# Introduction to Complex Geometry

## Chapter 1 Complex Manifolds and Vector Bundles

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# Outline

# §1.1 Complex manifolds

# Defining a complex manifold: the 1st condition

Roughly speaking, a complex manifold is a topological space $X$ on which we can talk about "holomorphic" functions. Since we know what does a holomorphic function means in Euclidean spaces, the first condition we impose on $X$ is:

Condition 1:(existence of coordinate charts) $X$ is locally homeomorphic to open sets of $\mathbb{C}^n$. To be precise, we require that there is an open covering $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of $X$ such that for each $U_i$ we have a homeomorphism $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^n$ onto an open set $\varphi_i(U_i)$ of $\mathbb{C}^n$.

# Defining a complex manifold: the 2nd condition

Given these coordinates, we should define a function $f : \Omega \to \mathbb{C}$ to be holomorphic if all its coordinate-representations $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i \cap \Omega))$. But is this a well-defined notion? For example if $\Omega \subset U_i \cap U_j \neq \emptyset$, then on $\Omega$ we have two sets of coordinates. Is it possible that $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i \cap \Omega))$ but $f \circ \varphi_j^{-1} \notin \mathscr{O}(\varphi_j(U_j \cap \Omega))$? To avoid this, note that $f \circ \varphi_j^{-1} = \left(f \circ \varphi_i^{-1}\right) \circ \left(\varphi_i \circ \varphi_j^{-1}\right)$, so we require:

Condition 2:(compatibility) Coordinate changes of Condition 1 should be holomorphic. To be precise, we require that whenever $U_i \cap U_j \neq \emptyset$, we have $\varphi_i \circ \varphi_j^{-1}$ is a biholomorphic map from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$.

# Defining a complex manifold: the 3rd condition

Given these 2 conditions, one can check easily that the notion of “holomorphic function” makes perfect sense. However, to avoid pathology and use more analytic tools such as metrics and integration, we also require a complex manifold to be a nice topological space:

Condition 3: $X$ satisfies $T_2$ and $C_2$ axioms, i.e. $X$ is a Hausdorff space, and has a countable topological basis.

# The definition

## Definition

- A complex (analytic) manifold of dimension $n$ is a topological space $X$ satisfying Conditions 1,2,3 above. A 1-dimensional complex manifold is also known as a "Riemann surface". A map $f: X \to \mathbb{C}$ from a complex manifold $X$ is called a "holomorphic function", if $f \circ \varphi_i^{-1} \in \mathscr{O}(\varphi_i(U_i))$ for all $i \in \Lambda$. In this case, we write $f \in \mathscr{O}(X)$.

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- If $X, Y$ are both complex manifolds of dimensions $n$ and $m$ respectively, a map $F : X \to Y$ is called "holomorphic", if for all coordinate charts $(U, \varphi)$ of $X$ and $(V, \psi)$ of $Y$, the map $\psi \circ F \circ \varphi^{-1}$ is a holomorphic map on $\varphi\big(U \cap F^{-1}(V)\big) \subset \mathbb{C}^n$ whenever $U \cap F^{-1}(V) \neq \emptyset$. A holomorphic map with a holomorphic inverse is called "biholomorphic".

# A technical remark

## Remark

In standard textbooks, the set of coordinate charts $\{(U_i, \varphi_i)\}_{i\in\Lambda}$ is assumed to be maximal, i.e., whenever a homeomorphism from an open set $V$, $\psi : V \to \psi(V) \subset \mathbb{C}^n$ is compatible with $(U_i, \varphi_i)$ for all $U_i \cap V \neq \emptyset$, we have $(V, \psi) \in \{(U_i, \varphi_i)\}_{i\in\Lambda}$. It is easy to check that from the coordinate charts in our definition, one can always enlarge it to a unique maximal one satisfying the compatibility condition.

## Example

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2. Let $\{e_1, \ldots, e_{2n}\}$ be any fixed $\mathbb{R}$-basis of $\mathbb{C}^n$, and let $\Lambda := \{m_1 e_1 + \cdots + m_{2n} e_{2n} | \ m_i \in \mathbb{Z}\}$ be a lattice of rank $2n$. Then we can define the quotient space $\mathbb{C}^n/\Lambda$, it is a compact Hausdorff space equipped with quotient topology. There is a natural complex manifold structure on $\mathbb{C}^n/\Lambda$, we call this complex manifold a "complex torus".

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3. Let $P \in \mathbb{C}[z, w]$ be a polynomial of degree $d$. Define

$$C := \{(z, w) | \ P(z, w) = 0\}.$$

We call it an "affine plane algebraic curve". Assume $P$ is irreducible and $\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}$ have no common zeroes on $C$. Then $C$ is a natural complex manifold.

# More about example 3

The coordinates can be chosen in the following way: if $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$, then we can apply the (holomorphic version of) implicit function theorem to find a neighborhood $\Delta(z_0, \epsilon) \times \Delta(w_0, \delta)$ and a holomorphic function $g(z)$ such that $U := C \cap \Big(\Delta(z_0, \epsilon) \times \Delta(w_0, \delta)\Big) = \{(z, w) |\ z \in \Delta(z_0, \epsilon), w = g(z)\}$. We choose $\varphi : U \to \mathbb{C}$ to be $\varphi(z, w) = z$. If $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$, we use $w$ as local coordinate.

Exercise: what's the coordinates transformation function?

