Introduction to Complex Geometry

Chapter 2 Sheaf Theory

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Outline

§2.1 Presheaves and sheaves

Presheaf

- A presheaf $\mathscr{F}$ of abelian groups over a topological space X is a rule assigning an abelian group $\mathscr{F}(U)$ for each open set $U \subset X$, and for each pair $V \subset U$ a homomorphism $r_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ (called "restriction homomorphism"), satisfying $r_U^U = id$ and for any $W \subset V \subset U$, we have $r_W^U = r_W^V \circ r_V^U$.

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- An element of $\mathscr{F}(U)$ is usually called a "section" of $\mathscr{F}$ over U. We also defined the stalk of $\mathscr{F}$ at a point $p \in X$ to be

$$\mathscr{F}_p := \varinjlim \mathscr{F}(U),$$

where the direct limit is taken with respect to open sets $p \in U$. This is $\coprod_{U \ni p} \mathscr{F}(U)/\sim$, with $s \in \mathscr{F}(U)$ equivalent to $t \in \mathscr{F}(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $r_W^U(s) = r_W^V(t)$. The image of $s \in \mathscr{F}(U)$ in $\mathscr{F}_p$ is denoted by s_p.

Caution

When the elements of $\mathscr{F}(U)$ are functions and the restriction homomorphisms are indeed restrictions, we need to be careful with stalks and germs: $s_p = t_p$ does not mean $s(p) = t(p)$! Instead, it is a much stronger condition, means that we can find a neighborhood V of p such that $s|_V \equiv t|_V$.

Morphism between presheaves

By a morphism f between two presheaves $\mathscr{F}$ and $\mathscr{G}$ over X, we mean for each U open, we are given a homomorphism of abelian groups $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$, such that whenever we have open sets $V \subset U$, we have a commutative diagram:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "f_U"] \arrow[d, "r_V^U"'] & \mathscr{G}(U) \arrow[d, "\rho_V^U"] \\
\mathscr{F}(V) \arrow[r, "f_V"'] & \mathscr{G}(V).
\end{tikzcd}$$

Sheaf

Definition

A presheaf of abelian groups $\mathscr{F}$ over X is called a sheaf, if it satisfies the following two properties:

(S1) Assume we have a family of open sets $U_i \subset U, i \in \Lambda$ and $\cup_i U_i = U$. If $s \in \mathscr{F}(U)$ satisfies $r^U_{U_i}(s) = 0, \forall i \in \Lambda$, then $s = 0$.

(S2) Assume we have a family of open sets $U_i \subset U, i \in \Lambda$ and $\cup_i U_i = U$. If we also have a family of sections $s_i \in \mathscr{F}(U_i), \forall i \in \Lambda$, satisfying $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$ whenever $U_i \cap U_j \neq \emptyset$, then there is a section $s \in \mathscr{F}(U)$ such that $r^U_{U_i}(s) = s_i, \forall i \in \Lambda$.

A morphism between two sheaves is just a morphism between presheaves.

Note that by (S1), the section in (S2) is also unique.

Example

- Let X be a complex manifold, then $\mathscr{O}_X$ is a sheaf of commutative rings over X. We call it the "structure sheaf" of X.
- We can also define other sheaves on X. For example, define $\mathscr{E}(U) := C^\infty(U; \mathbb{C})$, then it is easy to see that $\mathscr{E}$ is a sheaf, called the "sheaf of smooth functions". Similarly, we can define the sheaf of continuous functions on X.

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- If $E \to X$ is a holomorphic vector bundle, then $\mathscr{O}(E)(U)$ defines a sheaf of abelian groups. It can also be viewed as a sheaf of $\mathscr{O}_X$-modules. Similarly, we can define the sheaf of C^∞ sections $\mathscr{E}(E)$.

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- For $X = \mathbb{C}$, if we define $\mathscr{O}_b(U)$ to be the set of bounded holomorphic functions on $U \subset X$, then $\mathscr{O}_b$ is a presheaf over $\mathbb{C}$, but not a sheaf.
- Let G be a given abelian group, we define the constant presheaf over X to be $\underline{G}_{pre}(U) := G$ for any non-empty open set $U \subset X$, and $r_V^U = id$ for any non-empty pair $V \subset U$. Then it is in general not a sheaf.

