§3. Geodesics, Variation formula

§3.1. Geodesics, 1st variation formula.

• **Def 1.** (Length of curve). Let $\gamma : [a,b] \to M$ be a diff. curve, define:

$$L(\gamma) = \int_a^b |\dot\gamma(t)|\, dt, \quad \dot\gamma(t) := d\gamma_t\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M.$$

• The length is invariant under reparameterization:

if $\eta : [c,d] \to [a,b]$ is a diffeomorphism, then:

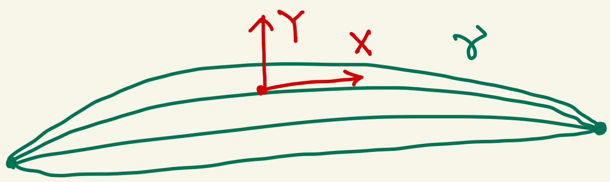
$$L(\gamma \circ \eta) = L(\gamma).$$

• **Def 2.** (Variation of curves).

• $\gamma : [0,\ell] \times (-\varepsilon, \varepsilon) \to M$, $\gamma_u(t) = \gamma(t,u)$, $u \in (-\varepsilon, \varepsilon)$, diff.

• $X = \dfrac{\partial \gamma}{\partial t} = \dfrac{\partial \gamma_u}{\partial t}$: tangent field of γ.

• $Y = \dfrac{\partial \gamma}{\partial u}$: variation field.

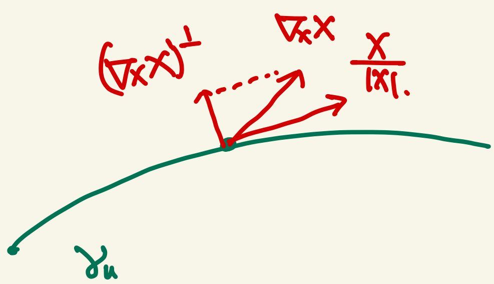


• **Prop. 3.** (1st variation formula). We have:

$$\frac{d}{du} L(\gamma_u) = \left\langle Y, \frac{X}{|X|} \right\rangle \Big|_{t=0}^{\ell} - \int_0^\ell \frac{1}{|X|} \cdot \left\langle Y, (\nabla_X X)^\perp \right\rangle dt.$$

where

$$(\nabla_X X)^\perp := \nabla_X X - \left\langle \nabla_X X, \frac{X}{|X|} \right\rangle \cdot \frac{X}{|X|}$$



is the normal part of $\nabla_X X$ along γ_u.

Proof: • Note that $[X, Y] = 0$, since:

$$[X, Y] = \left[d\gamma\left(\tfrac{\partial}{\partial t}\right), d\gamma\left(\tfrac{\partial}{\partial u}\right)\right] = d\gamma\left(\left[\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial u}\right]\right) = d\gamma(0) = 0.$$

For, $\forall f \in C^\infty(M)$,

$$[X, Y]f = X(Yf) - Y(Xf) = \frac{\partial^2(f\circ\gamma)}{\partial t \partial u} - \frac{\partial^2(f\circ\gamma)}{\partial u \partial t} = 0 .$$

• Then:

$$\frac{d}{du} L(\gamma_u) = \frac{d}{du} \int_0^\ell \langle X, X\rangle^{\frac{1}{2}} dt = \int_0^\ell \frac{\frac{\partial}{\partial u}\langle X, X\rangle}{2\langle X, X\rangle^{\frac{1}{2}}} dt.$$

$$= \int_0^\ell \frac{Y\langle X, X\rangle}{2|X|} dt = \int_0^\ell \frac{\langle \nabla_Y X, X\rangle}{|X|} dt = \int_0^\ell \frac{\langle \nabla_X Y, X\rangle}{|X|} dt.$$