# Complex (analytic) submanifolds

## Definition

A closed subset $Y$ of a $n$-dimensional complex manifold $X$ is called a (closed) "complex (analytic) submanifold" of dimension $k$, if for any $p \in Y$, we can find a compatible chart $(U, \varphi)$ of $X$ such that $p \in U$ and

$$\varphi(U \cap Y) = \{(z_1, \ldots, z_n) \in \varphi(U) | \, z_{k+1} = \cdots = z_n = 0\}.$$

One can check that the restriction of such charts (we call them "adapted charts") to $Y$ makes $Y$ a complex manifold and the inclusion $Y \subset X$ is a holomorphic map.

# Complex submanifolds of $\mathbb{C}^n$

A direct application of the maximum principle gives:

### Lemma

Any holomorphic function on a compact connected complex manifold should be a constant.

Let $M$ be a complex submanifolds of $\mathbb{C}^n$. Since the restriction of complex coordinate functions of $\mathbb{C}^n$ to $M$ are holomorphic functions on $M$, we get:

### Corollary

There are no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension.

### Remark

Those non-compact complex manifolds which admit proper holomorphic embeddings into $\mathbb{C}^N$ for some large $N$ are precisely "Stein manifolds" in complex analysis.

# The complex projective space

## Example

Define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n)$ iff $\exists \lambda \in \mathbb{C}^*$ such that $w_i = \lambda z_i, \forall i = 0, \ldots, n$. The equivalent class of $(z_0, \ldots, z_n)$ is denoted by $[z_0, \ldots, z_n]$. The n-dimensional complex projective space $\mathbb{C}P^n$ is defined to be the space of all equivalent classes, endowed with quotient topology. It is compact, Hausdorff. Choose holomorphic coordinate charts as follows: Define $U_i := \{[z_0, \ldots, z_n] \in \mathbb{C}P^n | \; z_i \neq 0\}, \quad i = 0, \ldots, n$. and define

$$\varphi_i : U_i \to \mathbb{C}^n, \quad \varphi_i([z_0, \ldots, z_n]) := (\frac{z_0}{z_i}, \ldots, \frac{\hat{z}_i}{z_i}, \ldots, \frac{z_n}{z_i}).$$

The checking of compatibility is left as an exercise. Also it is easy to check that $\mathbb{C}P^1$ is diffeomorphic to our familiar $S^2$.

# Projective algebraic manifolds

Let $F_1, \ldots, F_k \in \mathbb{C}[z_0, \ldots, z_n]$ be a set of irreducible homogeneous polynomials of degrees $d_1, \ldots, d_k$ respectively. Then the set

$$V(F_1, \ldots, F_k) := \{[z_0, \ldots, z_n] | \, F_1(z_0, \ldots, z_n) = \cdots = F_k(z_0, \ldots, z_n) = 0\}$$

is well-defined and is called a (complex) projective algebraic variety. If we assume that $V(F_1, \ldots, F_k)$ is a complex submanifold of $\mathbb{C}P^n$, then it will be called a "projective algebraic manifold" (or "Hodge manifold").

## Example

Let $F \in \mathbb{C}[z_0, \ldots, z_n]$ be irreducible and homogeneous of degree $d$. If the only common zero of $\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ is $(0, \ldots, 0)$. Then $V(F)$ is a complex submanifold of dimension $n-1$. E.g., the "Fermat hypersurface" $V(z_0^d + \cdots + z_n^d)$.

## Proof

We check this on $U_0$. $V(F) \cap U_0$ is the zero locus of $F(1, z_1, \ldots, z_n) \in \mathscr{O}(U_0)$. Need to show that $\frac{\partial F}{\partial z_1}(1, z_1, \ldots, z_n), \ldots \frac{\partial F}{\partial z_n}(1, z_1, \ldots, z_n)$ have no common zeroes on $V(F) \cap U_0$.

Suppose $F(1, z_1^0, \ldots z_n^0) = \frac{\partial F}{\partial z_1}(1, z_1^0, \ldots, z_n^0) = \cdots = \frac{\partial F}{\partial z_n}(1, z_1^0, \ldots, z_n^0) = 0$. By Euler:

$$\frac{\partial F}{\partial z_0}(1, z_1^0, \ldots, z_n^0) + z_1^0 \frac{\partial F}{\partial z_1}(1, z_1^0, \ldots, z_n^0) + \cdots + z_n^0 \frac{\partial F}{\partial z_n}(1, z_1^0, \ldots, z_n^0) = dF = 0.$$

This implies $\frac{\partial F}{\partial z_0}(1, z_1^0, \ldots, z_n^0) = 0$, so $(1, z_1^0, \ldots, z_n^0)$ is a common zero of $\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}$ in $\mathbb{C}^{n+1}$ different from $(0, \ldots, 0)$.

# Analytic subvarieties

A generalization of submanifold is the following:

## Definition

A closed subset $A$ of a complex manifold $X$ is called an "analytic subvariety", if it is locally the common zeroes of finitely many holomorphic functions, i.e. $\forall p \in A$, there is an open set $U \subset X$ and $f_1, \ldots, f_k \in \mathscr{O}(U)$ such that

$$A \cap U = \{z \in U | \ f_1(z) = \cdots = f_k(z) = 0\}.$$

An analytic subvariety $A$ is called a "hypersurface" if it is locally the zero locus of a holomorphic function.

# Relations between submanifolds and subvarieties

- A complex submanifold is an analytic subvariety, we just choose $U$ to be the domain of the adapted chart and $f_i$ to be $z_{k+1}, \ldots, z_n$.

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- Let $A \subset X$ be an analytic subvariety. $p \in A$ is called a "regular point", if we can find open $U \subset X$ and $f_1, \ldots, f_k \in \mathscr{O}(U)$ s.t. $A \cap U = \{z \in U |\ f_1(z) = \cdots = f_k(z) = 0\}$ and $rank \frac{\partial(f_1,\ldots,f_k)}{\partial(z_1,\ldots,z_n)}(p) = k$. In this case, $A$ is locally near $p$ a complex submanifold of dimension $n - k$.

