# Introduction to Complex Geometry

## Chapter 3 Differential geometry of vector bundles

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# Outline

# §3.1 Metrics, connections and curvatures

# Hermitian metrics

## Definition

Let $E \to X$ be a complex ($C^\infty$) vector bundle of rank $r$ over a smooth manifold $X$. A smooth Hermitian metric on $E$ is an assignment of Hermitian inner products $h_p(\cdot,\cdot) = <\cdot,\cdot>_p$ on each fiber $E_p$, such that if $\xi, \eta$ are smooth sections of $E$ over an open set $U$, then $h(\xi,\eta) \in C^\infty(U;\mathbb{C})$.

If $U$ is a local triviliazation neighborhood of $E$ via $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$, then we can define $r$ smooth sections of $E$ over $U$:

$$e_\alpha(p) := \varphi_U^{-1}(p, 0, \ldots, 0, 1, 0 \ldots, 0).$$

Then at any point $p \in U$, $\{e_\alpha(p)\}_{\alpha=1}^r$ is a basis of $E_p$. We call $\{e_\alpha\}_{\alpha=1}^r$ a local frame of $E$ over $U$. Note that when $E$ is a holomorphic bundle and $(U, \varphi_U)$ a holomorphic trivialization, then these $e_\alpha$'s are also holomorphic sections, and we call it a holomorphic frame.

# Local representation of a metric

If $\xi$ is a smooth section over $U$, then we can write in a unique way $\xi = \xi^\alpha e_\alpha$, with $\xi_\alpha \in C^\infty(U; \mathbb{C}), \alpha = 1, \ldots, r$. If we define the (positive definite) Hermitian matrix-valuded smooth functions: $h_{\alpha\bar{\beta}} := h(e_\alpha, e_\beta)$, then we have

$$h(\xi, \eta) = h(\xi^\alpha e_\alpha, \eta^\beta e_\beta) = h_{\alpha\bar{\beta}} \xi^\alpha \bar{\eta}^\beta.$$

Sometimes, we also denote the matrix-valued smooth function $(h_{\alpha\bar{\beta}})$ by $h$.

**Notation:** We shall denote the space of smooth sections of $E$ over $U$ by $C^\infty(U; E)$. When $E$ is a holomorphic bundle, the set of holomorphic sections over $U$ is denoted by $\Gamma(U; E)$ or $\mathscr{O}(E)(U)$.

# Connection

## Definition

A connection on a smooth rank $r$ complex vector bundle over a manifold $X$ is a map $D : C^\infty(X; E) \to C^\infty(X, T^{*\mathbb{C}}X \otimes E)$ satisfying :

1. $D$ is $\mathbb{C}$-linear;
2. (Leibniz rule) $D(f\xi) = df \otimes \xi + fD\xi, \forall f \in C^\infty(X; \mathbb{C}), \xi \in C^\infty(X; E)$.

$D$ is a local operator: if $\xi \in C^\infty(X; E)$ vanishes on an open subset $U$, then so is $D\xi$:

For any $p \in U$, choose a smooth function $f$ on $X$ such that $f$ vanishes on a small neighborhood $V \subset U$ and $f|_{U^c} \equiv 1$. Then $f\xi = \xi$, so by the Leibniz rule, we have

$$(D\xi)(p) = \big(D(f\xi)\big)(p) = (df \otimes \xi + fD\xi)(p) = 0.$$

# Local representation of a connection

- Locality of $D$ implies that we can define $D\xi$ for $\xi \in C^\infty(U; E)$ when $U \subset X$ is an open subset: for any $p \in U$, choose open nbhd $V \subset U$ and a global section $\eta \in C^\infty(X; E)$ such that $\eta|_V \equiv \xi|_V$, then define $(D\xi)|_V := (D\eta)|_V$.

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- If $\{e_\alpha\}$ is a local frame, then we can define a family of local smooth 1-forms $\theta_\alpha^\beta \in A^1(U)$ satisfying:

$$De_\alpha = \theta_\alpha^\beta \otimes e_\beta.$$

  Sometimes we just write $De_\alpha = \theta_\alpha^\beta e_\beta$ for short. We call these $\{\theta_\alpha^\beta\}$ "connection one-forms".

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- For $\xi = \xi^\alpha e_\alpha \in C^\infty(U; E)$, we then have
$$D\xi = D(\xi^\alpha e_\alpha) = (d\xi^\alpha + \xi^\beta \theta_\beta^\alpha)e_\alpha.$$

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- Regard $\xi^\alpha$ as a column vector, and for $\theta_\beta^\alpha$ we always regard the upper index as line index and the lower index the column index. So if we identify $\xi$ with the column vector $\xi^\alpha$, then we can write $D = d + \theta$ (Physicists' notation).

# The curvature

We can extend the action of $D$ to bundle-valued differential forms. We write $A^k(X,E) := C^\infty(X;\Lambda^k T^{*\mathbb{C}}X \otimes E)$. Then we define $D : A^k(X,E) \to A^{k+1}(X,E)$ by

$$D(\varphi\xi) := (d\varphi)\xi + (-1)^k\varphi \wedge D\xi,$$

where $\varphi$ is a $\mathbb{C}$-valued $k$-form and $\xi$ is a smooth section of $E$.

## Definition

We define the curvature of $D$ to be $\Theta := D^2 : A^0(X;E) \to A^2(X,E)$.

# Curvature as a bundle-valued 2-form

If $f$ is a smooth function and $\xi \in A^0(X, E)$, we have

$$\begin{aligned}\Theta(f\xi) &= D(df\xi + fD\xi)\\ &= d(df)\xi - df \wedge D\xi + df \wedge D\xi + fD^2\xi\\ &= f\Theta(\xi).\end{aligned}$$

Locally if we define the 2-forms $\Theta_\alpha^\beta$ by

$$\Theta(e_\alpha) = \Theta_\alpha^\beta e_\beta.$$

Then we have

$$\Theta(\xi) = \Theta(\xi^\alpha e_\alpha) = \xi^\alpha \Theta(e_\alpha) = \Theta_\beta^\alpha \xi^\beta e_\alpha.$$

From this, we conclude that $\Theta \in A^2(X, End(E))$.

# Local representation of the curvature

We can also represent $\Theta^\alpha_\beta$ in terms of $\theta^\alpha_\beta$:

$$\begin{aligned}\Theta^\beta_\alpha e_\beta &= D(De_\alpha) = D(\theta^\gamma_\alpha e_\gamma)\\ &= d\theta^\gamma_\alpha e_\gamma - \theta^\gamma_\alpha \wedge De_\gamma\\ &= d\theta^\beta_\alpha e_\beta - \theta^\gamma_\alpha \wedge \theta^\beta_\gamma e_\beta\\ &= (d\theta^\beta_\alpha + \theta^\beta_\gamma \wedge \theta^\gamma_\alpha)e_\beta.\end{aligned}$$

So we get

$$\Theta^\alpha_\beta = d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta,$$

or $\Theta = d\theta + \theta \wedge \theta$ for short. Note that our sign convention is different from Griffiths-Harris, since they regard the upper index as the column index.