• But:

$$\frac{\langle \nabla_X Y, X\rangle}{|X|} = \frac{X\langle X, Y\rangle - \langle Y, \nabla_X X\rangle}{|X|}$$

$$= \frac{\partial}{\partial t}\left(\frac{\langle X, Y\rangle}{|X|}\right) - \langle X, Y\rangle \cdot \frac{\partial}{\partial t}\frac{1}{|X|} - \frac{\langle Y, \nabla_X X\rangle}{|X|}$$

where:

$$\frac{\partial}{\partial t}\frac{1}{|X|} = X\left(|X|^2\right)^{-\frac{1}{2}} = -\frac{X\langle X, X\rangle}{2|X|^3} = -\frac{\langle \nabla_X X, X\rangle}{|X|^3}$$

hence:

$$\frac{d}{du} L(\gamma_u) = \left\langle Y, \frac{X}{|X|}\right\rangle\Big|_{t=0}^{\ell} + \int_0^\ell \left[\frac{\langle X, Y\rangle \cdot \langle \nabla_X X, X\rangle}{|X|^3} - \frac{\langle Y, \nabla_X X\rangle}{|X|}\right] dt.$$

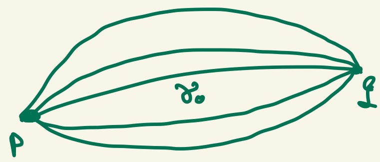
$$= \left\langle Y, \frac{X}{|X|}\right\rangle\Big|_{t=0}^{\ell} - \int_0^\ell \frac{1}{|X|}\cdot \left\langle Y, \nabla_X X - \left\langle \nabla_X X, \frac{X}{|X|}\right\rangle \cdot \frac{X}{|X|}\right\rangle dt.$$

$\boxed{///}$

Cor. 4. • Suppose γ_0 is of unit speed, i.e., $|\dot\gamma_0| = 1$.

(i). Assume $\gamma_u(0) \equiv p$, $\gamma_u(\ell) \equiv q$. Then:

$$\frac{d}{du}\Big|_{u=0} L(\gamma_u) = -\int_0^\ell \langle Y, \nabla_X X\rangle \, dt.$$



Hence, γ_0 is a critical point of any variation vanishing at endpoints, iff $\nabla_X X = 0$ along γ_0.

(ii). If $\nabla_X X = 0$ along γ_0, then:

$$\frac{d}{du}\Big|_{u=0} L(\gamma_u) = \langle X, Y\rangle(0,0) - \langle X, Y\rangle(\ell, 0).$$

• **Proof**: • From $|X|^2 = 1$, we have

$$0 = X|X|^2 = 2\langle \nabla_X X, X\rangle.$$

$$\Rightarrow (\nabla_X X)^\perp = \nabla_X X.$$

• If $\gamma_u(0) \equiv p$, $\gamma_u(\ell) \equiv q$, then:

$$Y(0,0) = 0 = Y(\ell, 0)$$

• If γ_0 is a critical point of any variation Y with $Y(0,0) = 0 = Y(\ell, 0)$,

$$0 = \frac{d}{du}\Big|_{u=0} L(\gamma_u) = -\int_0^\ell \langle Y, \nabla_X X\rangle \, dt$$

hence $\nabla_X X \equiv 0$. $\boxed{//}$

• A curve γ satisfying $\nabla_{\dot\gamma}\dot\gamma$ is called a *geodesic*.

- When γ is a geodesic, the speed must be constant:

$$\frac{d}{dt}|\dot\gamma(t)|^2 = \dot\gamma\langle\dot\gamma,\dot\gamma\rangle = 2\cdot\langle\nabla_{\dot\gamma}\dot\gamma,\dot\gamma\rangle = 0.$$

- $\nabla_{\dot\gamma}\dot\gamma$ is an ODE:
  - In local coord. $x=(x^1,\dots,x^n)$, write:

$$\gamma(t)=(\gamma^1(t),\dots,\gamma^n(t)),$$

then γ is a geodesic iff:

$$\frac{d^2}{dt^2}\gamma^k + \Gamma^k_{ij}\cdot\frac{d\gamma^i}{dt}\cdot\frac{d\gamma^j}{dt} \equiv 0, \quad \forall\, 1\le k\le n.$$

  - For any $p\in M$, $v\in T_pM$, $\exists!$ local geodesic $\gamma_v(t)$, $t\in(-\varepsilon,\varepsilon)$, s.t. $\gamma_v(0)=p$, $\dot\gamma_v(0)=v$.

