y - \nabla_Y X = [X,Y] = 0$.

- At $u=0$, we compute:

$$\frac{d^2}{du^2} L(\gamma_u) = \frac{d}{du}\int_0^l \frac{Y\langle X,X\rangle}{2|X|}dt = \frac{d}{du}\int_0^l \frac{\langle \nabla_X Y, X\rangle}{|X|}dt$$

$$= \int_0^l \left[\frac{Y\langle \nabla_X Y, X\rangle}{|X|} - \frac{\langle \nabla_X Y, X\rangle \cdot \langle \nabla_X Y, X\rangle}{|X|^3}\right] dt.$$

$$\overset{|X|=1}{=} \int_0^l \left[\langle \nabla_Y \nabla_X Y, X\rangle + \langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_X Y, X\rangle^2 \right] dt.$$

- For the 1st term, we have: • $[X,Y]=0 \Rightarrow R(X,Y)Y = \nabla_X\nabla_Y Y - \nabla_Y \nabla_X Y$.

$$\int_0^{\ell}\langle \nabla_Y\nabla_X Y, X\rangle\,dt = \int_0^{\ell}\langle \nabla_X\nabla_Y Y, X\rangle - \langle R(X,Y)Y, X\rangle\,dt$$

$$= \int_0^{\ell} \overset{\frac{d}{dt}}{X}\langle \nabla_Y Y, X\rangle\,dt - \int_0^{\ell} Rm(X,Y,X,Y)\,dt$$

$$= \langle \nabla_Y Y, X\rangle\Big|_{t=0}^{\ell} - \int_0^{\ell} Rm(X,Y,X,Y)\,dt.$$

• Also, we have:

$$\langle \nabla_X Y, \nabla_X Y\rangle - \langle \nabla_X Y, X\rangle^2 = |\nabla_X Y - \langle \nabla_X Y, X\rangle\cdot X|^2 = |(\nabla_X Y)^{\perp}|^2.$$

$\boxed{V(1)}$

• **Theorem 2** (Bonnet – Myers). • (M^n, g): complete, connected.

• $Ric \geq (n-1)K_0\cdot g$, $K_0 > 0$, a constant.

• Then: $\mathrm{Diam}(M,g) := \sup\{d(p,q) \mid p,q\in M\} \leq \dfrac{\pi}{\sqrt{K_0}}$

• **Proof**: • Let $p, q \in M$ be any two points, $\ell = d(p,q)$.

• Since M is complete, we can find a minimal geodesic $\gamma: [0,\ell]\to M$ with

$\gamma(0) = p$, $\gamma(\ell) = q$. then $|\dot\gamma| = 1$.

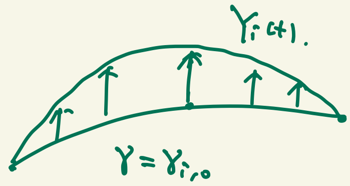
• At $p = \gamma(0)$, take $\{e_1, \cdots, e_{n-1}, e_n = \dot\gamma(0)\} \subset T_pM$ an orthonormal basis.

• Let $e_i(t)$ be parallel translation of e_i along γ, $i = 1, \cdots, n-1$, i.e.,

$$\nabla_{\dot\gamma(t)} e_i(t) = 0, \quad e_i(0) = e_i. \qquad (\text{ODE}, \exists!\ \text{solution}).$$

• Define variation fields $Y_i(t) := \sin\left(\frac{\pi}{\ell}\cdot t\right)\cdot e_i(t)$, $0\leq t\leq \ell$, $1\leq i\leq n-1$.

· For every $1 \le i \le n-1$, $\exists$ variation curves:



$$\gamma_{i,u}(t) = \gamma_i(t, u) := \exp_{\gamma(t)}\left(u \cdot Y_i(t)\right).$$

with $\gamma = \gamma_{i,0}$, $\left.\frac{\partial}{\partial u}\right|_{u=0} \gamma_i = Y_i$.

· Since $\forall\ 1 \le i \le n-1$, $\langle e_i(0), \dot\gamma(0) \rangle = 0$, and

$$\frac{d}{dt} \langle e_i(t), \dot\gamma(t) \rangle = \langle \nabla_{\dot\gamma(t)} e_i(t), \dot\gamma(t) \rangle + \langle e_i(t), \nabla_{\dot\gamma(t)} \dot\gamma(t) \rangle = 0,$$

we have: $\langle e_i(t), \dot\gamma(t) \rangle \equiv 0$, $t \in [0, \ell]$. Hence:

$$\langle \nabla_{\dot\gamma(t)} Y_i(t), \dot\gamma(t) \rangle = \left\langle \frac{\pi}{\ell} \cdot \cos\left(\frac{\pi}{\ell}\right) \cdot t \cdot e_i(t) + \sin\left(\frac{\pi}{\ell} \cdot t\right) \cdot \underset{=0}{\nabla_{\dot\gamma(t)} e_i(t)}, \dot\gamma(t) \right\rangle \equiv 0$$

Hence $\left(\nabla_{\dot\gamma(t)} Y_i\right)^{\perp} = \nabla_{\dot\gamma(t)} Y_i$.