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- The locus of regular points of $A$ is denoted by $A_{reg}$. Its complement in $A$ is called the "singular locus", and its elements are called "singular points of $A$".

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- The locus of regular points of $A$ is denoted by $A_{reg}$. Its complement in $A$ is called the "singular locus", and its elements are called "singular points of $A$".
- Chow's theorem: complex analytic subvarieties of $\mathbb{C}P^n$ are algebraic, i.e., the common zeroes of finitely many homogeneous polynomials.

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- However, for a given even dimensional oriented manifold, it is not always clear whether or not we can make it a complex manifold.
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- In this view, we give an example of complex structures on product of odd dimensional spheres:

## Example (Calabi-Eckmann manifolds)

We can make $S^{2p+1} \times S^{2q+1}$ into a complex manifold. The idea is that we can write

$$S^{2p+1} = \{z \in \mathbb{C}^{p+1} | \sum_{i=0}^{p} |z_i|^2 = 1\}, \quad S^{2q+1} = \{z \in \mathbb{C}^{q+1} | \sum_{j=0}^{q} |z_j|^2 = 1\},$$

and we have the Hopf fibration maps:

$$\pi_p : S^{2p+1} \to \mathbb{C}P^p, \quad \pi_q : S^{2q+1} \to \mathbb{C}P^q,$$

each with fiber $S^1$. So if we consider the map $\pi = (\pi_p, \pi_q) : S^{2p+1} \times S^{2q+1} \to \mathbb{C}P^p \times \mathbb{C}P^q$, then we can view $S^{2p+1} \times S^{2q+1}$ as a fiber bundle on $\mathbb{C}P^p \times \mathbb{C}P^q$, which is a complex manifold, with fiber $S^1 \times S^1 = T^2$, which can also be made a complex manifold.

## Example (Calabi-Eckmann manifolds (continued))

To be precise, fix a $\tau \in \mathbb{C}$ with $Im\tau > 0$. We donote by $T_\tau$ the complex torus $\mathbb{C}/ < 1, \tau >$. Consider the open sets:

$$U_{kj} := \{(z, z') \in S^{2p+1} \times S^{2q+1} | \, z_k z'_j \neq 0\},$$

and the map $h_{kj} : U_{kj} \to \mathbb{C}^{p+q} \times T_\tau$ given by

$$h_{kj}(z, z') = (\frac{z_0}{z_k}, \ldots, \frac{\hat{z}_k}{z_k}, \ldots, \frac{z_p}{z_k}, \frac{z'_0}{z'_j}, \ldots, \frac{\hat{z}'_j}{z'_j}, \ldots, \frac{z'_q}{z'_j}, t_{kj}),$$

where $t_{kj} := \frac{1}{2\pi\sqrt{-1}}(\log z_k + \tau \log z'_j) \mod < 1, \tau >$.

Exercise: check that these charts makes $S^{2p+1} \times S^{2q+1}$ a complex manifold.

## §1.2 Vector bundles

# Holomorphic vector bundle

Roughly speaking, a holomorphic vector bundle over a complex manifold is a family of vector spaces, varying holomorphically.

## Definition

A holomorphic vector bundle of rank $r$ over a $n$-dimensional complex manifold $X$ is a complex manifold $E$ of dimension $n+r$, together with a holomorphic surjective map $\pi : E \to X$ satisfying:

1. (Fiberwise linear) Each fiber $E_p := \pi^{-1}(p)$ has the structure of $r$-dimensional vector space over $\mathbb{C}$;

A vector bundle of rank 1 is usually called a "line bundle".

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1. (Fiberwise linear) Each fiber $E_p := \pi^{-1}(p)$ has the structure of $r$-dimensional vector space over $\mathbb{C}$;
2. (Locally trivial) There is an open cover of $X$, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ such that each $\pi^{-1}(U_i)$ is biholomorphic to $U_i \times \mathbb{C}^r$ via $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$, and $E_p \hookrightarrow \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ is a linear isomorphism onto $\{p\} \times \mathbb{C}^r$ for any $p \in U_i$. $\varphi_i$ is called a "local trivialization".

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# Cocycles of a vector bundle

In this case, whenever $U_i \cap U_j \neq \emptyset$, we have a holomorphic map, called the "transition map", $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ (viewed as an open subset of $\mathbb{C}^{r^2}$) such that $\varphi_i \circ \varphi_j^{-1}(z, v) = (z, \psi_{ij}(z)v)$. These families of transition maps satisfies the "cocycle condition":

(1) $\psi_{ij}\psi_{ji} = I_r$ on $U_i \cap U_j$;

(2) Whenever $U_i \cap U_j \cap U_k \neq \emptyset$, we have $\psi_{ij}\psi_{jk}\psi_{ki} = I_r$ on $U_i \cap U_j \cap U_k$.

The name "cocycle" is no coincidence. In fact we will see later that $\{\psi_{ij}\}$ above is indeed a cocycle in Čech's approach to sheaf cohomology theory.

# From cocycles to vector bundles

## Remark

On the other hand, if we are given a set of holomorphic transition maps $\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ satisfying the cocycle condition, we can construct a holomorphic vector bundle by setting $E = \coprod_{i \in \Lambda} (U_i \times \mathbb{C}^r)/\sim$, where $(z, v) \sim (z', w)$ for $(z, v) \in U_i \times \mathbb{C}^r$ and $(z', w) \in U_j \times \mathbb{C}^r$ if and only if $z = z'$ and $v = \psi_{ij}(z)w$. We leave the detail as an exercise.