- If $\gamma:[0,l]\to M$ is a geodesic, then $\forall\lambda>0$,

$$\tilde\gamma(t):=\gamma(\lambda t):\left[0,\frac{l}{\lambda}\right]\longrightarrow M \text{ is also a geodesic}$$

with $\dot{\tilde\gamma}(0)=\lambda\cdot\dot\gamma(0)$.

  - By uniqueness, $\gamma_{\lambda\cdot v}(t)=\gamma_v(\lambda\cdot t)$, $\forall v\in T_pM$, $\lambda>0$.

- Def 5 (Exponential map). $\forall p\in M$, $v\in T_pM$. let:

$$\exp_p(v):=\gamma_v(1), \quad \text{once the RHS exists.}$$

- ODE $\Rightarrow$ $\exp_p(v)$ can be defined whenever $|v|\ll 1$.

- If (M,g) is complete, then $\exp_p$ is defined on all T_pM.

- **Lemma 6**. $d(\exp_p)_0 = Id : T_pM \longrightarrow T_pM$.

Proof: · With identification, $\forall\, v \in T_pM$, choose $\sigma(t) = t\cdot v$, $t \in \mathbb{R}$, to be a curve in T_pM, with $\dot\sigma(0) = v$. Then:

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(\sigma(t)) = \frac{d}{dt}\Big|_{t=0} \exp_p(t\cdot v)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{tv}(1) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v.$$

$\blacksquare$

- Combining Lemma 6 and inverse function theorem, we have:

- **Corollary 7**: · (M,g): Riem. mfd, $p \in M$.

· **Then**, $\exists$ an open nbhd $\mathcal{V} \subset T_pM$ of $0 \in T_pM$, s.t. $\exp_p$ is defined on $\mathcal{V}$, and $\exp_p : \mathcal{V} \longrightarrow \exp_p(\mathcal{V}) \subset M$ is a diffeomorphism.

- Injectivity radius: $\forall\, p \in M$, define:

$$\mathrm{inj}_p := \sup\left\{ r>0 : \exp_p : \underbrace{B(0,r)}_{\subset T_pM} \longrightarrow M \text{ is a diffeomorphism onto its image} \right\}.$$

- Conjugate radius:

$$r_p := \sup\left\{ r>0 : \exp_p : B(0,r) \longrightarrow M \text{ non-degenerate everywhere} \right\}.$$

§3.2 Completeness, Hopf–Rinow theorem.

- Let (M^n, g) be a connected Riem. mfd. For any $p, q \in M$, define:

$$d(p,q) := \inf \left\{ L(\gamma) \,\middle|\, \gamma : [a,b] \to M \text{ smooth curve, with } \gamma(a) = p, \gamma(b) = q \right\}.$$

- d defines a metric on M:

  ① $d(p,q) \geq 0$. $d(p,q) = 0$ iff $p = q$.

  ② $d(p,q) = d(q,p)$.

  ③ $d(p_1, p_3) \leq d(p_1, p_2) + d(p_2, p_3)$, $\forall\, p_1, p_2, p_3 \in M$.

- The metric topology coincides with the original topology.

- **Def 8.** (complete Riem. mfd). A Riem. mfd (M, g) is called complete, if it is complete as a metric space (Any Cauchy sequence has a limit point).

- **Theorem 9.** (M, g): Riem. mfd, $p \in M$.
  - Assume $\exp_p$ is defined over all T_pM.
  - Then: $\forall\, q \in M$, $\exists$ minimal geodesic γ connecting p to q.

Proof: • First, we need:

- **Lemma 10.** (Gauss Lemma). • (M^n, g): Riem. mfd. $p \in M$.