Sheafification

Proposition

For any presheaf $\mathscr{F}$ over X, there is a unique (up to isomorphism) sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ satisfying the following "universal property": for any sheaf $\mathscr{G}$ over X and any morphism of presheaves $f : \mathscr{F} \to \mathscr{G}$, there is a unique morphism of sheaves $f^+ : \mathscr{F}^+ \to \mathscr{G}$ such that $f = f^+ \circ \theta$. If $\mathscr{F}$ is already a sheaf, then θ is an isomorphism. $\mathscr{F}^+$ is called the "sheafification" of $\mathscr{F}$. (By the universal property, if it exists, then must be unique up to isomorphisms.)

The most direct proof is to define $\mathscr{F}^+(U)$ explicitely: a map $\tilde{s} : U \to \coprod_{p\in U} \mathscr{F}_p$ is an element of $\mathscr{F}^+(U)$ if and only if:

1. $\pi \circ \tilde{s} = id_U$, i.e. $\tilde{s}(p) \in \mathscr{F}_p, \forall p \in U$;
2. For any $p \in U$, there is an open neighborhood $p \in V \subset U$ and a $s \in \mathscr{F}(V)$ such that for any $q \in V$, $\tilde{s}(q)$ equals s_q, the germ of s at q.

One can check that $\mathscr{F}^+$ is the sheafification of $\mathscr{F}$.

“étalé space” appraoch

From $\mathscr{F}$, we define a topological space, called the “étalé space” associated to $\mathscr{F}$:

$$\tilde{\mathscr{F}} := \coprod_{p \in X} \mathscr{F}_p.$$

We have a natural surjective projection map $\pi : \tilde{\mathscr{F}} \to X$. The topology on $\tilde{\mathscr{F}}$ is given as follows: If $s \in \mathscr{F}(U)$, then we have a natural map $\tilde{s} : U \to \tilde{\mathscr{F}}$, sending p to the germ of s at p, which is an element of $\mathscr{F}_p$. Then we require $\{\tilde{s}(U) | \, s \in \mathscr{F}(U), \forall U\}$ to be a topological basis for $\tilde{\mathscr{F}}$.
For any open $U \subset X$, define $\mathscr{F}^+(U) := \{s : U \to \tilde{\mathscr{F}} \text{ continuous} \big| \, \pi \circ s = id_U\}$.
The morphism θ is defined by $\theta_U : \mathscr{F}(U) \to \mathscr{F}^+(U)$, $\theta_U(s) := \tilde{s}$.

Example

- For the presheaf $\mathscr{O}_b$ of bounded holomorphic functions, its stalk at p $\mathscr{O}_{b,p}$ is isomorphic to the ring of convergent power series $\mathbb{C}\{z\}$ (i.e. power series with a positive convergent radius). Then it is easy to see that its sheafification is the sheaf of holomorphic functions $\mathscr{O}$.

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- For the constant presheaf $\underline{G}_{pre}$ over a manifold X, denote its sheafification by $\underline{G}$. Then the elements of $\underline{G}(U)$ consists of locally constant maps from U to the abelian group G. $\underline{G}$ is called "constant sheaf".

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- For the constant presheaf $\underline{G}_{pre}$ over a manifold X, denote its sheafification by $\underline{G}$. Then the elements of $\underline{G}(U)$ consists of locally constant maps from U to the abelian group G. $\underline{G}$ is called "constant sheaf".
- Let X be a complex manifold, we define a presheaf $\mathscr{M}_{pre}$ over X as follows: for open set $U \subset X$, elements of $\mathscr{M}_{pre}(U)$ are quotients of holomorphic functions on U, with denominator not identically zero on any connected component of U. Its sheafification $\mathscr{M}$ is the sheaf of meromorphic functions. Elements of $\mathscr{M}(U)$ are called meromorphic functions on U.

More about meromorphic functions

- In dim 1 case (Riemann surface), a meromorphic function= a holomorphic map to $\hat{\mathbb{C}} = \mathbb{C}P^1$. However, when dim$\geq 2$, there are meromorphic functions that can not be viewed as holomorphic maps to $\mathbb{C}P^1$, e.g., $\frac{z_1}{z_2} \in \mathscr{M}(\mathbb{C}^2)$.