# Changing the frame

- Suppose $\{\tilde{e}_\alpha\}$ is another local frame on $U$, then we can write $\tilde{e}_\alpha = a_\alpha^\beta e_\beta$, where $(a_\alpha^\beta)$ is a $GL(r, \mathbb{C})$-valued smooth function on $U$. (When both frames are local holomorphic frames of a holomorphic bundle, then $(a_\alpha^\beta)$ is a $GL(r, \mathbb{C})$-valued holomorphic function on $U$.)

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- The new connection forms and curvature forms are denoted by $\tilde{\theta}$ and $\tilde{\Theta}$. We have

$$\begin{aligned}\tilde{\theta}_\alpha^\gamma \tilde{e}_\gamma &= D\tilde{e}_\alpha = D(a_\alpha^\beta e_\beta)\\ &= da_\alpha^\beta e_\beta + a_\alpha^\beta \theta_\beta^\gamma e_\gamma\\ &= (da_\alpha^\beta + \theta_\gamma^\beta a_\alpha^\gamma) e_\beta.\end{aligned}$$

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$$\begin{aligned}\tilde{\theta}_\alpha^\gamma \tilde{e}_\gamma = D\tilde{e}_\alpha &= D(a_\alpha^\beta e_\beta)\\ &= da_\alpha^\beta e_\beta + a_\alpha^\beta \theta_\beta^\gamma e_\gamma\\ &= (da_\alpha^\beta + \theta_\gamma^\beta a_\alpha^\gamma)e_\beta.\end{aligned}$$

- On the other hand, the left equals $\tilde{\theta}_\alpha^\gamma a_\gamma^\beta e_\beta$. So we get $a\tilde{\theta} = da + \theta a$, or

$$\tilde{\theta} = a^{-1}da + a^{-1}\theta a.$$

# Change of curvature

From the above formula, we get

$$\begin{aligned}
\tilde{\Theta} &= d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta} \\
&= d(a^{-1}da + a^{-1}\theta a) + (a^{-1}da + a^{-1}\theta a) \wedge (a^{-1}da + a^{-1}\theta a) \\
&= -a^{-1}da \wedge a^{-1}da - a^{-1}da \wedge a^{-1}\theta a + a^{-1}d\theta a - a^{-1}\theta \wedge da \\
&\quad + a^{-1}da \wedge a^{-1}da + a^{-1}da \wedge a^{-1}\theta a + a^{-1}\theta \wedge da + a^{-1}\theta \wedge \theta a \\
&= a^{-1}(d\theta + \theta \wedge \theta)a.
\end{aligned}$$

So we conclude

$$\tilde{\Theta} = a^{-1}\Theta a.$$

From this, we can construct a family of globally defined differential forms:

$$c(E, D) := \det\left(I_r + \frac{\sqrt{-1}}{2\pi}\Theta\right) := 1 + c_1(E, D) + \cdots + c_r(E, D),$$

where $c_k(E, D) \in A^{2k}(X)$ is called the “k-th” Chern form of $E$ associated to $D$.

# Physicists' language

In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations. The "vector potential" and "scalar potential" together form the connection 1-form, and the curvature 2-form has 6 components, consisting the components of the electric field and the magnetic field.

# §3.2 Chern connection on holomorphic vector bundles

# Chern connection on a holomorphic vector bundle

In general, there is no "canonical connections" on a given vector bundle with a smooth Hermitian metric. However, if the bundle is a holomorphic vector bundle, there is indeed a canonical connection, called the "Chern connection":

## Theorem

*On a given holomorphic vector bundle $E$ with a smooth Hermitian metric $h$, there is a unique connection $D$, called the "Chern connection" satisfying the following two additional conditions:*

1. **(Compatibility with the metric)** *If $s, t$ are two smooth sections, then we have*

$$dh(s,t) = h(Ds,t) + h(s,Dt).$$

2. **(Compatibility with the complex structure)** *If $s$ is a holomorphic section of $E$, then $Ds$ is a $E$-valued $(1,0)$-form.*

# Proof (Uniqueness)

*Let* $\{e_\alpha\}_{\alpha=1}^r$ *be a local holomorphic frame, and the connection 1-form with respect to this frame is* $(\theta_\alpha^\beta)_{1\leq\alpha,\beta\leq r}$*, satisfying* $De_\alpha = \theta_\alpha^\beta e_\beta$*. By the compatibility with complex structure, each* $\theta_\alpha^\beta$ *is a smooth* $(1,0)$*-form. Now we use the compatibility with metric to get*

$$\begin{aligned} dh_{\alpha\bar\beta} &= h(De_\alpha, e_\beta) + h(e_\alpha, De_\beta) \\ &= \theta_\alpha^\gamma h_{\gamma\bar\beta} + \bar\theta_\beta^\gamma h_{\alpha\bar\gamma}. \end{aligned}$$

*On the other hand, we have* $dh_{\alpha\bar\beta} = \partial h_{\alpha\bar\beta} + \bar\partial h_{\alpha\bar\beta}$*. Comparing types, we get* $\partial h = \theta^t h$*, so* $\theta^t = \partial h \cdot h^{-1}$*. Denote* $h^{-1} = (h^{\bar\beta\alpha})$*, then we can rewrite this as*

$$\theta_\alpha^\beta = h^{\bar\gamma\beta}\partial h_{\alpha\bar\gamma}.$$

*Also, since* $\bar h^t = h$*, the* $(0,1)$*-part gives the same equation. This proves the uniqueness.*

## Proof (Existence)

*For existence, we simply set locally* $\theta_\alpha^\beta := h^{\bar\gamma\beta}\partial h_{\alpha\bar\gamma}$*, and define for* $s = f^\alpha e_\alpha$*:*

$$Ds := (df^\alpha + f^\beta\theta_\beta^\alpha)e_\alpha.$$

*We need to check that this is globally well-defined. For this, if* $\tilde e_\alpha = a_\alpha^\beta e_\beta$ *is another holomorphic frame on* $V$ *with* $U \cap V \neq \emptyset$*. Then* $a$ *is a holomorphic matrix. We have* $\tilde h = a^t h \bar a$*, so we have* $\tilde\theta := (\tilde h^t)^{-1}\partial\tilde h^t = a^{-1}\partial a + a^{-1}\theta a$*. Since* $s = \tilde f^\alpha \tilde e_\alpha = f^\alpha e_\alpha$*, we have* $\tilde f = a^{-1} f$*, so*

$$\begin{aligned}\tilde e(d\tilde f + \tilde\theta\tilde f) &= ea(-a^{-1}daa^{-1}f + a^{-1}df + a^{-1}\partial aa^{-1}f + a^{-1}\theta aa^{-1}f)\\ &= e(df + \theta f).\end{aligned}$$

