§1. Differential manifold.

§§1. Definition.

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- Schoen–Yau. 微分几何讲义.
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Def. 1. (n-dim. diff. mfd).

A set M, together with a family of injective mappings $\varphi_\alpha : U_\alpha \to M$, $U_\alpha \subset \mathbb{R}^n$, open, s.t.

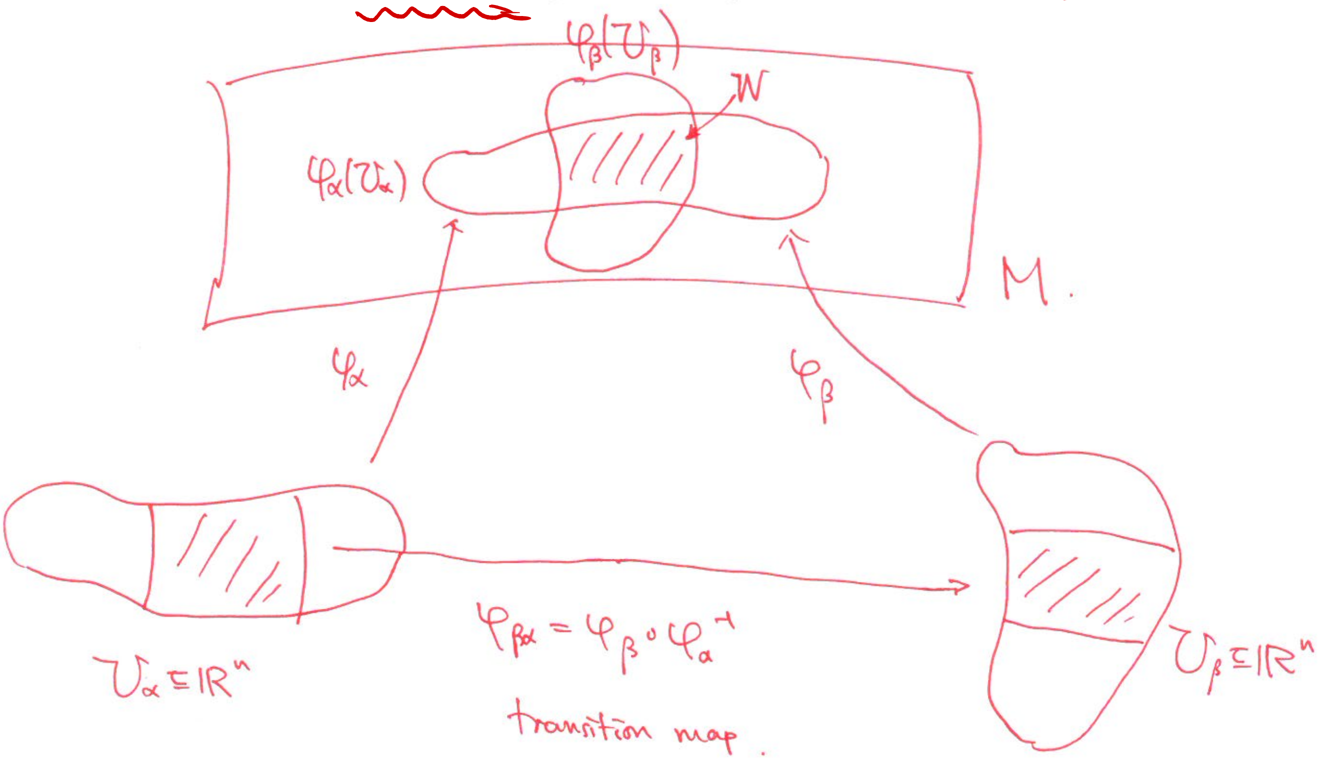
(i). $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$.

(ii). If $W = \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \phi$, then:

① $\varphi_\alpha^{-1}(W), \varphi_\beta^{-1}(W) \subset \mathbb{R}^n$ are open.

② the transition mappings $\varphi_{\beta\alpha} := \varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(W) \to \varphi_\beta^{-1}(W)$ is differentiable. (C^∞).

(iii). The family $\{(U_\alpha, \varphi_\alpha)\}$ satisfying (i) + (ii) is maximal.



• Such $\{(U_\alpha, \varphi_\alpha)\}$ in the def. is called a differential structure on M. #

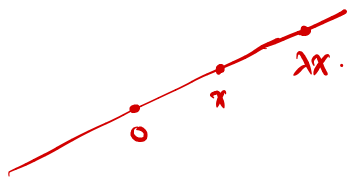
• Remark 2. • Given any family of coord.s $\{(U_\alpha, \varphi_\alpha)\}$ satisfying (i) & (ii) in Def. 1, $\exists$ differential structure compatible with each $(U_\alpha, \varphi_\alpha)$.

• The smallest compatible diff. structure is called the diff. structure determined by $\{(U_\alpha, \varphi_\alpha)\}$.

$Id: \mathbb{R}^n \to \mathbb{R}^n$. $B(0,1) \hookrightarrow \mathbb{R}^n$

• Example 3: Open subsets of $\mathbb{R}^n$, with canonical diff. structure induced from $\mathbb{R}^n$.

• Example 4. (Real projective space $\mathbb{RP}^n$).



• As set: $\mathbb{RP}^n = \{$lines of $\mathbb{R}^{n+1}$, passing through $0\}$

$$= \mathbb{R}^{n+1} \setminus \{0\} / \sim, \quad x \sim \lambda x, \ \lambda \in \mathbb{R}, \ \lambda \neq 0, \ x \in \mathbb{R}^{n+1} \setminus \{0\}.$$

• A point of $\mathbb{RP}^n$ is denoted by $[x^1, \dots, x^{n+1}] = [\lambda \cdot x^1, \dots, \lambda \cdot x^{n+1}]$. $\lambda \in \mathbb{R}$, $\neq 0$.