# $C^\infty$ and continuous vector bundles

**Remark**

We can similarly define $C^\infty$ (real or complex) vector bundles over a smooth manifold, and more generally continuous vector bundles over a topological space. There are similar characterizations using $C^\infty$ or continuous cocycles.

We leave all these details as exercises.

# Holomorphic sections of a vector bundle

## Definition (holomorphic section)

Let $\pi : E \to X$ be a holomorphic vector bundle over $X$. Let $U \subset X$ be an open set. A holomorphic section of $E$ over $U$ is a holomorphic map $s : U \to E$ such that $\pi \circ s = id_U$, i.e., $s(p) \in E_p$ for any $p \in U$. The set of holomorphic sections over $U$ is usually denoted by $\Gamma(U, \mathscr{O}(E))$ or $\mathscr{O}(E)(U)$.

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- A fundamental problem in the theory of holomorphic vector bundles: existence and construction of global holomorphic sections of a given bundle.
- Main difficulty: no “holomorphic partition of unity”.
- An important tool is the $L^2$-method for the $\bar{\partial}$-equation. It is interesting that whether or not we can solve the equation depends on the geometry, in particular, the curvature of the bundle.

# Classification of vector bundles

## Definition (bundle map)

Let $\pi^E : E \to X$ and $\pi^F : F \to X$ are holomorphic vector bundles of ranks $r$ and $s$ respectively. A bundle map from $E$ to $F$ is a holomorphic map $f : E \to F$ such that $f$ maps $E_p$ to $F_p$ for any $p \in X$ and $f|_{E_p} : E_p \to F_p$ is linear. When a bundle map has an inverse bundle map, we will say that these two bundles are isomorphic.

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- Another fundamental problem is the classification problem.
- One important tool is the theory of characteristic classes that we shall discuss later.
- Also the set of isomorphic classes of holomorphic vector bundles over a given complex manifold has rich structures and is an important invariant for the complex manifold.

# Examples of holomorphic vector bundles

## Example (trivial bundle)

$X \times \mathbb{C}^r$ with $\pi_1 : X \times \mathbb{C}^r \to X$ is a holomorphic vector bundle over $X$, called the "trivial bundle" over $X$, denoted by $\underline{\mathbb{C}^r}$.

## Example (holomorphic tangent bundle)

Let $X$ be a complex manifold of dimension $n$. We shall now construct its "holomorphic tangent bundle" $TX$ as follows:
Let $p \in X$, we first define the ring $\mathscr{O}_{X,p} := \lim_{\longrightarrow} \mathscr{O}_X(U)$,where the direct limit is taken with respect to open sets $p \in U$. For persons not familiar with direct limit, this is $\coprod_{U \ni p} \mathscr{O}_X(U)/\sim$, with $f \in \mathscr{O}_X(U)$ equivalent to $g \in \mathscr{O}_X(V)$ iff we can find another open set $p \in W \subset U \cap V$ such that $f|_W = g|_W$. As an exercise, we can see that $\mathscr{O}_{X,p}$ is isomorphic to the ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\}$. An element of $\mathscr{O}_{X,p}$ is called a "germ of holomorphic function" at $p$.

## Example (holomorphic tangent bundle (continued))

A tangent vector at $p$ is a derivation $v : \mathscr{O}_{X,p} \to \mathbb{C}$, i.e., a $\mathbb{C}$-linear map satisfying the Leibniz rule $v(fg) = v(f)g(p) + f(p)v(g)$. The set of tangent vectors at $p$ is easily seen to be a $\mathbb{C}$-vector space. We call it the (holomorphic) tangent space of $X$ at $p$, denoted by $T_pX$. If $\varphi : U_i \to \mathbb{C}^n$ is a holomorphic coordinate chart with $\varphi_i = (z_1, \ldots, z_n)$. Then we can define $\frac{\partial}{\partial z_i}|_p \in T_pX$ to be $\frac{\partial}{\partial z_i}|_p(f) := \frac{\partial(f \circ \varphi_i^{-1})}{\partial z_i}(\varphi_i(p))$. Then one can show that $\{\frac{\partial}{\partial z_i}|_p\}_{i=1}^n$ is a basis of $T_pX$. Let $TX := \coprod_{p \in X} T_pX$, and define $\pi : TX \to X$ in the obvious way. We can make it a holomorphic vector bundle of rank $n$ over $X$ as follows: Let $(U_i, \varphi_i)$ be a holomorphic chart. Then we can define the local trivialization $\tilde{\varphi}_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ to be $\tilde{\varphi}_i(q, \sum_i a_i \frac{\partial}{\partial z_i}|_q) := (q, a_1, \ldots, a_n)$. This gives a complex structure on $TX$ and at the same time gives a local trivialization of $TX$ over $U_i$.

A holomorphic section of $TX$ over $U$ is called a "holomorphic vector field" on $U$.

To stress that $TX$ is a holomorphic vector bundle, we shall write $T^hX$ sometimes.

## Example (holomorphic cotangent bundle)

Any $f \in \mathscr{O}_{X,p}$ defines a linear functional on $T_pX$ by $v \mapsto v(f)$. We call this $df|_p \in (T_pX)^* =: T_p^*X$. $T_p^*X$ is called the (holomorphic) cotangent space of $X$ at $p$. It is easy to see that if $(U_i, \varphi_i)$ is a holomorphic chart, then $\{dz_i|_p\}_{i=1}^n$ is the basis of $T_p^*X$ dual to $\{\frac{\partial}{\partial z_i}|_p\}_{i=1}^n$.

We can similarly give $T^*X := \coprod_{p\in X} T_p^*X$ a holomorphic bundle structure, called the "(holomorphic) cotangent bundle" of $X$. We leave this as an exercise.

