§2. Riemannian metric. Levi-Civita connection. Riem. curvature.

§2.1 Riem. metric.

- M^n: n-dim diff. mfd.
- $C^\infty(M) := \{ f: M \to \mathbb{R} \text{ , smooth function} \}$. $C^k(M)$: k-diff
- **Def 1** (Riem. metric). A Riem metric g on M, is given by an inner product $\langle \cdot, \cdot \rangle_p$ (or $g_p(\cdot,\cdot)$) on each T_pM, which varies smoothly in $p \in M$. That is,

$\langle X, Y \rangle \in C^\infty(M), \quad \forall X, Y \in \underbrace{\Gamma(TM)}_{\text{smooth sections of } \pi: TM \to M}$. #.

- The Riem metric determines a positive, symmetric (2,0)-tensor

$$g \in \Gamma(T^*M \otimes T^*M), \quad g(u,v) = \langle u, v \rangle_p, \quad \forall u, v \in T_pM.$$

g is called Riem. metric.

- In local coord. $\varphi: \overset{\subseteq \mathbb{R}^n}{U} \longrightarrow M$, $x = (x^1, \cdots, x^n)$, put:

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = g_{ji}.$$

Then $(g_{ij}^{(p)})_{n \times n}$ is positive, symmetric matrix, varies smoothly on M, and

$\forall X = X^i \cdot \frac{\partial}{\partial x^i}, \; Y = Y^j \cdot \frac{\partial}{\partial x^j} \in \Gamma(TM)$.

$$g(X,Y) = g_{ij} \cdot X^i \cdot Y^j$$

$g\left(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right)$

$= X^i Y^j \cdot g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$.

- Locally, $g = g_{ij} \cdot dx^i \otimes dx^j$.

· **Exp 2**: Euclidean Space $\mathbb{R}^n$ $(x^1,\dots,x^n)$, $g_{ij}=\delta_{ij}$,

$$g=\delta_{ij}\cdot dx^i\otimes dx^j = \sum_{i=1}^{n} dx^i\otimes dx^i .$$

\#

· **Lemma 3** (Existence). Any diff. mfd M admits a Riem. metric.

· **Proof**: · Take a countable, locally finite coord. covering $\{(U_\alpha \subseteq \mathbb{R}^n, \varphi_\alpha)\}$.

· Let $\{f_\alpha\}$ be a partition of unity subordinated to $\{(U_\alpha,\varphi_\alpha)\}$.

$\sum_\alpha f_\alpha = 1$

· On each $U_\alpha \underset{\mathbb{R}^n}{\cap}$, let g_α be the Euclidean metric on U_α. Then put:

$$g=\sum_\alpha f_\alpha \cdot g_\alpha .$$

$\boxed{///}$

· **Norm**: $|v|_g := \sqrt{\langle v,v\rangle} = \sqrt{g(v,v)}$, $\forall\, v\in T_pM$.

· Let $\gamma: [a,b] \to M$ $[0,l]$ be a diff. curve. C^∞- curve define its length by:

$$L(\gamma)=\int_a^b |\dot\gamma(t)|\,dt, \qquad \dot\gamma(t) := d\gamma_t\left(\frac{\partial}{\partial t}\right).$$

$\dot\gamma$, γ'

$\gamma:[a,b]\to M$

$d\gamma_t : T_t[a,b] \to T_{\gamma(t)}M$

$\frac{\partial}{\partial t}$

· §2.2. Levi-Civita connection.

· **Prop. 4** (Def. of Levi-Civita connection).

· On any Riem. mfd (M,g), $\exists!$ map:

gradient $\quad \nabla: \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$, $\quad (X,Y)\mapsto \nabla_X Y$.

satisfying: $\forall\, X,Y,Z\in\Gamma(TM)$, $f, h \in C^\infty(M)$,

(i). ①. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

②. $\nabla_{fX+hY} Z = f\cdot\nabla_X Z + h\cdot \nabla_Y Z$.

} linearity.

③. $\nabla_X(fY) = f\cdot\nabla_X Y + X(f)\cdot Y$. chain rule.

(ii). $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ compatible with metric.