· Now, from Prop. 1, $\forall\ 1 \le i \le n-1$, since $Y_i(0) = 0 = Y_i(\ell)$, and

$\gamma_{i,0} = \gamma$ is minimal geodesic,

$$0 \le \left.\frac{d^2}{du^2}\right|_{u=0} L(\gamma_{i,u}) = \int_0^{\ell} \left[|\nabla_{\dot\gamma} Y_i|^2 - Rm\left(\dot\gamma(t), Y_i(t), \dot\gamma(t), Y_i(t)\right) \right] dt$$

$$= \int_0^{\ell} \left[\left(\frac{\pi}{\ell}\right)^2 \cdot \cos^2\left(\frac{\pi}{\ell} \cdot t\right) - \sin^2\left(\frac{\pi}{\ell} \cdot t\right) \cdot Rm\left(\dot\gamma(t), e_i(t), \dot\gamma(t), e_i(t)\right) \right] dt$$

· Summing up $1 \le i \le n-1$,

$$0 \le \int_0^{\ell} \left[(n-1) \cdot \left(\frac{\pi}{\ell}\right)^2 \cdot \cos^2\left(\frac{\pi}{\ell} \cdot t\right) - \sin^2\left(\frac{\pi}{\ell} \cdot t\right) \cdot Ric\left(\dot\gamma(t), \dot\gamma(t)\right) \right] dt$$

$Ric \ge (n-1) K_0 \cdot g$, $|\dot\gamma| = 1$.

$$\le \int_0^{\ell} \left[(n-1) \cdot \left(\frac{\pi}{\ell}\right)^2 \cdot \cos^2\left(\frac{\pi}{\ell} \cdot t\right) - \sin^2\left(\frac{\pi}{\ell} \cdot t\right) \cdot (n-1) K_0 \right] dt$$

$$= (n-1)\cdot\left[\left(\frac{\pi}{\ell}\right)^2 - K_0\right]\cdot\int_0^\ell \sin^2\left(\frac{\pi}{\ell}\cdot t\right)dt.$$

Hence:

$$\left(\frac{\pi}{\ell}\right)^2 - K_0 \geq 0 \quad\Rightarrow\quad \ell \leq \frac{\pi}{\sqrt{K_0}}. \qquad \blacksquare$$

- **Theorem 3 (Synge).**

- (M^{2n}, g) : sectional curvature $\equiv$ constant $K_0 > 0$, compact.
- If M is orientable, then M is simply connected. $(\pi_1(M) = 0)$.

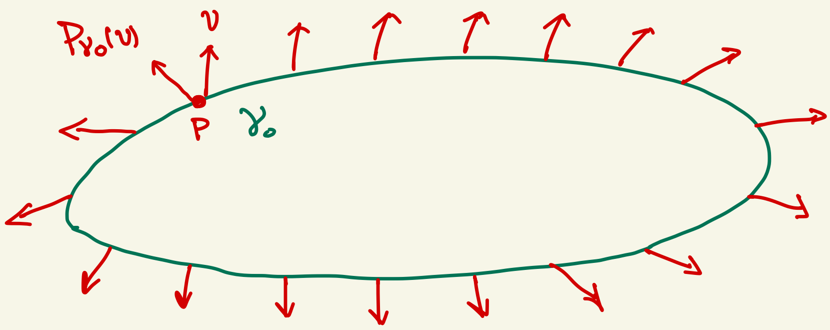
- **Proof:** • Suppose $\exists$ non-trivial homotopy class $\alpha \in \pi_1(M)$.

  Then by compactness, $\exists$ shortest representation $\gamma_0 \in \alpha$, a closed geodesic.

- Construct a parallel vector field Y along γ_0:

  - Fix $\gamma_0(0) = p \in M$, consider parallel translation:

$$P_{\gamma_0} : T_pM \longrightarrow T_pM, \quad P_{\gamma_0} \in SO(\underset{\mathbb{R}^{2n}}{\underset{\wr\wr}{T_pM}})$$



- Since γ_0 is a geodesic, $P_{\gamma_0}(\dot\gamma_0(0)) = \dot\gamma_0(0)$. Hence:

$$P_{\gamma_0} : \dot\gamma_0(0)^\perp \longrightarrow \dot\gamma_0(0)^\perp$$

is an isomorphism. Here $\dot\gamma_0(0)^\perp := \{v \in T_pM : \langle v, \dot\gamma_0(0)\rangle = 0\}$.