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- Let X be a compact complex manifold of dimension n, then $\mathscr{M}(X)$ is an extension field of $\mathbb{C}$. Its transcendental degree over $\mathbb{C}$, $a(X)$, is called the "algebraic dimension" of X. It is known (Serre, Siegel, Chow, Thimm...) that $a(X) \leq n$.

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- When X is projective algebraic, then $a(X) = n$. When $n = 2$, the converse is also true (Chow-Kodaira).
- There are compact complex manifolds such that there are no non-constant meromorphic functions. For example, the Hopf surface $X := \mathbb{C}^2 \setminus \{0\}/\sim$, where we identify (z_1, z_2) with $(\alpha_1 z_1, \alpha_2 z_2)$ where α_i are constants and $|\alpha_i| > 1$. Then X is diffeomorphic to $S^3 \times S^1$. For generic α_i, X has no non-constant meromorphic functions.

§2.2 Sheaf cohomology (Čech's approch)

We always assume X is a manifold and $\mathscr{F}$ is a sheaf of abelian groups.

Motivation: the Mittag-Leffler problem

Sheaf is a useful tool to describe the obstructions to solve global problems when we can always solve a local one.

To illustrate this point, we come back to the Mittag-Leffler problem on a Riemann surface M. Suppose we are given finitely many points $p_1, \ldots, p_m \in M$, and for each p_i we are given a Laurant polynomial $\sum_{k=1}^{n_i} \frac{c_k^{(i)}}{z^k}$. We can view this as an element of $\mathscr{M}_p/\mathscr{O}_p$. We want to find a meromorphic function on M whose poles are precisely those p_i's with the given Laurant polynomial as its principal part at p_i.

Čech cocycles in the Mittag-Leffler problem

- This problem is always solvable locally: we can find a locally finite open covering $\mathcal{U} = \{U_i | \ i \in \Lambda\}$ of M such that each U_i contains at most one of the p_i's, and $f_i \in \mathscr{M}(U_i)$ such that the only poles of f_i are those of $\{p_i\}$ contained in U_i with principal part equals the given Laurent polynomial.

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- The problem is that we can not patch them together: if $U_i \cap U_j \neq \emptyset$, there is no reason to have $f_i = f_j$. We have to define $f_{ij} := f_i - f_j$ and view the totality of these f_{ij}'s as the obstruction to solve the problem.

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- Now by our choice of f_i, $f_{ij} \in \mathscr{O}(U_i \cap U_j)$. Note that we have $f_{ij} + f_{ji} = 0$ on $U_i \cap U_j$ and whenever $U_i \cap U_j \cap U_k \neq \emptyset$, we have on $U_i \cap U_j \cap U_k$: $f_{ij} + f_{jk} + f_{ki} = 0$. We call this the "cocycle" condition and $\{f_{ij}\}$ is a "Čech cocycle" for the sheaf $\mathscr{O}$ with respect to the cover $\mathcal{U}$.

True obstruction to the Mittag-Leffler problem

- When can we solve the Mittag-Leffler problem on M? We can solve it if we can modify the f_i by a holomorphic function $h_i \in \mathscr{O}(U_i)$ such that $\tilde{f}_i := f_i - h_i$ will patch together.

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- This means that $\tilde{f}_i = \tilde{f}_j$ on $U_i \cap U_j$, equivalently, $f_{ij} = h_i - h_j$.

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- This means that $\tilde{f}_i = \tilde{f}_j$ on $U_i \cap U_j$, equivalently, $f_{ij} = h_i - h_j$.
- We call a cocycle of the form $\{h_i - h_j\}$ (where each h_i is holomorphic) a Čech coboundary. We get the conclusion that we can solve the Mittag-Leffler problem if the Čech cocycle $\{f_{ij}\}$ is a coboundary.