A holomorphic section of $T^*X$ over $U$ is called a "holomorphic 1-form" on $U$.

# Line bundles

Let $\pi : L \to X$ be a holomorphic line bundle and $\{U_i\}_{i\in\Lambda}$ an open cover by trivialization neighborhoods, and $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ the trivialization map. Since $GL(1,\mathbb{C}) = \mathbb{C}^*$, now the transition maps $\psi_{ij}$ become non-vanishing holomorphic functions on $U_i \cap U_j$. Let $s \in \Gamma(X, \mathscr{O}(L))$, then $\varphi_i \circ s|_{U_i} : U_i \to U_i \times \mathbb{C}$ could be represented by a holomorphic function $f_i \in \mathscr{O}(U_i)$, such that $\varphi_i \circ s|_{U_i}(p) = (p, f_i(p))$. When $U_i \cap U_j \neq \emptyset$, since $s|_{U_i} = s|_{U_j}$ on $U_i \cap U_j$, we have for any $p \in U_i \cap U_j$:

$$\begin{aligned}(p, f_i(p)) &= \varphi_i(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1}) \circ \varphi_j(s(p)) \\ &= (\varphi_i \circ \varphi_j^{-1})(p, f_j(p)) \\ &= (p, \psi_{ij}(p) f_j(p)).\end{aligned}$$

So we have $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$. On the other hand, it is direct to check that given a family of holomorphic functions $f_i \in \mathscr{O}(U_i)$, satisfying $f_i = \psi_{ij} f_j$ on $U_i \cap U_j$, then there corresponds a unique $s \in \Gamma(X, \mathscr{O}(L))$.

## Example (Universal line bundle (or "tautological bundle") over $\mathbb{C}P^n$)

- We define a holomorphic line bundle $U \to \mathbb{C}P^n$ as follows: As a set, $U = \{([z], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \ v \in [z]\} = \{([z], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \ v_i z_j - v_j z_i = 0, \forall i, j = 0, \ldots, n\}$.

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- Easy to see that $U$ is a complex submanifold of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. The projection onto $\mathbb{C}P^n$ is clearly holomorphic, with fiber the 1-dimensional linear subspace of $\mathbb{C}^{n+1}$ generated by $(z_0, \ldots, z_n)$.

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- Easy to see that $U$ is a complex submanifold of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. The projection onto $\mathbb{C}P^n$ is clearly holomorphic, with fiber the 1-dimensional linear subspace of $\mathbb{C}^{n+1}$ generated by $(z_0, \dots, z_n)$.
- For local triviality, we use the holomorphic charts $\{(U_i, \varphi_i)\}_{i=0}^n$ defined before. On $\pi^{-1}(U_i)$, each $v \in U_{[z]}$ can be uniquely write as $t \cdot (\frac{z_0}{z_i}, \dots, 1, \dots, \frac{z_n}{z_i})$, so we define $\tilde{\varphi}_i([z_0, \dots, z_n], t \cdot (\frac{z_0}{z_i}, \dots, 1, \dots, \frac{z_n}{z_i})) = ([z_0, \dots, z_n], t) \in U_i \times \mathbb{C}$. This is easily seen to be a biholomorphic map. And the transition functions are: $\psi_{ij}([z]) = \frac{z_i}{z_j}$. What are the global holomorphic sections of $U$? (exercise)

# Construct new bundles from old ones

The usual constructions in linear algebra all have counterparts in the category of vector bundles over $X$. Let $E, F$ be vector bundles over $X$ of rank $r$ and $s$ respectively.

- Direct sum
  The direct sum of $E$ and $F$ is a vector bundle of rank $r+s$ with fiber $E_p \oplus F_p$. To describe it, it suffices to write down the transition maps: if $\{U_i\}_{i\in\Lambda}$ is a common trivializing covering of $X$ for $E$ and $F$. The transition maps are $\psi_{ij}$ and $\eta_{ij}$ respectively, then the transition maps for $E \oplus F$ are precisely $diag(\psi_{ij}, \eta_{ij})$.

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- Tensor product
  The tensor product of $E$ and $F$ is a vector bundle of rank $rs$ with fiber $E_p \otimes F_p$. In this short course, we only use the tensor product of a line bundle $L$ with a general vector bundle $E$. In this case, if the transition maps for $E$ and $L$ with respect to a common trivializing covering are $\psi_{ij}$ and $\eta_{ij}$, then the transition maps of $E \otimes L$ are $\eta_{ij}\psi_{ij}$.

- $\underline{Hom}(E, F)$ is a vector bundle of rank $rs$ with fiber $Hom(E_p, F_p)$, the space of linear maps from $E_p$ to $F_p$. In particular, we define the dual of $E$ to be $E^* := \underline{Hom}(E, \underline{\mathbb{C}})$, whose fiber over $p$ is exactly the dual space of $E_p$, $(E_p)^*$. When $L \to X$ is a holomorphic line bundle, we can easily describe $L^*$ in terms of transition functions: if the transition functions of $L$ are $\psi_{ij}$, then the transition functions of $L^*$ are $\psi_{ij}^{-1}$. For this reason, we usually also write $L^{-1}$ for $L^*$.Exercise: (1), What's the transition function of $E^*$ in general? (2), Prove that $E^* \otimes F \cong \underline{Hom}(E, F)$.