(iii). $\nabla_X Y - \nabla_Y X = [X, Y]$. torsion free.

- The map ∇ is called Levi-Civita connection.

- **Proof**: - We define ∇ by: $\forall\ X, Y, Z \in \Gamma(TM)$,

(1). $$2\langle\nabla_X Y, Z\rangle := X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle.$$

- How to see (1): If we apply (ii), (iii) then: $X \leftarrow Y,\ Y \leftarrow Z,\ Z \leftarrow X$.

$$\left.\begin{array}{ll}(2) & X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \\ (3) & Y\langle Z, X\rangle = \langle\nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle. \\ (4). & Z\langle X, Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.\end{array}\right\}$$

apply (iii), (2)+(3)−(4) gives (1).

- We check (i) ③ for example:

  - Recall: $[fX, hY] = fh\cdot[X, Y] + f\cdot X(h)\cdot Y - h\cdot Y(f)\cdot X$.

  - then we can compute: $X(f\langle Y, Z\rangle)$

$$2\langle\nabla_X(fY), Z\rangle \overset{(1)}{=} X\langle fY, Z\rangle + fY\cdot\langle Z, X\rangle - Z\cdot\langle X, fY\rangle - \langle X, [fY, Z]\rangle + \langle fY, [Z, X]\rangle + \langle Z, [X, fY]\rangle$$

$$= \{X(f)\langle Y, Z\rangle + f\cdot X\langle Y, Z\rangle\} + f\cdot Y\langle Z, X\rangle - \{Z(f)\langle X, Y\rangle + f\cdot Z\langle X, Y\rangle\} - \langle X, f\cdot[Y, Z] - Z(f)\cdot Y\rangle + f\cdot\langle Y, [Z, X]\rangle + \langle Z, f\cdot[X, Y] + X(f)\cdot Y\rangle$$

$$\overset{(1)}{=} 2f\cdot\langle\nabla_X Y, Z\rangle + 2X(f)\cdot\langle Y, Z\rangle$$

$$= \langle 2f\cdot\nabla_X Y + 2X(f)\cdot Y,\ Z\rangle.$$

$$\Rightarrow\quad \nabla_X(fY) = f\cdot\nabla_X Y + X(f)\cdot Y. \qquad \blacksquare$$

- Calculate in local coord.: $\varphi: U \xrightarrow{\subseteq\mathbb{R}^n} M$, $x = (x^1, \dots, x^n)$.

  - Christoffel Symbols: $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}\cdot\frac{\partial}{\partial x^k}$ $\in \Gamma(TM)$.

  - For $X = X^i\cdot\frac{\partial}{\partial x^i}$, $Y = Y^j\cdot\frac{\partial}{\partial x^j}$,

$$\nabla_X Y = \nabla_{X^i\cdot\frac{\partial}{\partial x^i}}\left(Y^j\cdot\frac{\partial}{\partial x^j}\right) = X^i\cdot\nabla_{\frac{\partial}{\partial x^i}}\left(Y^j\cdot\frac{\partial}{\partial x^j}\right).$$

$$= X^i\cdot\left(\frac{\partial Y^j}{\partial x^i}\cdot\frac{\partial}{\partial x^j} + Y^j\cdot\Gamma^k_{ij}\cdot\frac{\partial}{\partial x^k}\right)$$

$$= X^i\cdot\left(\frac{\partial Y^k}{\partial x^i} + Y^j\cdot\Gamma^k_{ij}\right)\cdot\frac{\partial}{\partial x^k}.$$

- **Def. 5 (Covariant derivative).** • $X \in \Gamma(TM)$, $v \in T_pM$,

  define $\nabla_v X$ as follows:

  - Extend v to a vector field $V \in \Gamma(TM)$, then put:

$$\nabla_v(X) := \nabla_V X\,(p) \in T_pM. \qquad \#.$$

- Let $\gamma: I \to M$ ($[0,1]$) be a diff. curve, $X \in \Gamma(TM)$.