Definition of Čech cohomology: the space of cochains

This motivates the introduction of the following Čech cohomology of a sheaf $\mathscr{F}$ with respect to a locally finite cover $\mathcal{U}$ of X:We first define the chain groups:

$$
\begin{aligned}
C^0(\mathcal{U}, \mathscr{F}) &:= \prod_{i \in \Lambda} \mathscr{F}(U_i) \\
C^1(\mathcal{U}, \mathscr{F}) &\subset \prod_{(i,j) \in \Lambda^2} \mathscr{F}(U_i \cap U_j) \\
&\cdots \\
C^p(\mathcal{U}, \mathscr{F}) &\subset \prod_{(i_0, i_1, \ldots, i_p) \in \Lambda^{p+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \\
&\cdots
\end{aligned}
$$

where $\{\sigma_{i_0,\ldots,i_p}\}$ is in $C^p(\mathcal{U}, \mathscr{F})$ if and only if:

(1) Whenever $i_k = i_l$ for some $k \neq l$, we have $\sigma_{i_0,\ldots,i_p} = 0$;

(2) For any permutation $\tau \in S_{p+1}$, we have $\sigma_{i_{\tau(0)},\ldots,i_{\tau(p)}} = (-1)^{\tau} \sigma_{i_0,\ldots,i_p}$.

Note that we always define $\mathscr{F}(U) = \{0\}$ if $U = \emptyset$.

Definition of Čech cohomology: the coboundary maps

We define the coboundary operator $\delta : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ to be:

$$(\delta\sigma)_{i_0,\dots,i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\dots,\hat{i_j},\dots,i_{p+1}}|_{U_{i_0}\cap\dots\cap U_{i_{p+1}}}.$$

Here we use $\dots|_{\dots}$ to denote the restriction homomorphism of $\mathscr{F}$. It is direct to check that $\delta \circ \delta = 0$. So we have a cochain complex

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \dots \xrightarrow{\delta} C^p(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \dots$$

We can define the space of Čech p-cocycles

$$Z^p(\mathcal{U}, \mathscr{F}) = Ker\delta \subset C^p(\mathcal{U}, \mathscr{F}),$$

and the space of Čech p-coboundaries

$$B^p(\mathcal{U}, \mathscr{F}) = \delta C^{p-1}(\mathcal{U}, \mathscr{F}) \subset Z^p(\mathcal{U}, \mathscr{F}).$$

We define the Čech cohomology wrt $\mathcal{U}$ to be $H^p(\mathcal{U}, \mathscr{F}) := Z^p(\mathcal{U}, \mathscr{F})/B^p(\mathcal{U}, \mathscr{F})$.

Meanings of H^0, H^1

- An element of $H^0(\mathcal{U}, \mathscr{F})$ is given by a family of sections $f_i \in \mathscr{F}(U_i)$ such that $\delta\{f_i\} = 0$. This means precisely $r^{U_i}_{U_i \cap U_j}(f_i) = r^{U_j}_{U_i \cap U_j}(f_j)$ whenever $U_i \cap U_j \neq \emptyset$. By sheaf axiom (S2), we get a global section of $\mathscr{F}$ over X. So $H^0(\mathcal{U}, \mathscr{F})$ is in fact independent of $\mathcal{U}$ and we have a canonical isomorphism

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- When $p = 1$, $\{f_{ij}\} \in C^p(\mathcal{U}, \mathscr{F})$ is a cocycle if $f_{ij} + f_{ji} = 0$ and $f_{jk} - f_{ik} + f_{ij} = f_{ij} + f_{jk} + f_{ki} = 0$. This is precisely the "cocycle condition" we met before (Note that we use "+" for the group operation instead of "$\cdot$".). However, this time the cohomology may depend on the cover.

Refining the cover

Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Gamma}$ be a locally finite refinement of $\mathcal{U}$. This means we have a map $\tau : \Gamma \to \Lambda$ (not unique) such that $V_\alpha \subset U_{\tau(\alpha)}$. Then we have a homomorphism $\Phi^{\mathcal{U}}_{\mathcal{V}} : H^p(\mathcal{U}, \mathscr{F}) \to H^p(\mathcal{V}, \mathscr{F})$ induced by

$$\{\sigma_{i_0,\dots,i_p}\} \mapsto \{\sigma_{\tau(\alpha_0),\dots,\tau(\alpha_p)}|_{V_{\alpha_0}\cap\cdots\cap V_{\alpha_p}}\}.$$

One can prove that $\Phi^{\mathcal{U}}_{\mathcal{V}}$ is in fact independent of the choice of the map τ.