  (i). If $v \in T_pM$, $\exp_p$ is defined at v, then: $\forall\, w \in T_pM$,

$$\left\langle d(\exp_p)_v w,\ d(\exp_p)_v v \right\rangle_{g|_{\exp_p(v)}} = \langle w, v \rangle_{g|_p}.$$

(ii). $\forall p \in M$, $\forall\, 0 < r < \mathrm{inj}_p$, whenever $|v| < r$, $\exp_p(v)$ is well-defined, and $\gamma_v(t) := \exp_p(t\cdot v)$ is the unique minimal geodesic joining p and $\exp_p(v)$. In particular, $d(p, \exp_p(v)) = |v|$, $\forall\, |v| < r$.

- By Cor. 7 and Lemma 10 (ii), we can choose $\delta \in (0, \mathrm{inj}_p)$ small enough, s.t. $\exp_p : \underset{\subseteq T_pM}{B(0, 2\delta)} \longrightarrow B(p, 2\delta)$ is a diffeomorphism, and every point $q \in B(p, 2\delta)$ can be joined by an unique minimal geodesic from p. (If $q = \exp_p(v)$, then $\gamma(t) = \exp_p(t\cdot v)$, $0 \le t \le 1$).

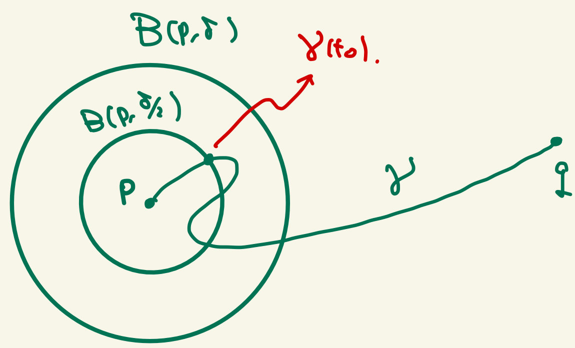
- Now, assume $q \notin B(p, 2\delta)$.

- Lemma 11: $\exists\, q_0 \in \partial B(p, \delta)$, such that:

$$d(p, q) = \delta + d(q_0, q).$$

Proof: • Let $\gamma : [0, 1] \to M$ be any smooth curve with:

$$\gamma(0) = p, \quad \gamma(1) = q.$$



- Since $q \notin B(p, 2\delta)$,

$$\gamma([0,1]) \cap \partial B(p, \delta) \neq \phi.$$

- Find minimal $t_0 \in (0, 1]$, s.t. $\gamma(t_0) \in \partial B(p, \delta)$. Then:

$$L(\gamma) = L(\gamma|_{[0, t_0]}) + L(\gamma|_{[t_0, 1]})$$

$$\ge \delta + d(\gamma(t_0), q) \ge \delta + \inf\{d(q', q) : q' \in \partial B(p, \delta)\}$$

Take infimum over all such γ, we have:

$$d(p,q) \geq \delta + \inf\{d(q',q) : q' \in \partial B(p,\delta)\}.$$

- On the other hand, $\forall\, q' \in \partial B(p,\delta)$,

$$d(p,q) \leq d(p,q') + d(q',q) \leq \delta + d(q',q).$$

Take infimum over q' gives:

$$d(p,q) \leq \delta + \inf\{d(q',q) : q' \in \partial B(p,\delta)\}.$$

- Hence

$$d(p,q) = \delta + \inf\{d(q',q) : q' \in \partial B(p,\delta)\}.$$

The lemma follows since $\partial B(p,\delta)$ is a compact subset. $\boxed{///}$

- By Lemma 11, find $q_0 \in \partial B(p,\delta)$, s.t.

$$d(p,q) = \delta + d(q_0,q).$$

- Since $q_0 \in B(p,2\delta)$, $\exists!\ \overset{\#}{v_0} \in T_pM$, s.t. $q_0 = \exp_p(v_0)$.

- Let $v := \dfrac{v_0}{|v_0|}$, then $v \in S_pM$. Since we assume $\exp_p$ is defined on whole T_pM, $\forall\, t \in [0,+\infty)$, $\exp_p(t \cdot v)$ is well-defined. Hence:

$$\gamma(t) := \exp_p(t \cdot v), \quad t \in [0,\infty)$$

is a well-defined geodesic of unit speed.