*So* $D$ *is globally defined. It is direct to check that* $D$ *is compatible with both the metric and the complex structure of the bundle.*

# Remark

- *If we define covariant derivatives of a smooth section* $s$ *with respect to a complex tangent vector* $\xi$ *at a given point* $p$ *by* $D_\xi s = Ds(\xi)$*, where we use the dual pairing of tangent vectors and differential 1-forms. Then the "compatibility with metric" takes the form*

$$\xi\big(h(s,t)\big) = h(D_\xi s, t) + h(s, D_{\bar{\xi}} t).$$

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- *If we write the* $(1,0)$ *and* $(0,1)$ *parts of* $D$ *as* $D^{1,0}$ *and* $D^{0,1}$*, so that* $D = D^{1,0} + D^{0,1}$*. then the compatibility with complex structure condition can be restated as* $D^{0,1} = \bar{\partial}$ *on smooth sections.*

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- *The line bundle case is particularly simple: if* $e$ *is a local holomorphic frame and we set* $h = h(e,e) > 0$. *Then the connection 1-form is* $\theta = h^{-1}\partial h = \partial \log h$. *Then the curvature is* $\Theta = d\theta + \theta \wedge \theta = d\theta = d\partial \log h = \bar{\partial}\partial \log h$. *It is already a globally defined closed* $(1,1)$*-form.*

# Property of the curvature of Chern connection

- In general, the curvature of Chern connection is locally given by:

$$\Theta = d\theta + \theta \wedge \theta = \bar{\partial}\theta + (\partial\theta + \theta \wedge \theta).$$

So $\Theta = \Theta^{2,0} + \Theta^{1,1}$, where $\Theta^{2,0} = \partial\theta + \theta \wedge \theta$ and $\Theta^{1,1} = \bar{\partial}\theta$.

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So $\Theta = \Theta^{2,0} + \Theta^{1,1}$, where $\Theta^{2,0} = \partial\theta + \theta \wedge \theta$ and $\Theta^{1,1} = \bar{\partial}\theta$.

- However, from the local expression of $\theta$, we get

$$\begin{aligned}
\Theta^{2,0} &= \partial\Big((h^t)^{-1}\partial h^t\Big) + (h^t)^{-1}\partial h^t \wedge (h^t)^{-1}\partial h^t \\
&= -(h^t)^{-1}\partial h^t (h^t)^{-1} \wedge \partial h^t + (h^t)^{-1}\partial^2 h^t + (h^t)^{-1}\partial h^t \wedge (h^t)^{-1}\partial h^t \\
&= 0.
\end{aligned}$$

# Property of the curvature of Chern connection(continued)

- So with respect to a local holomorphic frame we have $\Theta = \Theta^{1,1}$ is of type $(1,1)$, and

$$\Theta = \bar{\partial}\Big((h^t)^{-1}\partial h^t\Big).$$

# Property of the curvature of Chern connection(continued)

- So with respect to a local holomorphic frame we have $\Theta = \Theta^{1,1}$ is of type $(1,1)$, and
$$\Theta = \bar{\partial}\Big((h^t)^{-1}\partial h^t\Big).$$
- What happens if we choose a local $C^\infty$ frame? Let $\tilde{e}_\alpha = a_\alpha^\beta e_\beta$ be such a $C^\infty$ frame, and let $\tilde{\Theta}_\alpha^\beta$ be the corresponding curvature 2-forms, then we still have
$$\tilde{\Theta} = a^{-1}\Theta a,$$
which is also of type (1,1)!

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which is also of type (1,1)!
- Conclusion:For Chern connection on a holomorphic vector bundle, its curvature form is always of type (1,1), regardless of whether the frame is holomorphic or not!

# Koszul-Malgrange theorem

For a smooth vector bundle, the "compatibility with complex structure" condition does not make sense any more. But we can always find connections compatible with a given Hermitian metric. However, its curvature 2-form is never of type (1,1) unless it is a holomorphic vector bundle:

## Theorem

*Let $E \to X$ be a smooth complex vector bundle over a complex manifold. If we can define a linear operator $\bar{\partial}_E : C^\infty(X, E) \to A^{0,1}(X, E)$ satisfying*

$$\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f \bar{\partial}_E s$$

*and $\bar{\partial}_E^2 = 0$. Then we can make $E$ into a holomorphic vector bundle, so that the $\bar{\partial}$ operator of this holomorphic bundle is precisely $\bar{\partial}_E$.*

Ref: Atiyah-Hitchin-Singer: Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London 362 (1978), 425–461. or Donaldson-Kronheimer.

## Example

Consider the universal bundle $U \to \mathbb{C}P^n$. Recall that

$$U = \{([z], v) \mid v \in [z]\} \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}.$$

We can define a very natural Hermitian metric on $U$:

$$h_{[z]}(v, w) := \langle v, w \rangle_{\mathbb{C}^{n+1}}.$$

We now compute this metric and its curvature using local trivializations: Take $U_0 = \{[z] | \, z_0 \neq 0\}$ for example, the coordinates are $(\xi_1, \ldots, x_n) = (\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0})$. As a local frame of $U$ over $U_0$, we can choose $e([z]) := ([z], (1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}))$. So we get $h(e, e) = 1 + |\xi_1|^2 + \cdots + |\xi_n|^2$, and hence

$$\Theta(h) = \bar{\partial}\partial \log(1 + |\xi_1|^2 + \cdots + |\xi_n|^2) = -\Big(\frac{\delta_{ij}}{1 + |\xi|^2} - \frac{\bar{\xi}_i \xi_j}{(1 + |\xi|^2)^2}\Big) d\xi_i \wedge d\bar{\xi}_j.$$

# §3.3 Chern classes of a complex vector bundle

# Tools: trace and (super)-commutator

- We first define a trace map $tr : A^k(X, EndE) \to A^k(X)$. For a $EndE$-valued form $\eta \in A^k(X, EndE)$, the trace of $\eta$ is the $k$-form $tr(\eta)$ obtained by tracing out the $EndE$ factor. Locally, we can write $\eta$ as a matrix of $k$-forms, and $tr(\eta)$ is just the trace of this matrix. Or equivalently, we can write $\eta$ as $\sum_i \omega_i \otimes A_i$ with $\omega_i$ a family of $k$-forms and $A_i$ a family of local sections of $EndE$, and then $tr(\eta) = \sum_i tr(A_i)\omega_i$.

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- Another tool we shall use is the (super)-commutator, defined by $[\omega \otimes A, \eta \otimes B] := (\omega \wedge \eta) \otimes [A, B]$, where $\omega, \eta$ are locally defined forms and $A, B$ are local sections of $EndE$. It is easy to see that

$$[\omega \otimes A, \eta \otimes B] = \omega A \wedge \eta B - (-1)^{deg(\omega)deg(\eta)} \eta B \wedge \omega A.$$

The appearance of the extra factor $(-1)^{deg(\omega)deg(\eta)}$ is the reason why sometimes it is called a "super"-commutator. We sometimes extend the definition: we define for the connection $D$: $[D, \omega \otimes A]s := D(\omega \otimes As) - (-1)^{deg(\omega)} \omega \otimes A \wedge Ds$.