• But $\dim \dot{\gamma}_0(0)^\perp = 2n-1$, is odd, hence $\exists$ fixed point

$0 \neq v \in \dot{\gamma}_0(0)^\perp$ of P_{γ_0}, i.e., $v \neq 0$, $P_{\gamma_0}(v) = v$.

Γ. Since M is orientable, and $P_{\gamma_0}(\dot{\gamma}_0(0)) = \dot{\gamma}_0(0)$, hence $P_{\gamma_0} \in SO(\mathbb{R}^{2n})$ must have another eigenvalue $= 1$, whose eigenvector v $(\neq 0)$ is another fixed point of P_{γ_0}. $\rfloor$

• Let $Y(t)$ be the parallel vector field of v along γ_0.

• Find variation curves $\gamma_u(t) = \gamma(t,u)$ of $Y(t)$. (i.e., $\frac{\partial}{\partial u}\Big|_{u=0} \gamma = Y$).

Then from <u>Prop. 1</u> since γ_0 is a minimal geodesic,

$$0 \leq \frac{d^2}{du^2}\Big|_{u=0} L(\gamma_u) = \int_0^l \Big[|\underbrace{\nabla_{\dot{\gamma}_0} Y}_{=0}|^2 - Rm\big(\dot{\gamma}_0(t), Y(t), \dot{\gamma}_0(t), Y(t)\big)\Big]\, dt$$

$|\dot{\gamma}_0| = 1 = |Y|$, $\langle \dot{\gamma}_0, Y\rangle = 0$

$$= -\int_0^l K_0\, dt = -K_0 \cdot l < 0. \qquad \blacksquare$$

§4.2 Jacobi fields.

• Let $\gamma : [0,l] \times (-\varepsilon, \varepsilon) \to M$ be a variation of geodesics, i.e., each $\gamma_u(t) = \gamma(t,u)$ is a geodesic.

• Let $X = \frac{\partial}{\partial t}\gamma$, $Y = \frac{\partial}{\partial u}\gamma$, $[X, Y] = 0$.

• Then, since γ_u is a geodesic, $\nabla_X X = 0$, hence:

$$0 = \nabla_Y \nabla_X X = \nabla_X \nabla_Y X - R(X,Y)X = \nabla_X \nabla_X Y - R(X,Y)X.$$

• **Def.4 (Jacobi field).** Let γ be a geodesic. A vector field J along γ is called a Jacobi field if:

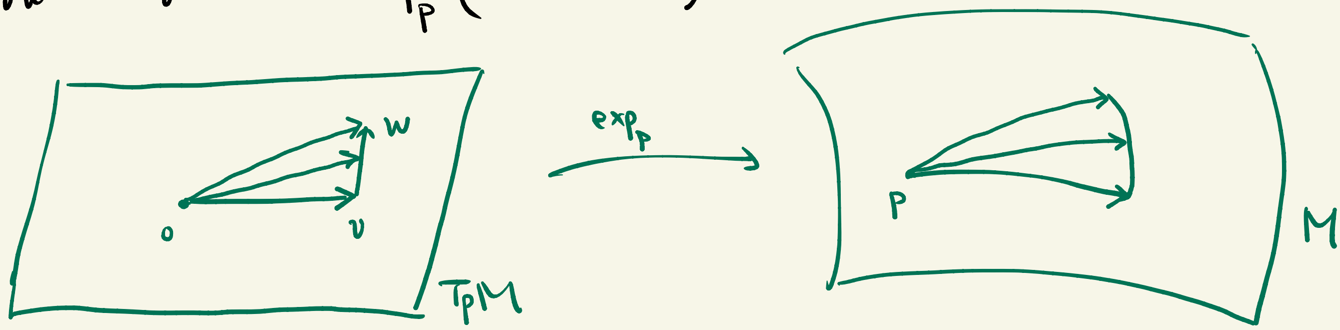
(1). $$\nabla_{\dot\gamma}\nabla_{\dot\gamma} J - R(\dot\gamma, J)\dot\gamma = 0. \qquad \#$$

• (1) is a 2nd order ODE along γ, it can be uniquely solved according to the initial value $J(0)$ & $J'(0) := \nabla_{\dot\gamma(0)} J(0)$.