Čech cohomology of X

The cohomology of X with coefficients sheaf $\mathscr{F}$ is defined to be the direct limit:

$$H^p(X,\mathscr{F}) := \varinjlim H^p(\mathcal{U},\mathscr{F}) = \coprod_{\mathcal{U}} H^p(\mathcal{U},\mathscr{F})/\sim$$

where two cohomology classes $[\{\sigma_{i_0,\ldots,i_p}\}] \in H^p(\mathcal{U},\mathscr{F})$ and $[\{\eta_{j_0,\ldots,j_p}\}] \in H^p(\mathcal{V},\mathscr{F})$ are equivalent if we can find a common refinement $\mathcal{W}$ of $\mathcal{U},\mathcal{V}$ such that

$$\Phi^{\mathcal{U}}_{\mathcal{W}}([\{\sigma_{i_0,\ldots,i_p}\}]) = \Phi^{\mathcal{V}}_{\mathcal{W}}([\{\eta_{j_0,\ldots,j_p}\}]).$$

Thus an element of $H^p(X,\mathscr{F})$ is an equivalent class of Čech cohomology classes, represented by an element of $H^p(\mathcal{U},\mathscr{F})$, for some cover $\mathcal{U}$. But in many cases, in particular all the sheaves we use in this course, there exists sufficiently fine cover $\mathcal{U}$ such that $H^p(\mathcal{U},\mathscr{F}) \cong H^p(X,\mathscr{F})$.

A detailed study of $H^1(\mathcal{U}, \mathscr{F})$ and $H^1(X, \mathscr{F})$

Proposition

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\Phi^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U}, \mathscr{F}) \to H^1(\mathcal{V}, \mathscr{F})$ is injective, and hence so is the induced homomorphism $H^1(\mathcal{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$. We can simply write $H^1(X, \mathscr{F}) = \cup_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{F})$.

Proof

Let $\mathcal{U} = \{U_i\}_{i \in \Gamma}$, $\mathcal{V} = \{V_\alpha\}_{\alpha \in \Lambda}$ and $\tau : \Gamma \to \Lambda$ be a map such that $U_i \subset V_{\tau(i)}$. Consider a common refinement of $\mathcal{U}, \mathcal{V}$, $\mathcal{W} := \{W_{i\alpha} := U_i \cap V_\alpha \neq \emptyset \mid i \in \Gamma, \alpha \in \Lambda\}$. Suppose $[\{f_{ij}\}] \in H^1(\mathcal{U}, \mathscr{F})$ satisfies $\Phi^{\mathcal{U}}_{\mathcal{V}}\big([\{f_{ij}\}]\big) = 0$. Then we also have $\Phi^{\mathcal{U}}_{\mathcal{W}}\big([\{f_{ij}\}]\big) = 0$. This mean that $\{f_{ij}\}$ is a cocycle and $\{f_{ij}|_{W_{i\alpha} \cap W_{j\beta}}\}$ is a coboundary. So we can find $h_{i\alpha} \in \mathscr{F}(W_{i\alpha})$ such that on $W_{i\alpha} \cap W_{j\beta}$, we have

$$f_{ij}|_{W_{i\alpha} \cap W_{j\beta}} = h_{j\beta} - h_{i\alpha}.$$

Proof (continued)

Since $f_{ii} = 0$, *we must have* $0 = h_{i\alpha}|_{W_{i\alpha}\cap W_{i\beta}} - h_{i\beta}|_{W_{i\alpha}\cap W_{i\beta}}$. *Since* $\{W_{i\alpha}\}_{\alpha\in\Lambda}$ *is an open covering of* U_i, *by sheaf axiom (S2), we can find a* $h_i \in \mathscr{F}(U_i)$ *such that* $h_i|_{W_{i\alpha}} = h_{i\alpha}$.

Now consider the open covering of $U_i \cap U_j$ *by* $U_i \cap U_j \cap V_\alpha = W_{i\alpha} \cap W_{j\alpha}$. *Since*

$$f_{ij}|_{W_{i\alpha}\cap W_{j\alpha}} = h_j|_{W_{i\alpha}\cap W_{j\alpha}} - h_i|_{W_{i\alpha}\cap W_{j\alpha}} = (h_j|_{U_i\cap U_j} - h_i|_{U_i\cap U_j})|_{W_{i\alpha}\cap W_{j\alpha}}.$$

This means $\delta\{h_i\} = \{f_{ij}\}$, *equivalently*, $[\{f_{ij}\}] = 0$. *This implies that* $\Phi^{\mathcal{U}}_{\mathcal{V}}$ *is injective.*

Picard group as sheaf cohomology

Theorem

Let X be a complex manifold, then we have $Pic(X) \cong H^1(X, \mathscr{O}^*)$, where $\mathscr{O}^*$ is the sheaf of nowhere vanishing holomorphic functions.