# Three simple lemmas

## Lemma

*If* $\tilde{D}$ *is another connection on* $E$*, then* $\tilde{D} - D \in A^1(X, EndE)$.

## Lemma

*If* $P, Q$ *are both* $EndE$*-valued differential forms, then* $tr[P, Q] = 0$.

The first nontrivial lemma is:

## Lemma (Bianchi identity)

*We have* $[D, \Theta^k] = 0$*, for any* $k \in \mathbb{N}$.

## Proof

*Simply note that* $\Theta = D^2$*, so* $[D, \Theta^k] = [D, D^{2k}] = 0$.

# The key lemma

## Lemma

*For $A \in A^k(X, EndE)$, we have $d\ tr(A) = tr[D, A]$.*

## Proof

*First note that the left hand side is obviously independent of the connection. For the right hand side, if we use another connection $\tilde{D}$, by Lemma10 and Lemma11, we have $tr[\tilde{D}, A] = tr[\tilde{D} - D, A] + tr[D, A] = tr[D, A]$. So the right hand side is also independent of the connection.*
*So we can in fact choose a trivial connection locally to carry out the computation: Let $D_0 = d$ be a trivial connection on $E|_U \to U$, then*

$$[D_0, A]s = D_0(As) - (-1)^{deg(A)} A \wedge D_0 s = d(A^\beta_\alpha f^\alpha) e_\beta - (-1)^{deg(A)} A^\beta_\alpha \wedge df^\alpha e_\beta = dA^\beta_\alpha f^\alpha e_\beta.$$

*Hence $tr[D_0, A] = d\ tr(A)$.*

# Chern-Weil Theorem

For any formal power series in one variable $f(x) = a_0 + a_1 x + \dots$, we define $f(\Theta) := a_0 + a_1\Theta + \cdots + a_n\Theta^n \in A^*(X)$.

## Theorem (Chern-Weil)

*For $f$ as above, we have:*

1. $d\, trf(\Theta) = 0$;
2. *If $\tilde{D}$ is another connection with curvature $\tilde{\Theta}$, there is a differential form $\eta \in A^*(X)$ such that $tr\ f(\tilde{\Theta}) - tr\ f(\Theta) = d\eta$.*

*So the cohomology class of $tr\ f(\Theta)$ is independent of the connection. We call it the "characteristic class" of $E$ associated to $f$, and $tr\ f(\Theta)$ the corresponding "characteristic form" of $E$ associated to $f$ and $D$.*

# Chern classes

## Example

Since $det(I_r + \frac{\sqrt{-1}}{2\pi}\Theta) = \exp\left(tr \log(I_r + \frac{\sqrt{-1}}{2\pi}\Theta)\right)$. So $c_i(E, D) \in A^{2i}(X)$ are all closed forms, whose cohomology classes are all independent of $D$. These are called "Chern classes". For example we have

$$c_1(E, D) = \frac{\sqrt{-1}}{2\pi} tr\Theta, \quad c_2(E, D) = \frac{1}{8\pi^2}\left(tr(\Theta^2) - (tr\Theta)^2\right).$$

## Proof (Proof of Chern-Weil Theorem:)

- *For the first conclusion, we have*

$$d\ trf(\Theta) = tr[D, f(\Theta)] = \sum_k a_k tr[D, \Theta^k] = 0,$$

*where we used Bianchi identity in the last step.*

# Proof (Proof of Chern-Weil Theorem:)

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*where we used Bianchi identity in the last step.*

- *For the second one, we choose a family of connections* $D_t := t\tilde{D} + (1-t)D$. *Then* $\dot{D}_t := \frac{dD_t}{dt} = \tilde{D} - D \in A^1(X, EndE)$, *and*

$$\dot{\Theta}_t := \frac{d\Theta_t}{dt} = \frac{dD_t}{dt} D_t + D_t \frac{dD_t}{dt} = [D_t, \frac{dD_t}{dt}] = [D_t, \dot{D}_t].$$

## Proof of Chern-Weil Theorem:(continued).

- So we have (we can change the positions of $\Theta$ and $\dot{\Theta}$ by previous lemmas)

$$\frac{d}{dt} tr\ f(\Theta_t) = tr(\dot{\Theta}_t f'(\Theta_t)) = tr([D_t, \dot{D}_t] f'(\Theta_t))$$

$$\overset{\text{Bianchi}}{=} tr[D_t, \dot{D}_t f'(\Theta_t)]$$

$$= d\ tr\left(\dot{D}_t f'(\Theta_t)\right).$$

□

## Proof of Chern-Weil Theorem:(continued).

- So we have (we can change the positions of $\Theta$ and $\dot{\Theta}$ by previous lemmas)

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$$\overset{\text{Bianchi}}{=} tr[D_t, \dot{D}_t f'(\Theta_t)]$$
$$= d\ tr\Big(\dot{D}_t f'(\Theta_t)\Big).$$

- So we conclude that $tr\ f(\tilde{\Theta}) - tr\ f(\Theta) = d \int_0^1 tr\Big(\dot{D}_t f'(\Theta_t)\Big) dt.$

□

# Chern classes are real

Let $E \to X$ be a complex vector bundle over $X$, we know that the Chern classes $c_k(E)$ are independent of the connection, so we can choose a metric $h$ and require that $D$ is compatible with the metric. Choose a local unitary frame, so that $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$. Then we have

$$0 = dh_{\alpha\bar\beta} = \theta_\alpha^\gamma \delta_{\gamma\beta} + \delta_{\alpha\gamma}\overline{\theta}_\beta^\gamma = \theta_\alpha^\beta + \overline{\theta}_\beta^\alpha.$$

In short, $\bar\theta^t = -\theta$. This in turn implies that $\bar\Theta^t = -\Theta$, and so

$$\overline{\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)}^t = -\frac{\sqrt{-1}}{2\pi}\bar\Theta^t = \frac{\sqrt{-1}}{2\pi}\Theta.$$

We have $\overline{c(E,D)} = \overline{\det(I_r + \frac{\sqrt{-1}}{2\pi}\Theta)} = \det(I_r + \overline{\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)}^t) = \det(I_r + \frac{\sqrt{-1}}{2\pi}\Theta) = c(E,D)$.