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- Wedge product For $k \in \mathbb{N}$ and $k \leq r$, the degree $k$ wedge product of $E$ is a vector bundle $\Lambda^k E$ with fiber $\Lambda^k E_p$ at $p$. The highest degree wedge product $\Lambda^r E$ is also called the "determinant line bundle" of $E$, since its transition functions are precisely $\det \psi_{ij}$. $\Omega^p(X) := \Lambda^p T^* X$ is the bundle of holomorphic $p$-forms. The determinant line bundle of the holomorphic cotangent bundle $T^* X$ of a complex manifold $X$ is called the "canonical line bundle" of $X$, denoted by $K_X$.

- Pull back via holomorphic map Let $E \to X$ be a holomorphic vector bundle of rank $r$, $f : Y \to X$ be a holomorphic map between complex manifolds, then we can define a "pull back" holomorphic vector $f^*E$ over $Y$. As a set, we define $f^*E := \{(y,(x,v)) \in Y \times E |\, x = f(y)\}$, and $p : f^*E \to Y$ is just the projection to its first component.

  We can also describe $f^*E$ via transition maps: if $\{U_i\}_{i\in\Lambda}$ is a trivializing covering of $X$ for $E$ with transition maps $\psi_{ij} : U_i \cap U_j \to GL(r,\mathbb{C})$, and we choose an open covering $\{V_\alpha\}_{\alpha\in I}$ such that $f(V_\alpha) \subset U_i$ for some $i \in \Lambda$. We fix a map $\tau : I \to \Lambda$ such that $f(V_\alpha) \subset U_{\tau(\alpha)}$. Then the transition maps for $f^*E$ with respect to $\{V_\alpha\}_{\alpha\in I}$ are just $f^*\psi_{\tau(\alpha)\tau(\beta)} = \psi_{\tau(\alpha)\tau(\beta)} \circ f : V_\alpha \cap V_\beta \to GL(r,\mathbb{C})$.

## Example (The hyperplane bundle)

Let $U \to \mathbb{C}P^n$ be the universal bundle, its dual is usually denoted by $H$, we call it the "hyperplane line bundle". (Reason for this name will be explained later.) Another common notation for $H$ is $\mathscr{O}(1)$. We also write the $H^k$, or $\mathscr{O}(k)$, short for the $k$-times tensor product of $H$, $H^k := H^{\otimes k} = H \otimes \cdots \otimes H$, and $\mathscr{O}(-k) := H^{-k} := U^{\otimes k}$.

We now study the holomorphic sections of $H^k$ for $k > 0$. Let $s \in \Gamma(\mathbb{C}P^n, \mathscr{O}(H^k))$, $s$ can be represented by $f_\alpha \in \mathscr{O}(U_\alpha)$, where $U_\alpha = \{[z] \in \mathbb{C}P^n | \; z_\alpha \neq 0\}$. These $f_\alpha$'s satisfy: $f_\alpha([z]) = \left(\frac{z_\beta}{z_\alpha}\right)^k f_\beta([z])$ on $U_\alpha \cap U_\beta$. Pulling back to $\mathbb{C}^{n+1} \setminus \{0\}$, we can view $z_\alpha^k f_\alpha([z])$ as a homogeneous function of degree $k$ on $\mathbb{C}^{n+1} \setminus \{z_\alpha = 0\}$, which is also holomorphic. Now the above compatibility condition means that these $z_\alpha f_\alpha([z])$'s could be "glued" together to form a holomorphic function $F$ on $\mathbb{C}^{n+1} \setminus \{0\}$, homogeneous of degree $k$.

## Example (The hyperplane bundle (continued))

By Hartogs extension theorem , it extends to a holomorphic function $F \in \mathscr{O}(\mathbb{C}^{n+1})$. We necessarily have $F(0) = 0$ by homogeneity and continuity. From this we easily conclude that $F$ is a homogeneous polynomial of degree $k$.

On the other hand, it is easy to see that any homogeneous polynomial of degree $k$ in $\mathbb{C}[z_0, \dots, z_n]$ determines uniquely a holomorphic section of $H^k$. So we have

$$\dim_{\mathbb{C}} \Gamma(\mathbb{C}P^n, \mathscr{O}(H^k)) = \binom{n+k}{n}.$$

Exercise: Prove that when $k < 0$, $\Gamma(\mathbb{C}P^n, \mathscr{O}(H^k)) = \{0\}$.

# The Picard group

## Definition

The isomorphic classes of holomorphic line bundles over $X$ is called the "Picard group" of $X$, denoted by $Pic(X)$.

$Pic(X)$ is indeed a group: we define $[L_i] \cdot [L_2] := [L_1 \otimes L_2]$, then $\underline{\mathbb{C}}$ is the identity element and $[L]^{-1}$ is just $[L^*]$.

For $\mathbb{C}P^n$, we have $Pic(\mathbb{C}P^n) \cong \mathbb{Z}$, and any holomorphic line bundle is isomorphic to $\mathscr{O}(k)$ for some $k \in \mathbb{Z}$. We shall prove this next week.

# §1.3 Almost complex structure and $\bar{\partial}$-operator

# From real tangent bundle to holomorphic tangent bundle

Recall: a $n$-dimensional complex manifold $X$ is also a $2n$-dimensional orientable differential manifold. For $p \in X$, we can define a real tangent vector at $p$ and the corresponding real tangent space at $p$, $T_p^{\mathbb{R}}X$. In terms of coordinate chart $\varphi = (z_1, \ldots, z_n)$, we have

$$T_p^{\mathbb{R}}X = Span_{\mathbb{R}}\left\{\frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial y_i}\Big|_p\right\}_{i=1}^n.$$

We can give $\coprod_{p \in X} T_p^{\mathbb{R}}X$ a structure of $\mathbb{R}$-vector bundle of rank $2n$, called the "real tangent bundle" of $X$, and denoted by $T^{\mathbb{R}}X$. Similarly, we can define the real cotangent bundle $T^{*\mathbb{R}}X$.