Proof

- *Given a holomorphic line bundle L with local trivializing covering $\mathcal{U}$, we get a cocycle $\{\psi_{ij}\}$ and hence a cohomology class $[\{\psi_{ij}\}] \in H^1(\mathcal{U}, \mathscr{O}^*) \subset H^1(X, \mathscr{O}^*)$. Easy to see that it is well-defined, and is surjective.*

Picard group as sheaf cohomology

Theorem

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Proof

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- *If L is isomorphic to L', we can assume that they have common trivializing coverings $\mathcal{U}$, with cocycles $\{\psi_{ij}\}$ and $\{\psi'_{ij}\}$ respectively. The bundle isomorphism map gives $\lambda_i \in \mathscr{O}^*(U_i)$ such that $\psi'_{ij}\lambda_j = \lambda_i\psi_{ij}$. This implies that $\{\psi'_{ij}\psi_{ij}^{-1}\}$ is a coboundary, so $[\{\psi_{ij}\}] = [\{\psi'_{ij}\}] \in H^1(X, \mathscr{O}^*)$. So we get a map $Pic(X) \to H^1(X, \mathscr{O}^*)$, which is easily seen to be a group isomorphism.*

§2.3 Fundamental results for sheaf cohomology

Short exact sequence

Recall that a morphism $f : \mathscr{F} \to \mathscr{G}$ of sheaves over X induces for each point $p \in X$ a homomorphism of stalks: $f_p : \mathscr{F}_p \to \mathscr{G}_p$. We call a sequence of morphisms of sheaves an "exact sequence" if the induced sequence on stalks is so for each pint p.

Theorem

If we have a short exact sequence for sheaves of abelian groups over X

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0,$$

then we have a long exact sequence for cohomologies

$$\begin{aligned} 0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to \dots \\ \dots \to H^p(X, \mathscr{H}) \to H^{p+1}(X, \mathscr{F}) \to H^{p+1}(X, \mathscr{G}) \to \dots \end{aligned}$$

Explanation

For the given short exact sequence, we always get an exact sequence

$$0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X),$$

(Exercise: for any open set U, the sequence $0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is always exact.)but the last homomorphism is in general not surjective:

Given $\sigma \in \mathscr{H}(X)$, can we find $\eta \in \mathscr{G}(X)$ such that $g_X(\eta) = \sigma$? We know that $0 \to \mathscr{F}_p \xrightarrow{f_p} \mathscr{G}_p \xrightarrow{g_p} \mathscr{H}_p \to 0$ is exact, so we can always find $\eta_p \in \mathscr{G}_p$ s.t. $g_p(\eta_p) = \sigma_p$. This means that we can find a cover $\mathcal{U} = \{U_i\}$ of X and a sequence $\eta_i \in \mathscr{G}(U_i)$ s.t. $g_{U_i}(\eta_i) = \sigma|_{U_i}$. If all the $\eta_{ij} := \eta_j - \eta_i = 0$ on $U_i \cap U_j$, we can patch these η_i's together, then we solve the problem. Need to modify η_i! Since $g_{U_i \cap U_j}(\eta_{ij}) = 0$, we can find $\mu_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $f_{U_i \cap U_j}(\mu_{ij}) = \eta_{ij}$. By the injectivity of f, we in fact get a cocycle $\{\mu_{ij}\} \in C^1(\mathcal{U}, \mathscr{F})$. So we get a homomorphism $\mathscr{H}(X) \to H^1(X, \mathscr{F})$. It is easy to check that if σ goes to 0 in $H^1(X, \mathscr{F})$, then we can modify η_i properly (on a refinement of $\mathcal{U}$) such that they patch together to get an element of $\mathscr{G}(X)$.