# Chern classes as obstructions

We shall prove that Chern classes are obstructions to the existence of global linearly independent smooth sections:

## Theorem

*If $E \to X$ is a smooth complex vector bundle of rank $r$. If there are $k$ smooth sections $s_1, \ldots, s_k \in C^\infty(X; E)$ such that $\{s_i(p)\}_{i=1}^k$ are linearly independent everywhere, then we have $c_i(E) = 0$ for $i > r - k$.*

## Proof

*We first consider the $k = r$ case. Then the assumption implies that $E$ is a trivial bundle. We use a trivial connection $D = d$ on $E$, then $\Theta \equiv 0$ and we have $c(E, D) = 1$, and hence $c_i(E) = 0$ for $i = 1, \ldots, r$.*

# Proof (continued)

*In general, the assumption implies that there is a rank $k$ trivial sub-bundle $T$ of $E$, generated by these $k$ sections. Using a metric on $E$, we can define the orthogonal complement of $T$ in $E$, it is also a sub-bundle of $E$, denoted by $E'$, so we get $E = T \oplus E'$.*

*Now we choose connections $D_T$ and $D_{E'}$ respectively, where $D_T$ is the trivial connection, and form the connection $D := D_T \oplus D_{E'}$ on $E$. Then locally we have $\theta = diag\{\theta_T, \theta_{E'}\}$, and consequently $\theta = diag\{\theta_T, \theta_{E'}\} = diag\{0, \theta_{E'}\}$. Consequently, we have $\Theta = diag\{0, \Theta_{E'}\}$. Note that $c(T, D_T) = 1$, so we get*

$$
\begin{aligned}
c(E, D) &= \det(I_r + \frac{\sqrt{-1}}{2\pi}\Theta) = \det I_k \cdot \det(I_{r-k} + \frac{\sqrt{-1}}{2\pi}\Theta_{E'}) \\
&= 1 + c_1(E', D_{E'}) + \cdots + c_{r-k}(E', D_{E'}).
\end{aligned}
$$

*This implies $c_i(E) = 0$ for $i > r - k$.*

# Comparing two definitions of first Chern classes

Recall: let $X$ be a complex manifold, using the short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathscr{O}^* \to 1$$

we get the exact sequence

$$\cdots \to H^1(X, \mathscr{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \ldots.$$

We call $\delta : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$ the "first Chern class" map.

# First Chern class classifies line bundles

Instead of holomorphic line bundles, we can consider $C^\infty$ line bundles. These bundles are classified by $H^1(X, \mathcal{E}^*)$. Similarly, we have short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{E} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathcal{E}^* \to 1,$$

and consequently a short exact sequence:

$$\cdots \to H^1(X, \mathcal{E}) \cdots \to H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{E}) \to \ldots.$$

Since $\mathcal{E}$ is a fine sheaf, we have $H^p(X, \mathcal{E}) = 0$ whenever $p \geq 1$. So $\delta : H^1(X, \mathcal{E}^*) \to H^2(X, \mathbb{Z})$ is an isomorphism (also called "first Chern class map"). This means that *complex line bundles are determined up to $C^\infty$ isomorphisms by their first Chern class.*

# The problem

- On the other hand, we can use a connection on a given $C^\infty$ complex line bundle $L$, and use the curvature form $\Theta$ to define

$$c_1(L) := \left[\frac{\sqrt{-1}}{2\pi}\Theta\right] \in H^2_{dR}(X;\mathbb{R}) \cong H^2(X,\mathbb{R}).$$

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- Since we have a natural homomorphism $\Phi : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$ using the sheaf morphism $\underline{\mathbb{Z}} \to \underline{\mathbb{R}}$. We shall explore the relation between $\Phi\big(\delta([L])\big) \in H^2(X,\mathbb{R})$ and $c_1(L) \in H^2_{dR}(X,\mathbb{R})$.

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- For simplicity, in the following we assume $L$ is a holomorphic line bundle with Hermitian metric $h$. We leave the necessary modification in the general complex line bundle case as an exercise.

# Computing $\Phi\big(\delta([L])\big)$

- First recall the construction of $\delta : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$. Let $L$ be a complex line bundle. We use sufficiently fine locally finite trivializations $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ such that each $U_\alpha$ is simply connected and $H^*(X, \mathscr{O}^*)$ is isomorphic to $H^*(\mathcal{U}, \mathscr{O}^*)$. Then $[L] \in H^1(X, \mathscr{O}^*)$ is determined by the Čech cocycle $\{\psi_{\alpha\beta}\}$, $\psi_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$. We define $\phi_{\alpha\beta} := \frac{1}{2\pi\sqrt{-1}} \log \psi_{\alpha\beta}$. Note that this is not a well-defined Čech cochain: *log* is a multi-valued function!

# Computing $\Phi\big(\delta([L])\big)$

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- However, since $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$, we get

$$z_{\alpha\beta\gamma} := \phi_{\alpha\beta} + \phi_{\beta\gamma} - \phi_{\alpha\gamma} \in \underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma).$$

  This defines a Čech cocycle, whose cohomology class defines $\delta([L])$. Then $\Phi\big(\delta([L])\big)$ is also defined by $\{z_{\alpha\beta\gamma}\}$, just viewing $\underline{\mathbb{Z}}$ as a subsheaf of $\underline{\mathbb{R}}$.

# A closer look at the de Rham isomorphism

To compare we first break the resolution $0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathscr{A}^1 \to \dots$ into short exact sequences:

$$0 \to \underline{\mathbb{R}} \to \mathscr{A}^0 \to \mathcal{K}_1 \to 0, \quad 0 \to \mathcal{K}_1 \to \mathscr{A}^1 \to \mathcal{K}_2 \to 0, \quad \dots$$

where $\mathcal{K}_i$ is the sheaf of closed $i$-forms. We get exact sequence for cohomology:

$$0 \to H^1(X, \mathcal{K}_1) \to H^2(X, \mathbb{R}) \to 0, \quad A^1(X) \to \mathcal{K}_2(X) \to H^1(X, \mathcal{K}_1) \to 0.$$

The first one gives $\delta_2 : H^1(X, \mathcal{K}_1) \cong H^2(X, \mathbb{R})$ and the second gives $\delta_1 : H^2_{dR}(X) \cong H^1(X, \mathcal{K}_1)$.
Our de Rham class is given by $\frac{\sqrt{-1}}{2\pi}\Theta(h) \in \mathcal{K}_2(X)$. Locally, we have $\Theta = d\theta_\alpha$, where $\theta_\alpha = \partial \log h_\alpha$, $h_\alpha = h(e_\alpha, e_\alpha)$. Then $\delta_1\left(\left[\frac{\sqrt{-1}}{2\pi}\Theta(h)\right]\right)$ is given by $[\{\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha)\}]$.

Now

$$e_\beta(p) = \varphi_\beta^{-1}(p,1) = \varphi_\alpha^{-1} \circ (\varphi_\alpha \circ \varphi_\beta^{-1})(p,1) = \varphi_\alpha^{-1}(p, \psi_{\alpha\beta}(p)) = \psi_{\alpha\beta}(p) e_\alpha(p).$$

So we get $h_\beta = h_\alpha |\psi_{\alpha\beta}|^2$, and hence $\log h_\beta = \log h_\alpha + \log|\psi_{\alpha\beta}|^2$. So on $U_\alpha \cap U_\beta$, we have

$$\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha) = \frac{\sqrt{-1}}{2\pi}\partial \log|\psi_{\alpha\beta}|^2 = \frac{\sqrt{-1}}{2\pi}\partial \log \psi_{\alpha\beta} = \frac{\sqrt{-1}}{2\pi} d \log \psi_{\alpha\beta}.$$

Then $\delta_2\Big([\{\frac{\sqrt{-1}}{2\pi}(\theta_\beta - \theta_\alpha)\}]\Big)$ is represented by

$$\left\{\frac{\sqrt{-1}}{2\pi}\Big(\log\psi_{\beta\gamma} - \log\psi_{\alpha\gamma} + \log\psi_{\alpha\beta}\Big)\right\}.$$

This is precisely $-\{z_{\alpha\beta\gamma}\}$.