There are two equivalent ways to get from this real tangent bundle to our previous holomorphic tangent and cotangent bundles.——Cause of most of the troubles for beginners !

# The "intrinsic" way

Recall that any real vector space $V$ of dimension $2n$ can be regarded as $\mathbb{C}$-vector space of dimension $n$ once we know what does it mean to multiply $\sqrt{-1}$ to an element of $V$. This is equivalent to giving a $\mathbb{R}$-linear map $J : V \to V$ such that $J^2 := J \circ J = -id$. We call such a $J$ a "complex structure" on $V$. In this case, $V$ can be regarded as a $\mathbb{C}$-vector space by defining $(\alpha + \sqrt{-1}\beta)v := \alpha v + \beta J v$, $\forall \alpha, \beta \in \mathbb{R}, \forall v \in V$.

## Definition

Let $M$ be a real orientable differential manifold of dimension $2n$. An almost complex structure on $M$ is a bundle map $J : TM \to TM$ satisfying $J^2 = -id$.

A complex manifold $X$ has a natural almost complex structure: just define $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$, $\quad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$. Then $(T_p^{\mathbb{R}}X, J_p)$ can be viewed as a $\mathbb{C}$-vector space, isomorphic to the holomorphic tangent space $T_p^h X$, identifying $\frac{\partial}{\partial x_i}$ with $\frac{\partial}{\partial z_i}$.

# Examples

If an almost complex structure is induced from a complex structure as above, we will call it "integrable".

## Example

For $S^2$, we can define $J : TS^2 \to TS^2$ as follows: we identify $T_xS^2$ with the subspace of $\mathbb{R}^3$:

$$T_xS^2 \cong \{y \in R^3 | \ x \cdot y = 0\}.$$

Then we define $J_x : T_xS^2 \to T_xS^2$ by

$$J_x(y) := x \times y.$$

On can check that this is an integrable almost complex structure, induced by the complex structure of $S^2 \cong \mathbb{C}P^1$.

# An almost complex structure on $S^6$

## Example

For $S^6$, we have a similar almost complex structure given by "wedge product" in $\mathbb{R}^7$. Note that the wedge product in $\mathbb{R}^3$ can be defined as the product of purely imaginary quaternions. To define this wedge product in $\mathbb{R}^7$, we shall use Cayley's octonions.

We write $\mathbb{H} \cong \mathbb{R}^4$ the space of quaternions $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$, satisfying $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. Then this multiplication is still associative but not commutative. For $q \in \mathbb{H}$, we define $\bar{q} := a - bi - cj - dk$, then $|q|^2 = q\bar{q}$.

Now we define the space of octonians, $\mathbb{O} \cong \mathbb{R}^8$, as $\mathbb{O} := \{x = (q_1, q_2) | \, q_1, q_2 \in \mathbb{H}\}$. The multiplication is defined by

$$(q_1, q_2)(q_1', q_2') := (q_1 q_1' - \bar{q}_2' q_2, q_2' q_1 + q_2 \bar{q}_1').$$

## Example ($S^6$ continued)

We also define $\bar{x} := (\bar{q}_1, -q_2)$. Then we still have $x\bar{x} = x \cdot x = |x|^2$, here the $\cdot$ means the usual inner product in $\mathbb{R}^8$. Note that this multiplication is even not associative.

We identify $\mathbb{R}^7$ as the space of purely imaginary octonians. If $x, x' \in \mathbb{R}^7$, we define $x \times x'$ as the imaginary part of $xx'$. Then one can check that $xx = -|x|^2$, $x \times x' = -x' \times x$, and $(x \times x') \cdot x'' = x \cdot (x' \times x'')$.

From this, one can define an almost complex structure on $S^6 \subset \mathbb{R}^7$ in a similar way as $S^2$: identify $T_x S^6$ with $\{y \in \mathbb{R}^7 | \, x \cdot y = 0\}$, then define

$$J_x(y) := x \times y.$$

One can prove that this almost complex structure is not integrable. (Ref: Calabi: Construction and properties of some 6-dimensional almost complex manifolds )

# A famous open problem

## Remark

For $S^{2n}$, it is known (Borel-Serre, Ehresmann, Wu) that there are no almost complex structures unless $n = 1, 3$. (Ref.: P. May's Concise course in algebraic topology). It is generally believed that there are no integrable almost complex structures on $S^6$, however S.T. Yau has a different conjecture saying that one can make $S^6$ into a complex manifold. This is still open.

Interested readers can visit the journal Differential Geometry and its Applications Vol. 57, 2018 for a set of survey papers on this problem.

# The “extrinsic” way

The second approach also uses $J$. Let again $V$ be a real vector space with complex structure $J$. But now we simply complexify $V$ to get $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. We also extend $J$ $\mathbb{C}$-linearly to $V_{\mathbb{C}}$, again $J^2 = -id$. There is a direct sum decomposition of $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, which are $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $J$ respectively. In fact we have a very precise description of $V^{1,0}$ and $V^{0,1}$:

$$V^{1,0} = \{v - \sqrt{-1}Jv |\ v \in V\}, \quad V^{0,1} = \{v + \sqrt{-1}Jv |\ v \in V\}.$$

It is direct to check that they are both $\mathbb{C}$-linear subspaces of $V_{\mathbb{C}}$ and $V^{0,1} = \overline{V^{1,0}}$.

Now apply this to $(T^{\mathbb{R}}X, J)$ for a manifold with an almost complex structure: define the complexified tangent bundle to be $T^{\mathbb{C}}X := T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ and we have the decomposition $T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$, which are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $J$, respectively. When $J$ is integrable, $T^{1,0}X$ is locally generated by $\{\frac{\partial}{\partial z_i}\}_{i=1}^{n}$, so we can again identify it with $T^hX$.