Γ • unit speed: By Gauss Lemma (i),

$$|\dot{\gamma}(t)|^2 = \left| d(\exp_p)_{tv}(v) \right|^2 = |v|^2 = 1. \quad \lrcorner$$

• Let $l := d(p, q)$. We want to prove: $\gamma(l) = q$. (then $\gamma|_{[0,l]}$ is a minimal geodesic from p to q).

• Consider the set $A := \{ t \in [0, l] : d(\gamma(t), q) = l - t \}$.

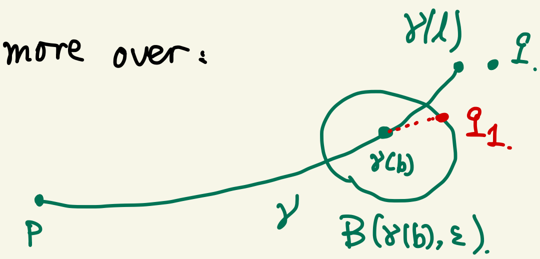
• First, $0 \in A$, and $\gamma(l) = q \iff l \in A$.

• By continuity of distance function d and γ, $A \subset [0, l]$ is a closed subset. Hence:

$$b := \sup A \in A.$$

• If $b \neq l$, then $b < l$. Again, we can choose $0 < \varepsilon < l - b$, such that Cor. 7 and Lemma 10.(ii) holds, and moreover:

$$q \notin B(\gamma(b), \varepsilon).$$



• From the definition of $b \in A$ and Lemma 11, $\exists\, q_1 \in \partial B(\gamma(b), \varepsilon)$, s.t.

(1). $$l - b = d(\gamma(b), q) = \varepsilon + d(q_1, q).$$

• Claim: $q_1 = \gamma(b + \varepsilon)$.

• First, by triangle inequality,

$d(p,q_1) \geq d(p,q) - d(q_1,q) = \ell - (\ell - b - \varepsilon) = b+\varepsilon.$

- Next, let γ_1 be a minimal geodesic from $\gamma(b)$ to q_1, then

$\tilde{\gamma} := \gamma|_{[0,b]} * \gamma_1$ is a piecewise smooth curve of length:

$$b+\varepsilon = L(\gamma|_{[0,b]}) + L(\gamma_1) = L(\tilde{\gamma}) \geq d(p,q_1) \geq b+\varepsilon.$$

$$\Rightarrow L(\tilde{\gamma}) = d(p,q_1) = b+\varepsilon.$$

- Hence $\tilde{\gamma}$ is a minimal geodesic from p to q_1, hence it's smooth.

By the uniqueness of geodesic, $\tilde{\gamma} = \gamma|_{[0,b+\varepsilon]}$, hence:

$$\gamma(b+\varepsilon) = \tilde{\gamma}(b+\varepsilon) = \gamma_1(\varepsilon) = q_1. \quad \Rightarrow \text{Claim.}$$

$$\Rightarrow d(\gamma(b+\varepsilon), q) = d(q_1, q) \overset{(1).}{=} \ell - b - \varepsilon = \ell - (b+\varepsilon).$$

- By the def. of the set A, $b+\varepsilon \in A$, contradicts to the maximality of b.

- Hence $b = \ell$, hence $\ell \in A$. This proves the thm. $\blacksquare$

• Theorem 12 (Hopf – Rinow).

- (M,g): Riem. mfd. The followings are equivalent:

(a). $\forall p \in M$, $\exp_p$ is defined on T_pM.

(b). $\exists p \in M$, s.t. $\exp_p$ is defined on T_pM.

(c). Every bounded closed subset of (M, d) is a compact subset.

(d). (M, g) is complete.

Proof: • (a) $\Rightarrow$ (b) is clear.

• (b) $\Rightarrow$ (c): Let A be a bounded closed subset of (M, d).

• Then, $\exists$ radius $R < +\infty$, s.t. $A \subset B(p, R)$.