# §3.4 Hermitian metrics and Kähler metrics

# Hermitian metric as a special Riemannian metric

Let $X$ be a complex manifold of dimension $n$. We denote the canonical almost complex structure by $J$. A Riemannian metric $g$ on $X$ is called "Hermitian", if $g$ is $J$-invariant, i.e.

$$g(Ju, Jv) = g(u, v), \quad \forall u, v \in T_x^{\mathbb{R}} X, \forall x \in X.$$

As before, we extend $g$ to $T^{\mathbb{C}} X$ as a complex bilinear form. For simplicity, we also denote this bilinear form by $g$. Then we have

$$g(T^{1,0}, T^{1,0}) = 0 = g(T^{0,1}, T^{0,1})$$

and $\langle Z, W \rangle := g(Z, \bar{W})$ defines an Hermitian metric on the rank $n$ holomorphic vector bundle $T^{1,0}X$. Conversely, any Hermitian metric on $T^{1,0}X$ determines uniquely a $J$-invariant Riemannian metric on $X$.

# Kähler form and Kähler metric

For an Hermitian metric $g$ on $(X, J)$, we define the associated Kähler form $\omega_g$ by

$$\omega_g(u, v) := g(Ju, v).$$

It is direct to check that $\omega_g$ is a real 2-form on $X$.

## Definition

An Hermitian metric $g$ on $X$ is called a Kähler metric, if $d\omega_g = 0$. Its cohomology class in $H^2_{dR}(X)$ is call the "Kähler class" of $g$. If a (compact) complex manifold admits a Kähler metric, we call it a "Kähler manifold".

# Local representation

Locally, if $(z_1, \ldots, z_n)$ is a holomorphic coordinate system, then $g$ is determined by $g_{i\bar{j}} := g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$, since $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Then we have $\omega_g = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, where Einstein's summation convention is always used. Now we have

$$
\begin{aligned}
0 = d\omega_g = & \sqrt{-1} dg_{i\bar{j}} dz_i \wedge d\bar{z}_j \\
= & \sqrt{-1} \frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j - \sqrt{-1} \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} dz_i \wedge d\bar{z}_l \wedge d\bar{z}_j \\
= & \sqrt{-1} \Big[ \sum_j \sum_{k<i} \Big( \frac{\partial g_{i\bar{j}}}{\partial z_k} - \frac{\partial g_{k\bar{j}}}{\partial z_i} \Big) dz_k \wedge dz_i \wedge d\bar{z}_j + \sum_i \sum_{j<l} \Big( \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} - \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j} \Big) dz_i \wedge d\bar{z}_j \wedge d\bar{z}_l \Big].
\end{aligned}
$$

So being Kähler mean that $g_{i\bar{j}}$ have the additional symmetries:

$$
\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} = \frac{\partial g_{i\bar{l}}}{\partial \bar{z}_j}, \quad \forall i, j, k, l.
$$

## Example

The Euclidean metric $g = \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i)$ is a Kähler metric, since we have $\omega_g = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$, which is obviously closed.

For more examples, note that to define a Kähler metrics, it suffices to define its associated Kähler form, since we have $g(u, v) = g(Ju, Jv) = \omega_g(u, Jv)$.

## Example

Let $X = B(1) \subset \mathbb{C}^n$ be the unit ball in $\mathbb{C}^n$. We define a Kähler metric:

$$\omega_g := \sqrt{-1}\partial\bar{\partial} \log \frac{1}{1 - |z|^2} = \sqrt{-1}\partial\Big(\frac{z_j d\bar{z}_j}{1 - |z|^2}\Big).$$

Here, we have $(g_{i\bar{j}}) = (\frac{\delta_{ij}}{1-|z|^2} + \frac{\bar{z}_i z_j}{(1-|z|^2)^2})$, which is positive definite. So it is indeed a Kähler metric. This is called the "complex hyperbolic metric".

## Example

Let $X = \mathbb{C}P^n$ with homogeneous coordinates $[Z_0, \dots, Z_n]$, we define a Kähler metric:

$$\omega_g := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|Z_0|^2 + \cdots + |Z_n|^2).$$

It is easy to check that this is well-defined and equals $-\frac{\sqrt{-1}}{2\pi}\Theta(h)$, where $h$ is the natural metric on the universal bundle. It is called the “Fubini-Study metric”.

Not every compact complex manifold is Kähler, since, for example, $H^2_{dR}(X)$ must be non-trivial. For if not, $\omega_g$ will be exact, so $\int_X \omega_g^n = 0$ by Stokes theorem. But this is impossible since $\int_X \omega_g^n > 0$. So Calabi-Eckmann manifolds are never Kähler.

# Submanifold of a Kähler manifold

## Lemma

*If $X$ is Kähler and $Y$ is a complex analytic submanifold of $X$, then $Y$ is also Kähler.*

## Proof (Outline)

*Let $g$ be a Kähler metric on $X$ and $\iota : Y \to X$ be the embedding map, then $\iota^* g$ is a Kähler metric on $Y$ and the associated Kähler form is just $\iota^* \omega_g$.*

By this lemma, all projective algebraic manifolds are Kähler.

# Kähler normal coordinates

In Riemannian geometry, normal coordinates are very useful in tensor calculations. The next lemma shows that being Kähler is both necessary and sufficient for the existence of complex analogue of normal coordinates.