  Then $\nabla_{\dot\gamma} X$ depend only on values of X along γ, since:

$$\langle \nabla_{\dot\gamma}X, Y\rangle = \dot\gamma\,\underbrace{\langle X, Y\rangle}_{\text{function.}} - \langle X, \nabla_{\dot\gamma}Y\rangle$$

$$= \frac{d}{dt}\langle X, Y\rangle(\gamma(t)) - \langle X, \nabla_{\dot\gamma}Y\rangle.$$

- **Lemma 6.** • In local coord. $\varphi: U \xrightarrow{\subseteq\mathbb{R}^n} M$, $x = (x^1, \dots, x^n)$.

$$\begin{cases} \Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \cdot \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \\ \partial_k g_{ij} = g_{il} \cdot \Gamma_{jk}^{l} + g_{jl} \cdot \Gamma_{ik}^{l}. \end{cases}$$

where $(g^{ij})_{n\times n} = (g_{ij})_{n\times n}^{-1}$ denotes the inverse matrix.

• Proof: • $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial^2}{\partial x^j \partial x^i} \equiv 0.$

• Then, from Eqn. (1).

$$\left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l} \right\rangle = \frac{1}{2} \cdot \left\{ \frac{\partial}{\partial x^i} \underbrace{\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l} \right\rangle}_{= g_{jl}} + \frac{\partial}{\partial x^j} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^l} \right\rangle}_{= g_{il}} - \frac{\partial}{\partial x^l} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle}_{= g_{ij}} \right\}.$$

$$= \left\langle \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle = g_{kl} \cdot \Gamma_{ij}^{k} \, .$$

$$g_{kl} \cdot \Gamma_{ij}^{k} = \frac{1}{2} \cdot \left\{ \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right\}. \quad \Rightarrow \text{1st identity.}$$

• $2 g_{kl} \cdot \Gamma_{ij}^{k} = \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}$
 $+$
 $j \leftrightarrow l$: $2 g_{kj} \cdot \Gamma_{il}^{k} = \partial_i g_{jl} + \partial_l g_{ij} - \partial_j g_{il}$

$\Rightarrow$ 2nd identity. $\square$

• §2.3. Riem. curvature.

• Def 7. (Riem. curvature). Define:

$$R: \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM),$$

$$(X, Y, Z) \mapsto R(X,Y)Z := \nabla_X (\nabla_Y Z) - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

• Or:

$$Rm: \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M).$$

$$(X, Y, Z, W) \mapsto Rm(X, Y, \underline{Z, W}) := \langle R(X,Y)W, Z \rangle$$

Prop 8. (i). R and Rm is C^∞- linear in each variable.

(ii). $Rm(X,Y,Z,W) = -Rm(Y,X,Z,W)$

$$= -Rm(X,Y,W,Z) = Rm(Z,W,X,Y).$$

(iii). (1st Bianchi identity).

$\boxed{X \mapsto Y,\ Y \mapsto Z,\ Z \mapsto W.}$

- $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$
- $Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0.$

Proof:

(i). $f \in C^\infty(M)$, then:

$$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX} Z - \nabla_{[fX,Y]} Z$$

$$= f\cdot\nabla_X\nabla_Y Z - \nabla_Y(f\cdot\nabla_X Z) - \nabla_{f\cdot[X,Y]-Y(f)\cdot X} Z$$

$$= f\cdot\nabla_X\nabla_Y Z - \left(f\cdot\nabla_Y\nabla_X Z + \cancel{Y(f)\cdot\nabla_X Z}\right) - \left(f\cdot\nabla_{[X,Y]} Z - \cancel{Y(f)\cdot\nabla_X Z}\right)$$

$$= f\cdot R(X,Y)Z.$$

(ii). We can compute:

$\langle\nabla_X\nabla_Y W, Z\rangle = X\langle\nabla_Y W, Z\rangle - \langle\nabla_Y W, \nabla_X Z\rangle$

$$Rm(X,Y,Z,W) + Rm(X,Y,W,Z)$$

$$= \langle\nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W, Z\rangle + \langle\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle$$

$$= \left\{X\langle\nabla_Y W, Z\rangle - \cancel{\langle\nabla_Y W, \nabla_X Z\rangle}\right\} - \left\{Y\langle\nabla_X W, Z\rangle - \cancel{\langle\nabla_X W, \nabla_Y Z\rangle}\right\} - \langle\nabla_{[X,Y]}W, Z\rangle$$

$$+ \left\{X\langle\nabla_Y Z, W\rangle - \cancel{\langle\nabla_Y Z, \nabla_X W\rangle}\right\} - \left\{Y\langle\nabla_X Z, W\rangle - \cancel{\langle\nabla_X Z, \nabla_Y W\rangle}\right\} - \langle\nabla_{[X,Y]}Z, W\rangle$$

$$= XY\langle Z,W\rangle - YX\langle Z,W\rangle - [X,Y]\langle Z,W\rangle = 0.$$

(iii). $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$

• torsion free: $\nabla_X Y - \nabla_Y X = [X,Y]$

$$= \left(\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right) + \left(\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X\right)$$

$$+ \left(\nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y\right)$$

$$= \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$$

$$= 0 \longleftarrow \text{Jacobi identity.}$$

$\blacksquare$

• In local coord., denote:

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = R_{ij}{}^k{}_l \cdot \frac{\partial}{\partial x^k}.$$

then:

$$R_{ijkl} = Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = g_{ks} \cdot R_{ij}{}^s{}_l.$$

From Prop. 8, we have:

$i \leftarrow j,\ j \leftarrow l,\ l \leftarrow i$

$i \leftarrow j,\ j \leftarrow k,\ k \leftarrow i$

• $R_{ijkl} + R_{jkil} + R_{kijl} = 0.$

• $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$

• For vector fields, $Rm(X,Y,Z,W) = R_{ijkl} \cdot X^i \cdot Y^j Z^k W^l.$

• $C^\infty(M)$-linearity implies that R defines a $(3,1)$-tensor (Rm a $(4,0)$-tensor).

• **Def 9.** (Sectional curvature).

• Let $p \in M$, $P \subset T_pM$ be a plane, $u, v \in P$ linear indep. vectors.

• Define the sectional curvature of P:

$$K(\bar{P}) := \frac{Rm(u,v,u,v)}{|u\wedge v|^2}$$

here:

$$|u\wedge v|^2 = |u|^2\cdot|v|^2 - \langle u,v\rangle^2 = \det\begin{pmatrix} \langle u,u\rangle, & \langle u,v\rangle \\ \langle u,v\rangle, & \langle v,v\rangle \end{pmatrix}.$$

- $K(\bar{P})$ does not depend on the choice of u and v.

g_p inner product in T_pM

Proof: • Choose orthonormal basis $e_1, e_2 \in \bar{P}$, then write:

$$u = u_1 e_1 + u_2 e_2, \qquad v = v_1 e_1 + v_2 e_2.$$

. Then we can compute:

$$Rm(u,v,u,v) = Rm\left(u_1e_1+u_2e_2,\ v_1e_1+v_2e_2,\ u_1e_1+u_2e_2,\ v_1e_1+v_2e_2\right)$$

$$= R_{1212}\, u_1^2 v_2^2 + \underbrace{R_{1221}}_{=\,-R_{1212}} u_1u_2v_1v_2 + \underbrace{R_{2112}}_{=\,-R_{1212}} u_1u_2v_1v_2 + \underbrace{R_{2121}}_{=\,R_{1212}} u_2^2\cdot v_1^2$$

$$= R_{1212}\cdot\left\{ u_1^2v_2^2 + u_2^2v_1^2 - 2u_1u_2v_1v_2 \right\}.$$

$$= R_{1212}\cdot\left\{ (u_1^2+u_2^2)\cdot(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 \right\}.$$

$$= R_{1212}\cdot|u\wedge v|^2.$$

Hence $K(\bar{P})$ does not depend on the choice of u and v. $\square$

• **Lemma 10** We have:

s, t $\qquad \frac{\partial^2}{\partial s\partial t}(s) = 0.$ $\qquad \frac{\partial^2}{\partial s\partial t}(s^2\cdot t)\Big|_{s=0} = s\Big|_{s=0} = 0.$

$$6Rm(X,Y,Z,W) = \frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0}\Big(Rm(X+sZ, Y+tW, X+sZ, Y+tW) - Rm(X+sW, Y+tZ, X+sW, Y+tZ)\Big).$$

$Z \leftrightarrow W$.