"Abstract de Rham theorem"

A corollary of the above Theorem is the following "abstract de Rham theorem":

Theorem

Suppose we have an exact sequence of the form:

$$0 \to \mathscr{F} \to \mathscr{S}_0 \to \mathscr{S}_1 \to \cdots \to \mathscr{S}_r \to \dots$$

where each $\mathscr{S}_r$ satisfies $H^p(X, \mathscr{S}_r) = 0, \forall p \geq 1$. (This is called an "acyclic resolution of $\mathscr{F}$".) Then $H^*(X, \mathscr{F})$ is isomorphic to the cohomology of the cochain complex

$$0 \to \mathscr{S}_0(X) \to \mathscr{S}_1(X) \to \cdots \to \mathscr{S}_r(X) \to \dots$$

i.e., $H^*(X, \mathscr{F}) \cong H^*(\Gamma(X, \mathscr{S}^*))$.

Proof

We break the sheaf sequence into a sequence of short exact sequences for $p \geq 1$*:*

$$0 \to \mathscr{K}_{p-1} \to \mathscr{S}_{p-1} \to \mathscr{K}_p \to 0,$$

where $\mathscr{K}_p = Ker(\mathscr{S}_p \to \mathscr{S}_{p+1}) = Im(\mathscr{S}_{p-1} \to \mathscr{S}_p)$*. Note that* $\mathscr{K}_0 \cong \mathscr{F}$*. By the above theorem and the assumption for* $\mathscr{S}_p$*, we have an exact sequence*

$$0 \to \mathscr{K}_{p-1}(X) \to \mathscr{S}_{p-1}(X) \to \mathscr{K}_p(X) \to H^1(X, \mathscr{K}_{p-1}) \to 0.$$

Also note that $\mathscr{K}_p(X) \cong Ker(\mathscr{S}_p(X) \to \mathscr{S}_{p+1}(X))$*, so we get*

$$H^1(X, \mathscr{K}_{p-1}) \cong Ker(\mathscr{S}_p(X) \to \mathscr{S}_{p+1}(X))/Im(\mathscr{S}_{p-1}(X) \to \mathscr{K}_p(X)) = H^p(\Gamma(X, \mathscr{S}^*)).$$

We need to prove $H^1(X, \mathscr{K}_{p-1}) \cong H^p(X, \mathscr{F}) = H^p(X, \mathscr{K}_0)$*.*

Proof

For this, we only need to show for $2 \leq r \leq p$

$$H^{r-1}(X, \mathscr{K}_{p-r+1}) \cong H^r(X, \mathscr{K}_{p-r}).$$

But this again follows from the segment of long exact sequence:

$$\cdots \to H^{r-1}(X, \mathscr{S}_{p-r}) \to H^{r-1}(X, \mathscr{K}_{p-r+1}) \to H^r(X, \mathscr{K}_{p-r}) \to H^r(X, \mathscr{S}_{p-r}) \to \ldots.$$

Fine sheaves

When can we get an acyclic resolution? In particular, how can we find a lot of sheaves $\mathscr{S}_r$ such that $H^p(X, \mathscr{S}_r) = 0, \forall p \geq 1$?

Definition

A sheaf $\mathscr{F}$ over X is called a "fine sheaf", if for any locally finite open cover $\mathcal{U} = \{U_i\}$, we can find a family of morphisms $\eta_i : \mathscr{F} \to \mathscr{F}$ such that:

(1) For each i, $\eta_i(p) : \mathscr{F}_p \to \mathscr{F}_p$ equals 0 for p outside a compact set $W_i \subset U_i$;

(2) $\sum_i \eta_i = id_{\mathscr{F}}$.

It is obvious that in case we can use a smooth function to multiply the sections of $\mathscr{F}$, then a usual partition of unity will make $\mathscr{F}$ a fine sheaf.

Proposition

If $\mathscr{F}$ is a fine sheaf, then $H^p(X, \mathscr{F}) = 0, \forall p \geq 1$.