## Lemma

*For an Hermitian metric $g$ on $X$, the follows two properties are equivalent:*

(1) *$g$ is Kähler;*

(2) *For any point $p \in X$, there are local holomorphic coordinates $(z_1, \ldots, z_n)$ such that $z_i(p) = 0$, $g_{i\bar{j}}(p) = \delta_{ij}$ and $dg_{i\bar{j}}(p) = 0$.*

## Proof

(2) $\Longrightarrow$ (1): *For any given point* $p$*, we choose the coordinate in (2), then since first order derivatives of* $g_{i\bar{j}}$ *at* $p$ *vanish, we will have* $d\omega_g(p) = 0$*. This implies* $d\omega_g = 0$*, i.e.,* $g$ *is Kähler.*

(1) $\Longrightarrow$ (2): *Suppose* $g$ *is Kähler. Given any point* $p \in X$*, we can first choose local holomorphic coordinates* $(w_1, \ldots, w_n)$ *such that* $w_i(p) = 0$ *and* $g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j})(p) = \delta_{ij}$*. We want to find holomorphic coordinate transformation of the form* $w_i = z_i + \frac{1}{2}a_{ijk}z_j z_k$ *with* $a_{ijk} = a_{ikj}$ *such that*

$$\omega_g = \sqrt{-1}(\delta_{ij} + O(|z|^2))dz_i \wedge d\bar{z}_j.$$

# Proof (continued)

*Direct computation shows that*

$$\begin{aligned}\omega_g &= \sqrt{-1}\Big(\delta_{ij} + g_{i\bar{j},k}(0)w_k + g_{i\bar{j},\bar{l}}(0)\bar{w}_l + O(|w|^2)\Big)dw_i \wedge d\bar{w}_j \\ &= \sqrt{-1}\Big(\delta_{ij} + g_{i\bar{j},k}(0)z_k + g_{i\bar{j},\bar{l}}(0)\bar{z}_l + O(|z|^2)\Big)(dz_i + a_{ipq}z_p dz_q) \wedge (d\bar{z}_j + \bar{a}_{jst}\bar{z}_s d\bar{z}_t) \\ &= \sqrt{-1}\Big(\delta_{ij}dz_i \wedge d\bar{z}_j + \bar{a}_{ilj}\bar{z}_l dz_i \wedge d\bar{z}_j + a_{jki}z_k dz_i \wedge d\bar{z}_j \\ &\quad + (g_{i\bar{j},k}(0)z_k + g_{i\bar{j},\bar{l}}(0)\bar{z}_l)dz_i \wedge d\bar{z}_j + O(|z|^2)\Big).\end{aligned}$$

*So the condition we need is* $a_{jki} + g_{i\bar{j},k}(0) = 0$ *and* $\bar{a}_{ilj} + g_{i\bar{j},\bar{l}}(0) = 0$*. So we simply take* $a_{jki} := -\frac{\partial g_{i\bar{j}}}{\partial w_k}(0)$*. The Kähler condition makes sure that this is well-defined.*

# A useful corollary

## Corollary

*For a Kähler manifold* $(X, J, g)$*, we always have* $\nabla J = 0$.

## Proof

*For any given point* $p \in X$*, we compute using Kähler normal coordinates in Lemma 28. Now in complex coordinates,* $J$ *has constant coefficients, this implies* $\nabla J$ *vanishes at* $p$*. Since* $p$ *is arbitrary, we have* $\nabla J = 0$.

By definition, this implies that $\nabla(JX) = J\nabla X$.

# Comparing Chern and Levi-Civita

Recall that for a connection $\nabla$ on a vector bundle $E$, we can define the covariant derivative of a section $s$ with respect to a tangent vector $v \in T_pX$ by setting $\nabla_v s := \nabla s(v)$. If $e_\alpha$ is a local frame of $E$, then we have $\nabla e_\alpha = \omega_\alpha^\beta e_\beta$, and $\nabla_v e_\alpha = \omega_\alpha^\beta(v) e_\beta$. Another good feature of the Kähler condition is that if we complexify the usual Levi-Civita connection, we will automatically get the Chern connection on $T^{1,0}X$.

## Proposition

*Let $(X, J, g)$ be a Kähler manifold. Then the complexification of the Levi-Civita connection restricts to the Chern connection on $T^{1,0}X$.*

# Proof

- *Since the Levi-Civita connection* $\nabla$ *preserves the metric, it suffices to check that* $\nabla$ *is also compatible with the complex structure. If we use local holomorphic coordinates* $z = (z_1, \ldots, z_n)$, *write* $\partial_i := \frac{\partial}{\partial z_i}$ *and* $\partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}_j}$. *We only need to prove that* $\nabla \partial_i$ *is a* $T^{\mathbb{C}}X$*-valued (1,0)-form, i.e.* $\nabla_{\bar{j}} \partial_i := \nabla_{\partial_{\bar{j}}} \partial_i = 0$.

# Proof

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- *Now since* $\nabla J = 0$*, we have* $J(\nabla_{\bar{j}} \partial_i) = \nabla_{\bar{j}}(J \partial_i) = \sqrt{-1} \nabla_{\bar{j}} \partial_i$*. This implies that* $\nabla_{\bar{j}} \partial_i$ *is of type (1,0).*

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- *Now since* $\nabla J = 0$, *we have* $J(\nabla_{\bar{j}} \partial_i) = \nabla_{\bar{j}}(J \partial_i) = \sqrt{-1} \nabla_{\bar{j}} \partial_i$. *This implies that* $\nabla_{\bar{j}} \partial_i$ *is of type (1,0).*
- *On the other hand, since* $\nabla$ *is torsion free, we have* $\nabla_{\bar{j}} \partial_i = \nabla_i \partial_{\bar{j}}$, *so we have*

$$J(\nabla_{\bar{j}} \partial_i) = J(\nabla_i \partial_{\bar{j}}) = \nabla_i (J \partial_{\bar{j}}) = -\sqrt{-1} \nabla_i \partial_{\bar{j}} = -\sqrt{-1} \nabla_{\bar{j}} \partial_i.$$

*This implies that* $\nabla_{\bar{j}} \partial_i$ *is of type (0,1). So we must have* $\nabla_{\bar{j}} \partial_i = 0$.

# Local formula for the Christoffel symbols

- We already get $\nabla_i \partial_{\bar{j}} = 0 = \nabla_{\bar{j}} \partial_i$. Now since $J(\nabla_i \partial_j) = \nabla_i (J\partial_j) = \sqrt{-1}\nabla_i \partial_j$, we can assume that $\nabla_i \partial_j = \Gamma_{ij}^k \partial_k$. Similarly, we can assume $\nabla_{\bar{i}} \partial_{\bar{j}} = \Gamma_{\bar{i}\bar{j}}^{\bar{k}} \partial_{\bar{k}}$.

# Local formula for the Christoffel symbols

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- But we also have $\nabla_{\bar{i}} \partial_{\bar{j}} = \overline{\nabla_i \partial_j} = \overline{\Gamma_{ij}^k} \partial_{\bar{k}}$, we get $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k}$. All the nontrivial informations are contained in $\Gamma_{ij}^k$.

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- Now we compute, using compatibility with metric to get

$$\partial_i g_{j\bar{l}} = g(\nabla_i \partial_j, \partial_{\bar{l}}) + g(\partial_j, \nabla_i \partial_{\bar{l}}) = \Gamma_{ij}^k g_{k\bar{l}},$$

which implies that $\Gamma_{ij}^k = g^{\bar{l}k} \frac{\partial g_{j\bar{l}}}{\partial z_i} = g^{\bar{l}k} \frac{\partial g_{i\bar{l}}}{\partial z_j}$.