• **Proof**: • By linearity, coefficient of $s\cdot t$ on the RHS is:

$$\mathrm{Rm}(Z,W,X,Y)+\mathrm{Rm}(Z,Y,X,W)+\mathrm{Rm}(X,W,Z,Y)+\mathrm{Rm}(X,Y,Z,W) \quad Z\leftrightarrow W$$

$$-\mathrm{Rm}(W,Z,X,Y)-\mathrm{Rm}(W,Y,X,Z)-\mathrm{Rm}(X,Z,W,Y)-\mathrm{Rm}(X,Y,W,Z)$$

$+\mathrm{Rm}(Z,W,X,Y)=0$

$Y\leftarrow Z,\ Z\leftarrow W,\ W\leftarrow Y$

$$=4\,\mathrm{Rm}(X,Y,Z,W)-\big(\mathrm{Rm}(Y,Z,X,W)+\mathrm{Rm}(W,Y,X,Z)\big) \quad =-\mathrm{Rm}(Z,W,X,Y).$$

$$-\big(\mathrm{Rm}(W,X,Z,Y)+\mathrm{Rm}(W,Y,X,Z)\big). \quad =-\mathrm{Rm}(W,Z,Y,X).$$

$\boxed{\text{1st Bianchi identity.}}$

$$=6\,\mathrm{Rm}(X,Y,Z,W).$$

- All other terms are zero under $\left.\dfrac{\partial^2}{\partial s\partial t}\right|_{s=t=0}$. ∎

• **Corollary 11.** If for some $K_0\in\mathbb{R}$, $K\equiv K_0$ (sectional) at $p\in M$, then:

$$\mathrm{Rm}(X,Y,Z,W)=K_0\cdot\{\langle X,Z\rangle\cdot\langle Y,W\rangle-\langle X,W\rangle\cdot\langle Y,Z\rangle\}.$$

Proof: • From the def. of sectional curvature K, we have:

$$\mathrm{Rm}(X,Y,X,Y)=K_0\cdot|X\wedge Y|^2=K_0\cdot\big(|X|^2\cdot|Y|^2-\langle X,Y\rangle^2\big).$$

• Hence, the RHS of Lemma 10 is:

$$\mathrm{Rm}(X+sZ,Y+tW,X+sZ,Y+tW)-\mathrm{Rm}(X+sW,Y+tZ,X+sW,Y+tZ)$$

$$=K_0\cdot\{|X+sZ|^2\cdot|Y+tW|^2-\langle X+sZ,Y+tW\rangle^2\}$$

$Z\leftrightarrow W$

$$-K_0\cdot\{|X+sW|^2\cdot|Y+tZ|^2-\langle X+sW,Y+tZ\rangle^2\}.$$

- $|X+sZ|^2\cdot|Y+tW|^2=\big(|X|^2+2s\langle X,Z\rangle+s^2\cdot|Z|^2\big)\cdot\big(|Y|^2+2t\cdot\langle Y,W\rangle+t^2\cdot|W|^2\big).$
- $\langle X+sZ,Y+tW\rangle^2=\big(\langle X,Y\rangle+t\cdot\langle X,W\rangle+s\cdot\langle Z,Y\rangle+st\cdot\langle W,Z\rangle\big)^2.$

$$= s\cdot t\cdot \Big[4\langle X,Z\rangle\cdot\langle Y,W\rangle - 2\cdot\langle X,W\rangle\cdot\langle Y,Z\rangle - 2\cdot\langle X,Y\rangle\cdot\langle W,Z\rangle \quad \updownarrow\ Z\leftrightarrow W.$$

$$- 4\langle X,W\rangle\cdot\langle Y,Z\rangle + 2\cdot\langle X,Z\rangle\cdot\langle Y,W\rangle + 2\cdot\langle X,Y\rangle\cdot\langle W,Z\rangle \Big]$$

$$+ \text{other order terms}$$

$$= s\cdot t\cdot \Big[6\cdot\langle X,Z\rangle\cdot\langle Y,W\rangle - 6\cdot\langle X,W\rangle\cdot\langle Y,Z\rangle \Big] + \text{other order terms.}$$

$$=: f(s,t).$$

- Hence from Lemma 10,

$$6\,Rm(X,Y,Z,W) = \frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0} f(s,t)$$

$$= 6\cdot\Big(\langle X,Z\rangle\cdot\langle Y,W\rangle - \langle X,W\rangle\cdot\langle Y,Z\rangle \Big).$$