• **Construction of Jacobi field:**

• Take $v, w \in T_pM$, let:

$$\gamma_u(t) := \gamma(t,u) := \exp_p\big(t\cdot(v+u\cdot w)\big)$$



then $J(t) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma(t,u)$ is Jacobi field along γ_0.

• $\gamma_u(0) \equiv p$, fixed point $\Rightarrow$ $J(0)=0$.

• $J'(0) = \nabla_{\dot\gamma_0(0)} J(0) = w$:

$$\nabla_{\dot\gamma_0(t)} J(t) = \nabla_{\dot\gamma_0(t)}\left[d(\exp_p)_{tv}(tw)\right] = \nabla_{\dot\gamma_0(t)}\left[t\cdot d(\exp_p)_{tv}(w)\right]$$

$$= d(\exp_p)_{tv}(w) + t\cdot\nabla_{\dot\gamma_0(t)}\left[d(\exp_p)_{tv}(w)\right]. \qquad \#$$

• **Lemma 5** For any Jacobi field J along geodesic γ, we have:

$$\langle J, \dot\gamma\rangle(t) = \langle J, \dot\gamma\rangle(0) + \langle J', \dot\gamma\rangle(0)\cdot t.$$

Proof: • From the Jacobi eqn. (1), we have

$$\frac{d^2}{dt^2}\langle J, \dot\gamma\rangle = \frac{d}{dt}\langle \nabla_{\dot\gamma} J, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J, \dot\gamma\rangle$$

$$\overset{(1)}{=} \langle R(\dot\gamma, J)\dot\gamma, \dot\gamma\rangle = Rm(\dot\gamma, J, \dot\gamma, \dot\gamma) = 0$$

anti-symmetry of Rm

• Next, we have

$$\left.\frac{d}{dt}\right|_{t=0}\langle J, \dot\gamma\rangle = \langle \nabla_{\dot\gamma} J, \dot\gamma\rangle(0).$$

$\square$

• Note that $(at+b)\cdot\dot\gamma(t)$ is always a Jacobi field, $\forall\, a, b \in \mathbb{R}$.

• From Lemma 5, any Jacobi field J admits a decomposition:

$$J = J^{\perp} + J^{\parallel}$$

where

• $J^{\parallel}(t) = \langle J, \dot\gamma\rangle\dot\gamma(t) = \left[\langle J, \dot\gamma\rangle(0) + \langle J', \dot\gamma\rangle(0)\cdot t\right]\cdot\dot\gamma(t).$

• $J^{\perp}$ is orthogonal to $\dot\gamma$ pointwisely along γ.

• Let $\mathcal{J} = \mathcal{J}_\gamma := \{\text{Jacobi fields } J \text{ along } \gamma\}.$

$$\mathcal{J}^{\perp} := \{J \in \mathcal{J} : J \perp \dot\gamma\}.$$

$$\mathcal{J}_0 := \{ J \in \mathcal{J} : J(0) = 0 \}.$$

$$\mathcal{J}_0^\perp := \mathcal{J}^\perp \cap \mathcal{J}_0.$$

- $$\mathcal{J} = \mathcal{J}^\perp \oplus \{ (at+b)\cdot\dot{\gamma}(t) : a, b \in \mathbb{R} \}.$$

  $$\mathcal{J}_0 = \mathcal{J}_0^\perp \oplus \{ a\cdot t\cdot\dot{\gamma}(t) : a \in \mathbb{R} \}.$$

  $$\dim_{\mathbb{R}} \mathcal{J} = n^2, \quad \dim_{\mathbb{R}} \mathcal{J}_0 = n, \quad \dim_{\mathbb{R}} \mathcal{J}_0^\perp = n-1.$$

- Def. 6 (Conjugate point). • Let $\gamma : [0, \ell] \to M$ be a unit-speed geodesic.
  - Let $p = \gamma(0)$. A point $q = \gamma(t_0)$, $t_0 \neq 0$, is called a conjugate point of p along γ, if $\exists$ non-trivial ($\neq 0$) Jacobi field J along γ, s.t.

  $$J(0) = J(t_0) = 0.$$ #

- q conjugate to p along γ $\Leftrightarrow$ p conjugate to q along γ^{-1}.

- Let $q = \exp_p(t_0 \cdot v)$, $v \in T_pM$, $\gamma = \gamma_v(t) = \exp_p(t\cdot v)$.
  - Let $w = J'(0) = \nabla_{\dot{\gamma}(0)} J(0) \neq 0$, then:

  $$\gamma(t,u) := \exp_p(t\cdot(v + u\cdot w)), \quad J(t) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma(t,u).$$

  - Hence from $J(t_0) = 0$, we have:

  $$0 = J(t_0) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma(t_0, u) = \frac{\partial}{\partial u}\Big|_{u=0} \exp_p(t_0\cdot(v + u\cdot w))$$

  $$= d(\exp_p)_{t_0 v}(t_0 w) = t_0 \cdot d(\exp_p)_{t_0 v}(w).$$

hence $\exp_p$ degenerate at $t_0 v$.