# Special symmetries of curvature tensor

For curvature, we also extend the curvature tensor $\mathbb{C}$-linearly to the complexified tangent bundle. Then this curvature tensor automatically satisfies the Bianchi identities. The Kähler condition also implies that the curvature tensor has more symmetries, and hence has much simpler formula. For the curvature $R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$, we have $R(X,Y)JZ=JR(X,Y)Z$. Also, by symmetry of curvature tensor, we have

$$\langle R(JX,JY)Z,W\rangle=\langle R(Z,W)JX,JY\rangle=\langle R(Z,W)X,Y\rangle=\langle R(X,Y)Z,W\rangle.$$

Since $W$ is arbitrary, we also have $R(JX,JY)Z=R(X,Y)Z$. Moreover, we have:

## Proposition

*For the curvature tensor of the Kähler metric $g$, we have $\langle R(\partial_i,\partial_j)\cdot,\cdot\rangle=0$ $=\langle R(\partial_{\bar{i}},\partial_{\bar{j}})\cdot,\cdot\rangle$, and the only essentially non-trivial term is $R_{i\bar{j}k\bar{l}}:=\langle R(\partial_i,\partial_{\bar{j}})\partial_k,\partial_{\bar{l}}\rangle$ $=-\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\bar{z}_l}+g^{\bar{q}p}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial\bar{z}_l}$. In particular, $R_{i\bar{j}k\bar{l}}=R_{i\bar{l}k\bar{j}}=R_{k\bar{j}i\bar{l}}$.*

## Proof

*We compute by definition:*

$$
\begin{aligned}
R_{i\bar{j}k\bar{l}} &= \langle(\nabla_i\nabla_{\bar{j}} - \nabla_{\bar{j}}\nabla_i)\partial_k, \partial_{\bar{l}}\rangle = -\langle\nabla_{\bar{j}}(\Gamma^p_{ik}\partial_p), \partial_{\bar{l}}\rangle \\
&= -\partial_{\bar{j}}\Gamma^p_{ik}g_{p\bar{l}} = -\partial_{\bar{j}}(g^{\bar{q}p}\frac{\partial g_{k\bar{q}}}{\partial z_i})g_{p\bar{l}} \\
&= -g^{\bar{q}p}\frac{\partial^2 g_{k\bar{q}}}{\partial z_i \partial z_{\bar{j}}}g_{p\bar{l}} + g^{\bar{q}s}g^{\bar{t}p}\frac{\partial g_{s\bar{t}}}{\partial z_{\bar{j}}}\frac{\partial g_{k\bar{q}}}{\partial z_i}g_{p\bar{l}} \\
&= -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial z_{\bar{j}}} + g^{\bar{q}s}\frac{\partial g_{s\bar{l}}}{\partial z_{\bar{j}}}\frac{\partial g_{k\bar{q}}}{\partial z_i}.
\end{aligned}
$$

*The first conclusion follows by Kähler metric's special symmetry.*

# Special properties of Ricci

## Proposition

*The Ricci curvature $Rc$ of a Kähler metric is also $J$-invariant, and the 2-form $Ric(\omega_g) := Rc(J\cdot,\cdot)$ is called the Ricci form, and we have $Ric(\omega_g) = \sqrt{-1}R_{i\bar{j}}dz_i \wedge d\bar{z}_j$, with*

$$R_{i\bar{j}} = Rc(\partial_i, \partial_{\bar{j}}) = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\det(g_{p\bar{q}}).$$

# Proof

*We choose a local orthonormal frame* $\{e_i\}_{i=1}^{2n}$ *to compute:*

$$\begin{aligned} Rc(JX, JY) &= \sum_{i=1}^{2n} \langle R(JX, e_i)e_i, JY\rangle = \sum_{i=1}^{2n} \langle JR(JX, e_i)e_i, J^2 Y\rangle \\ &= -\sum_{i=1}^{2n} \langle R(JX, e_i)Je_i, Y\rangle = -\sum_{i=1}^{2n} \langle R(J^2 X, Je_i)Je_i, Y\rangle \\ &= \sum_{i=1}^{2n} \langle R(X, Je_i)Je_i, Y\rangle = Rc(X, Y), \end{aligned}$$

*since* $\{Je_i\}_{i=1}^{2n}$ *is also an orthonormal frame. As the computation for* $\omega_g$*, we easily get the formula*

$$Ric(\omega_g) = \sqrt{-1} R_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

# Proof (continued)

*Finally, we calculate* $R_{i\bar{j}}$*: Choose a local orthonormal frame of the form* $\{e_\alpha, Je_\alpha\}_{\alpha=1}^n$ *at one point, and write* $Z_\alpha := e_\alpha - \sqrt{-1}Je_\alpha$*. Then we have*

$$
\begin{aligned}
R_{i\bar{j}} = Rc(\partial_i, \partial_{\bar{j}}) &= \sum_\alpha \langle R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}\rangle + \sum_\alpha \langle R(\partial_i, Je_\alpha)Je_\alpha, \partial_{\bar{j}}\rangle \\
&= \sum_\alpha \langle R(\partial_i, e_\alpha)e_\alpha, \partial_{\bar{j}}\rangle + \sqrt{-1}\sum_\alpha \langle R(\partial_i, Je_\alpha)e_\alpha, \partial_{\bar{j}}\rangle \\
&= \sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)e_\alpha, \partial_{\bar{j}}\rangle \\
&= \frac{1}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)e_\alpha, \partial_{\bar{j}}\rangle - \frac{\sqrt{-1}}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)Je_\alpha, \partial_{\bar{j}}\rangle \\
&= \frac{1}{2}\sum_\alpha \langle R(\partial_i, \bar{Z}_\alpha)Z_\alpha, \partial_{\bar{j}}\rangle.
\end{aligned}
$$

# Proof (continued)

*On the other hand, we have* $Z_\alpha = a^\mu_\alpha \partial_\mu$ *and* $\partial_\mu = b^\alpha_\mu Z_\alpha$*, with* $a^\mu_\alpha b^\beta_\mu = \delta^\beta_\alpha$*, so at the given point, we have*

$$2\delta_{\alpha\beta} = g(Z_\alpha, \bar{Z}_\beta) = a^\mu_\alpha \bar{a}^\nu_\beta g_{\mu\bar{\nu}},$$

*which implies that* $g^{\bar{\beta}\alpha} = \frac{1}{2}\bar{a}^\beta_\mu a^\alpha_\mu$*, and so*

$$\begin{aligned} R_{i\bar{j}} &= \frac{1}{2}\bar{a}^\nu_\alpha a^\mu_\alpha R_{i\bar{\nu}\mu\bar{j}} = g^{\bar{l}k} R_{i\bar{l}k\bar{j}} = g^{\bar{l}k} R_{i\bar{j}k\bar{l}} \\ &= -g^{\bar{l}k}\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{\bar{l}k} g^{\bar{q}p} \frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j} = \frac{\partial}{\partial z_i}\Big( - g^{\bar{l}k}\frac{\partial g_{k\bar{l}}}{\partial \bar{z}_j}\Big) \\ &= -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{p\bar{q}}). \end{aligned}$$