$\boxed{///}$

• Prop.12. In local coord. $x=(x^1,\cdots,x^n)$, we have

$R_{ijkl} = g_{ks}\cdot R_{ij}{}^{s}{}_{l}$

~~$R_{ijkl} = g^{ks}$~~

$$\cdot\ R_{ij}{}^{k}{}_{l} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^r_{jl}\cdot\Gamma^k_{ir} - \Gamma^r_{il}\cdot\Gamma^k_{jr}$$

$$\cdot\ R_{ijkl} = -\frac{1}{2}\cdot\left(\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{il}}{\partial x^j\partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i\partial x^l} \right)$$

$$- g_{rs}\cdot\Big(\Gamma^r_{ik}\cdot\Gamma^s_{jl} - \Gamma^r_{il}\cdot\Gamma^s_{jk} \Big).$$

• Proof: • From $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]=0$, we have

$$R_{ij}{}^{k}{}_{l}\cdot\frac{\partial}{\partial x^k} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}$$

$$= \nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}$$

$$= \nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^k_{jl}\cdot\frac{\partial}{\partial x^k}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^k_{il}\cdot\frac{\partial}{\partial x^k}\right)$$

$$= \frac{\partial \Gamma^k_{jl}}{\partial x^i}\cdot\frac{\partial}{\partial x^k} + \Gamma^k_{jl}\cdot \underset{k \leftrightarrow r}{\Gamma^r_{ik}}\cdot\frac{\partial}{\partial x^r} - \frac{\partial \Gamma^k_{il}}{\partial x^j}\cdot\frac{\partial}{\partial x^k} - \Gamma^k_{il}\cdot \underset{k \leftrightarrow r}{\Gamma^r_{jk}}\cdot\frac{\partial}{\partial x^r}$$

$$= \left(\frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^r_{jl}\cdot\Gamma^k_{ir} - \Gamma^r_{il}\cdot\Gamma^k_{jr} \right)\cdot \frac{\partial}{\partial x^k}.$$

- Next,

$$R_{ijkl} = Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$= \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right\rangle$$

$$= \left\langle \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right\rangle$$

$\downarrow$ compatible

$$= \frac{\partial}{\partial x^i}\left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right\rangle - \frac{\partial}{\partial x^j}\left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right\rangle \quad \text{------ ①}$$

$$- \left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right\rangle \quad \text{---- ②}$$

- ② is Γ-terms.

$$② = -\Gamma^r_{jl}\cdot\Gamma^s_{ik}\cdot g_{rs} + \Gamma^r_{il}\cdot\Gamma^s_{jk}\cdot g_{rs}.$$

- $$① = \frac{\partial}{\partial x^i}\left\langle \Gamma^r_{jl}\cdot\frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^k} \right\rangle - \frac{\partial}{\partial x^j}\left\langle \Gamma^r_{il}\cdot\frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^k} \right\rangle$$

$$= \frac{\partial}{\partial x^i}\left(g_{kr}\cdot\Gamma^r_{jl} \right) - \frac{\partial}{\partial x^j}\left(g_{kr}\cdot\Gamma^r_{il} \right)$$

$$= \frac{\partial}{\partial x^i}\frac{1}{2}\cdot\left(\partial_l g_{jk} + \partial_j \underline{g_{lk}} - \partial_k g_{jl} \right) - \frac{\partial}{\partial x^j}\frac{1}{2}\cdot\left(\partial_l g_{ik} + \partial_i \underline{g_{lk}} - \partial_k g_{il} \right).$$

$$= \frac{1}{2}\cdot\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} \right).$$