• By condition (b) and Thm 9, $\forall q \in B(p,R)$, $\exists$ minimal geodesic γ on M, connecting p to q. Hence, $\exists\, v \in S_pM$, s.t.

$$q = \exp_p(l \cdot v), \qquad l = d(p, q) < R.$$

Since $l \cdot v \in B(0, R) \subseteq T_pM$, this means:

$$B(p,R) \subseteq \exp_p\left(B(0,R)\right) \subseteq \exp_p\left(\overline{B(0,R)}\right).$$

• Hence $A \subseteq \exp_p\left(\overline{B(0,R)}\right)$. But $\exp_p$ is a continuous map, $\overline{B(0,R)} \subset T_pM$ is compact, hence $\exp_p\left(\overline{B(0,R)}\right)$ is compact. Since A is closed, hence A is compact too.

• (c) $\Rightarrow$ (d): • Let $\{p_i\}$ be a Cauchy sequence in (M, d), i.e., $\forall \varepsilon > 0$, $\exists N < +\infty$, s.t. $\forall i, j > N$,

$$d(p_i, p_j) < \varepsilon.$$

• If $\{p_i\}$ has an accumulation point p_∞, i.e., $\exists$ subsequence p_{i_k}, s.t. $\lim\limits_{k\to\infty} d(p_{i_k}, p_\infty) = 0$, then $\lim\limits_{i\to\infty} d(p_i, p_\infty) = 0$.

- Assume $\{p_i\}$ does not have accumulation point. Then $\{p_i\}_{i=1}^{\infty} \subset M$ is a bounded closed subset.

- We may assume WLOG that all $\{p_i\}$ are mutually different.

- For each i, $\exists$ an open nbhd U_i of p_i, s.t.

$$U_k \cap \{p_i\}_{i=1}^{\infty} = \{p_k\}.$$

- Hence we have an open cover of $\{p_i\}_{i=1}^{\infty}$:

$$\mathcal{U} = \{U_i : 1 \leq i < \infty\}.$$

But according to condition (c), $\{p_i\}_{i=1}^{\infty} \subseteq M$ is compact, hence $\mathcal{U}$ must have a finite cover of $\{p_i\}_{i=1}^{\infty}$. Contradiction.

(d) $\Rightarrow$ (a): • Assume $\exists\, p \in M$, $v \in S_pM$, $0 < l < +\infty$, s.t. $\gamma(t) = \exp_p(t \cdot v)$ is defined on $[0, l)$, but not on l.

- Let $t_i := l - \frac{1}{i}$, $t_i \uparrow l$. Since γ is of unit speed,

$$d(\gamma(t_i), \gamma(t_j)) \leq L(\gamma(t_i), \gamma(t_j)) = t_i - t_j < \varepsilon,$$

whenever $i \geq j$ large enough, s.t. $0 \leq t_i - t_j < \varepsilon$.

- Hence $\{\gamma(t_i)\}$ is a Cauchy sequence of (M, d), by condition (d), (M, d) is complete, hence $\{\gamma(t_i)\}$ has a limit point $q \in M$.

- $\forall\, t \in [0, \ell)$, $\exists\, N$ large enough, s.t. when $i \geq N$, $t_i \in (t, \ell)$, and:

$$d(\gamma(t), q) \leq d(\gamma(t), \gamma(t_i)) + d(\gamma(t_i), q)$$

$$\leq \ell - t + d(\gamma(t_i), q).$$

- Letting $i \to \infty$, we obtain

$$d(\gamma(t), q) \leq \ell - t \implies \lim_{t \to \ell} \gamma(t) = q.$$

- Hence we can extend γ to $t = \ell$, contradicts to the def. of ℓ. $\boxed{///}$

- The following is corollary of the Hopf–Rinow thm.

- Cor. 13:

(a). On a complete Riem. mfd, every two points can be joined by a minimizing geodesic.

(b). Compact connected Riem. mfd must be complete.

(c). Isometry will keep the completeness of a Riem. mfd.