# Complex differential forms

We define $T^{*1,0}X$ to be the subspace of $T^{*\mathbb{C}}X := T^{*\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ that annihilates $T^{0,1}X$. And similarly define $T^{*0,1}X$. Then

$$T^{*\mathbb{C}}X = T^{*1,0}X \oplus T^{*0,1}X.$$

(When $J$ is integrable, $T^{*1,0}X$ is locally generated by $\{dz_i\}_{1\leq i\leq n}$ and $T^{*0,1}X$ is generated by $\{d\bar{z}_i\}_{1\leq i\leq n}$. )We define $\Lambda^{p,q}T^*X$, the $C^\infty$ bundle of $(p,q)$-forms to be the sub-bundle of $\Lambda^{p+q}T^{*\mathbb{C}}X$, generated by $\Lambda^p T^{*1,0}X$ and $\Lambda^q T^{*0,1}X$. Then we have

$$\Lambda^k T^{*\mathbb{C}}X = \bigoplus_{p=0}^{k} \Lambda^{p,k-p}T^*X,$$

and we denote the projection map of $\Lambda^{p+q}T^{*\mathbb{C}}X$ onto $\Lambda^{p,q}T^*X$ by $\Pi_{p,q}$. The set of smooth sections of $\Lambda^{p,q}T^*X$ (or $\Lambda^k T^{*\mathbb{C}}X$) over an open set $U$ is denoted by $A^{p,q}(U)$ (or $A^k(U)$).

# The operators $\partial$ and $\bar{\partial}$

The exterior differential operator $d$ extends $\mathbb{C}$-linearly to $d : A^k(U) \to A^{k+1}(U)$. We define the operators

$$\partial := \Pi_{p+1,q} \circ d : A^{p,q}(U) \to A^{p+1,q}(U),$$

and

$$\bar{\partial} := \Pi_{p,q+1} \circ d : A^{p,q}(U) \to A^{p,q+1}(U).$$

# Integrable case

When $J$ is integrable, a smooth section of $\Lambda^{p,q}T^*X$ over a coordinate open set $U$ is of the forms

$$\eta = \sum_{1\le i_1<\cdots<i_p\le n, 1\le j_1<\cdots<j_q\le n} a_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q},$$

where $a_{i_1\ldots i_p,\bar{j}_1\ldots\bar{j}_q}\in C^\infty(U;\mathbb{C})$. We write $\eta=\sum_{|I|=p,|J|=q} a_{I\bar{J}}dz_I\wedge d\bar{z}_J\in A^{p,q}(U)$ for short.

In this case, we have

$$\begin{aligned} d\eta &= \sum_{I,J} da_{I\bar{J}}\wedge dz_I\wedge d\bar{z}_J \\ &= \sum_{I,J}\partial a_{I\bar{J}}\wedge dz_I\wedge d\bar{z}_J + \sum_{I,J}\bar{\partial} a_{I\bar{J}}\wedge dz_I\wedge d\bar{z}_J \in A^{p+1,q}(U)\oplus A^{p,q+1}(U). \end{aligned}$$

So we always have $d=\partial+\bar{\partial}$ in the integrable case.

# The Newlander-Nirenberg Theorem

**Theorem (Newlander-Nirenberg)**

An almost complex structure is integrable if and only if $d = \partial + \bar{\partial}$ for any $A^{p,q}(U)$ (equivalently, $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$).

- Note that on 0-forms (smooth functions) $d = \partial + \bar{\partial}$ always holds. The first non-trivial situation is on 1-forms. Since a $k$-forms are linear combinations of wedge products of 1-forms, it also suffices to check $d = \partial + \bar{\partial}$ on 1-forms.
- Besides the original proof of Newlander-Nirenberg, there is another proof by J.J. Kohn based on techniques for solving the "$\bar{\partial}$-equation", which can be found in Hörmander's book.

# Dolbeault cohomology

In the following, we always assume $X$ is a complex manifold. Now $d = \partial + \bar{\partial}$. Since we always have $d^2 = 0$, we have $0 = \partial^2 + \bar{\partial}^2 + (\partial\bar{\partial} + \bar{\partial}\partial)$, acting on $A^{p,q}(X)$.
Comparing types, we get $\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0$.
We can define from these identities several differential cochain complexes:

- The de Rham complex $0 \to A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \dots \xrightarrow{d} A^{2n}(X) \to 0$. From this we can define the de Rham cohomology (with coefficient $\mathbb{C}$) $H^k_{dR}(X, \mathbb{C}) := Ker\ d|_{A^k(X)}/dA^{k-1}(X)$. Its dimension $b_k$ is called the "k-th Betti number" of $X$.

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Comparing types, we get $\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$
We can define from these identities several differential cochain complexes:

- The de Rham complex $0 \to A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \dots \xrightarrow{d} A^{2n}(X) \to 0$. From this we can define the de Rham cohomology (with coefficient $\mathbb{C}$) $H^k_{dR}(X, \mathbb{C}) := Ker\ d|_{A^k(X)}/dA^{k-1}(X)$. Its dimension $b_k$ is called the "k-th Betti number" of $X$.
- The Dolbeault complex $0 \to A^{p,0}(X) \xrightarrow{\bar{\partial}} A^{p,1}(X) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} A^{p,n}(X) \to 0$. We define the Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(X) := Ker\ \bar{\partial}|_{A^{p,q}(X)}/\bar{\partial}A^{p,q-1}(X)$. Its dimension is denoted by $h^{p,q}$. They are important invariants of the complex manifold.