• Ricci curvature: Let $\{e_i\}_{i=1}^n$ be an orthonormal basis. Define:

$$Ric(X,Y) := \sum_{i=1}^n Rm(X, e_i, Y, e_i)$$

• In local coord $x = (x^1, \dots, x^n)$,

$$Ric(X,Y) = g^{ij} \cdot Rm\left(X, \frac{\partial}{\partial x^i}, Y, \frac{\partial}{\partial x^j}\right).$$

• Write:

$$R_{ij} = Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g^{pq} \cdot R_{ipqj}. \qquad \#$$

• A Riem. mfd (M, g) satisfying:

$$Ric \equiv \lambda \cdot g, \qquad \lambda \in \mathbb{R}.$$

is called an Einstein mfd.

• Scalar curvature: $S := g^{ij} \cdot R_{ij}$.

• Exp 13: Euclidean space $\mathbb{R}^n$:

$$g_{ij} = \delta_{ij}, \quad \Gamma_{ij}^k = 0, \quad Rm \equiv 0. \qquad \#$$

• $f \in C^\infty(M)$:

• ∇f: gradient field of f, defined by:

$$\langle \nabla f, X \rangle = X(f), \quad \forall X \in \Gamma(TM)$$

Locally, $\nabla f = g^{ij} \cdot \frac{\partial f}{\partial x^i} \cdot \frac{\partial}{\partial x^j}$. $\qquad df = \frac{\partial f}{\partial x^i} dx^i, \quad |\nabla f|^2 = |df|^2$

• $\mathrm{Hess}(f)$: Hessian tensor, defined by:

$$\mathrm{Hess}(f)(X,Y) = X(Yf) - (\nabla_X Y)f, \quad \forall X, Y \in \Gamma(TM).$$

.. $\mathrm{Hess}(f)$ is C^∞-linear in X, Y.

.. $\mathrm{Hess}(f)$ is symmetric in X, Y : $\mathrm{Hess}(f)(X,Y) = \mathrm{Hess}(f)(Y,X)$.

$$\mathrm{Hess}(f)(X,Y) - \mathrm{Hess}(f)(Y,X)$$

$$= X(Yf) - Y(Xf) - (\nabla_X Y - \nabla_Y X)f \overset{\text{torsion free.}}{=} 0.$$

.. Locally, we write:

$$\boxed{\nabla_i := \nabla_{\frac{\partial}{\partial x^i}}}$$

$$\mathrm{Hess}(f) = \nabla_i \nabla_j f \cdot dx^i \otimes dx^j,$$

where: $$\nabla_i \nabla_j f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \cdot \frac{\partial f}{\partial x^k}.$$

• Δf : $$\Delta f := \mathrm{tr}_g \mathrm{Hess}(f) = g^{ij} \cdot \nabla_i \nabla_j f.$$

• Exp 14 (Conformal change).

• g: Riem. metric, $f \in C^\infty(M)$, $\tilde{g} := e^f \cdot g$ is also a Riem. metric.

• In local coord. $x = (x^1, \cdots, x^n)$, $g_{ij}, \tilde{g}_{ij}, \Gamma_{ij}^k, \tilde{\Gamma}_{ij}^k, Rm, \widetilde{Rm}, \cdots$

• $$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2} \cdot \left(f_i \cdot \delta_{jk} + f_j \cdot \delta_{ik} - f_l \cdot g^{kl} \cdot g_{ij} \right).$$

• $$\widetilde{Rm} = e^f \cdot \left[Rm - \frac{1}{2} \cdot \left(\mathrm{Hess}(f) - \frac{df \otimes df}{2} \right) ⓧ g - \frac{|\nabla f|^2}{8} \cdot g ⓧ g \right].$$

here, ⓧ is the Kulkani–Nomizu product:

for symmetric $(2,0)$ tensors α, β, define $\alpha ⓧ \beta$ to be the $(4,0)$-tensor:

$$\alpha \wedge \beta\,(u, v, w, z) = \alpha(u, w)\cdot\beta(v, z) - \alpha(u, z)\cdot\beta(v, w)$$
$$+\,\alpha(v, z)\cdot\beta(u, w) - \alpha(v, w)\cdot\beta(u, z).$$

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