Proof

For any p-cocycle $\{\sigma_{i_0,\dots,i_p}\} \in C^p(\mathcal{U}, \mathscr{F})$ for a locally finite cover $\mathcal{U} = \{U_i\}_{i\in\Lambda}$. Let η_i be the above morphisms in the definition. We define a $p-1$ cochain $\{\psi_{i_0,\dots,i_{p-1}}\}$ as follows:

$$\psi_{i_0,\dots,i_{p-1}} := \sum_i \eta_i(\sigma_{i,i_0,\dots,i_{p-1}}).$$

Then (using the fact that $\delta\{\sigma_{\dots}\} = 0$)

$$(\delta\psi)_{i_0,\dots,i_p} = \sum_{j=0}^{p}(-1)^j \psi_{i_0,\dots,\hat{i}_j,\dots,i_p} = \sum_j \sum_i (-1)^j \eta_i(\sigma_{i,i_0,\dots,\hat{i}_j,\dots,i_p}) = \sum_i \eta_i(\sigma_{i_0,\dots,i_p}) = \sigma_{i_0,\dots,i_p}.$$

§2.4 Applications

Cohomology of constant sheaves

- Let G be a given abelian group, we can define the constant sheaf $\underline{G}$ over X by $\underline{G}(U) = \{\text{locally constant maps } U \to G\}$, then we usually denote $H^p(X, \underline{G})$ by $H^p(X, G)$.

Cohomology of constant sheaves

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- One can show that when X is a manifold, this is isomorphic to the singular cohomology or simplicial cohomology. But we won't prove this. For the isomorphism to simplicial cohomology when $G = \mathbb{Z}$, one can read Chapter 0 of Griffiths-Harris.

de Rham theorem and Dolbeault theorem

- We use the de Rham resolution of $\underline{\mathbb{C}}$:

$$0 \to \underline{\mathbb{C}} \to \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathscr{A}^{2n} \to 0$$

to get de Rham isomorphism:

$$H^p(X, \mathbb{C}) \cong H^p_{dR}(X, \mathbb{C}), \quad p = 0, \dots, 2n.$$

The reason for this to be a resolution is Poincaré's Lemma.

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  The reason for this to be a resolution is Poincaré's Lemma.
- Similarly, we have a Dolbeault-Grothendieck Lemma, which says that a $\bar{\partial}$-closed form is locally $\bar{\partial}$-exact. So we get a fine resolution for any $0 \le p \le n$:

$$0 \to \Omega^p \to \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathscr{A}^{p,n} \to 0,$$

  so we get

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(X).$$

  Also for a holomorphic vector bundle E, we have $H^q(X, \Omega^p(E)) \cong H^{p,q}_{\bar{\partial}}(X, E)$.

Divisors

Recall: we define the sheaf of meromorphic functions $\mathscr{M}$ on X, where X is a compact complex manifold, to be the sheafification of the presheaf

$$U \mapsto \text{quotient field of } \mathscr{O}(U).$$

We define $\mathscr{M}^*$ to be the sheaf of meromorphic functions that are not identically 0, and let $\mathscr{O}^*$ be the subsheaf of $\mathscr{M}^*$, consisting of no-where vanishing holomorphic functions. The short exact sequence

$$1 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{M}^*/\mathscr{O}^* \to 1$$

gives us a long exact sequence, starting with

$$\{1\} \to \mathbb{C}^* \to \mathscr{M}^*(X) \to \mathscr{M}^*/\mathscr{O}^*(X) \to H^1(X, \mathscr{O}^*) \to \dots.$$

The global section of $\mathscr{M}^*/\mathscr{O}^*(X)$ can be equivalently described as a finite formal sum $\sum_i a_i D_i$, where $a_i \in \mathbb{Z}$ and D_i is codimension 1 irreducible analytic subvariety of X. This is called a "divisor".

Line bundles associated to a divisor

We define the groups of divisor classes by

$$Div(X) := \left(\mathscr{M}^*/\mathscr{O}^*(X)\right)/\mathscr{M}^*(X).$$

Two divisors are called linearly equivalent, if their difference is a divisor of a global meromorphic function.

The map $\mathscr{M}^*/\mathscr{O}^*(X) \to H^1(X, \mathscr{O}^*)$ is given as follows: locally we can cover X by $\{U_i\}$ such that an element of $\mathscr{M}^*/\mathscr{O}^*(X)$ is given by $f_i \in \mathscr{M}^*(U_i)$. Then $g_{ij} := f_i/f_j$ defines a class in $H^1(X, \mathscr{O}^*)$.

First Chern class of a line bundle

A very useful exact sequence is the following

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathscr{O}^* \to 1.$$

We get the exact sequence

$$\cdots \to H^1(X, \mathscr{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \ldots.$$

We call $c_1 : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$ the "first Chern class" map. We shall use differential forms to give another characterization of Chern classes in the next chapter.