- Suppose p has no conjugate point on time interval $(0, l]$.

Since $\dim_{\mathbb{R}} \mathcal{J}_0^{\perp} = n-1$, let $J_1, \dots, J_{n-1} \in \mathcal{J}_0^{\perp}$ be a basis.

Then for each $t \in (0, l]$, $\{J_i(t)\}_{i=1}^{n-1}$ is a basis of $\dot{\gamma}(t)^{\perp}$.

- If for some $t_0 \in (0, l]$, $a_1, \dots, a_{n-1} \in \mathbb{R}$, s.t. $\sum_{i=1}^{n-1} a_i J_i(t_0) = 0$.

Then $\tilde{J} := \sum_{i=1}^{n-1} a_i J_i$ is Jacobi field along γ with $\tilde{J}(0) = 0 = \tilde{J}(t_0)$.

Hence $\tilde{J} \equiv 0$, hence $a_1 = \dots = a_{n-1} = 0$.

§4.3. Space forms.

- Hyperbolic space $\mathbb{H}^n$, Poincaré model.

  - $B_1 \subseteq \mathbb{R}^n$, unit ball. $B_1 := \{x = (x_1, \dots, x_n) : x_1^2 + \dots + x_n^2 < 1\}$.

$$g_{\mathbb{H}^n} := \frac{4(dx_1^2 + \dots + dx_n^2)}{(1-|x|^2)}, \quad \mathbb{H}^n := (B_1, g_{\mathbb{H}^n}).$$

  - Upper half space model.

$$\mathbb{R}_n^+ := \{x = (x_1, \dots, x_n) : x_n > 0\},$$

$$g_{\mathbb{H}^n} = \frac{dx_1^2 + \dots + dx_n^2}{x_n^2}$$

  - $g_{\mathbb{H}^n}$ is of constant sectional curvature -1.

• Theorem 7. (Uniformization).

• If (M^n, g) is a complete, simply connected Riem. mfd of constant sectional curvature λ, then:

(i). If $\lambda = 0$, then $(M, g) \cong (\mathbb{R}^n, g_{Euc})$, is the flat n-dim. Euclidean space.

(ii). If $\lambda > 0$, there is a diffeomorphism $\phi : M \to \mathbb{S}^n$ such that $\lambda^{-1} \cdot \phi^*(g_{\mathbb{S}^n})$, where $g_{\mathbb{S}^n}$ is the usual metric on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$: $g_{\mathbb{S}^n} = g_{Euc}|_{\mathbb{S}^n}$.

(iii). If $\lambda < 0$, there is a diffeomorphism $\phi : M \to \mathbb{H}^n$ such that $g = |\lambda|^{-1} \cdot \phi^*(g_{\mathbb{H}^n})$.

• Jacobi field on space form.

• Suppose (M^n, g) has constant sectional curvature $K_0 \in \mathbb{R}$. Then:

$$Rm(X, Y, Z, W) = K_0 \cdot \left(\langle X, Z \rangle \cdot \langle Y, W \rangle - \langle X, W \rangle \cdot \langle Y, Z \rangle \right)$$

$$R(X, Y)Z = K_0 \cdot \left(\langle Y, Z \rangle X - \langle X, Z \rangle Y \right).$$

• Jacobi eqn becomes:

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} J - K_0 \cdot \left(\langle J, \dot\gamma \rangle \dot\gamma - J \right) = 0.$$

- If $J \in \mathcal{J}_0^{\perp}$, then $J'' + K_0 \cdot J = 0$, then:

$$J(t) = \begin{cases} a \cdot \dfrac{\sin(\sqrt{K_0} \cdot t)}{\sqrt{K_0}} \cdot e(t), \quad a \in \mathbb{R}, & \text{if } K_0 > 0. \\ a \cdot t \cdot e, \quad a \in \mathbb{R}, & \text{if } K_0 = 0 \\ a \cdot \dfrac{\sinh(\sqrt{-K_0} \cdot t)}{\sqrt{-K_0}} \cdot e(t), & \text{if } K_0 < 0. \end{cases}$$

for some unit parallel vector field $e(t)$ along γ with $e(0) \parallel J'(0) \perp \dot{\gamma}(0)$,

with initial value $J'(0) = a \cdot e(0)$.

- Recall: $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$.