• Homogeneous coord.s: chart $V_i := \{[x^1, \dots, x^{n+1}] \mid x^i \neq 0\}$. $i = 1, \dots, n+1$.

$1 \le i \le n+1$

$$\varphi_i : \mathbb{R}^n \to V_i$$

$\in \mathbb{R}^{n+1} \setminus \{0\}$

$$x = (x^1, \dots, x^n) \longmapsto [x^1, \dots, x^{i-1}, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, x^i, \dots, x^n].$$

• Transitions: $\forall\ i > j$,

$$V_i \cap V_j = \{[x^1, \dots, x^{n+1}] \mid x^i \neq 0, \ x^j \neq 0\}.$$

$\forall \quad x = (x^1, \dots, x^n), \qquad x^i \neq 0, \ x^j \neq 0,$

$$\varphi_j^{-1}\circ\varphi_i(x) = \varphi_j^{-1}\left([x^1, \dots, x^{i-1}, \overset{i\text{-th}}{1}, x^i, \dots, x^n]\right)$$

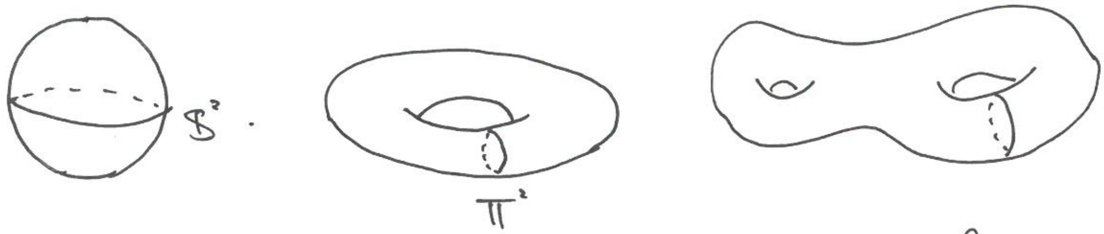
$$= \varphi_j^{-1}\left(\left[\frac{x^1}{x^j}, \dots, \frac{x^{j-1}}{x^j}, \overset{j\text{-th}}{1}, \frac{x^{j+1}}{x^j}, \dots, \frac{x^{i-1}}{x^j}, \overset{i\text{-th}}{\frac{1}{x^j}}, \frac{x^i}{x^j}, \dots, \frac{x^n}{x^j}\right]\right)$$

$$= \left(\frac{x^1}{x^j}, \dots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \dots, \frac{x^{i-1}}{x^j}, \frac{1}{x^j}, \frac{x^i}{x^j}, \dots, \frac{x^n}{x^j}\right).$$

• Exp 5 (1-dim). curves, line, circle. (all are flat).

S^1. $\mathbb{R}$.

• Exp 6 (2-dim). Uniformization Thm. orientable, closed.



S^2. T^2 ...

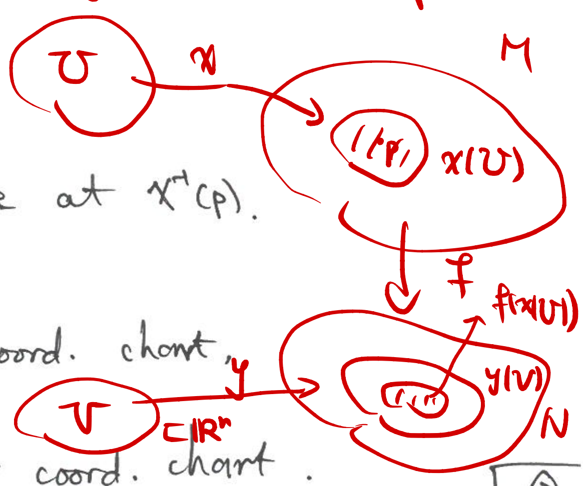
genus: 0 1 2. ...

• Def 7. (Diff. map) • M^m, N^n: diff. mfds. $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

• A map $f: M \to N$ is called differentiable at $p \in M$, if $\exists$ coord.s at p and $f(p)$, say $x: U \overset{\subseteq \mathbb{R}^m}{\longrightarrow} M$, $y: V \overset{\subseteq \mathbb{R}^n}{\longrightarrow} N$, s.t. $\underset{U}{\mathbb{R}^m}$ $\quad \varphi_\alpha: U_\alpha \to \varphi_\alpha(U_\alpha)$ $\quad p\in$

① $f(x(U)) \subseteq y(V)$.

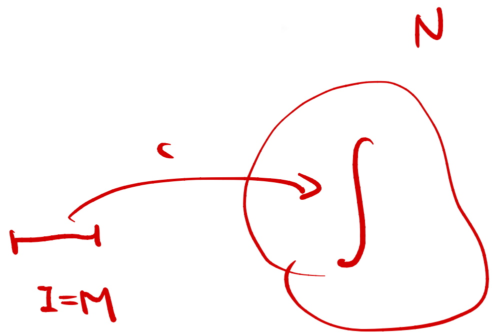
②. $y^{-1}\circ f\circ x: U \to V$ is differentiable at $x^{-1}(p)$.



• Rmk 8. • If f is diff. at p restricted on a coord. chart, then f is diff. at p restricted on any coord. chart.

• Exp 9. • diff. function $f: M \to \mathbb{R}$

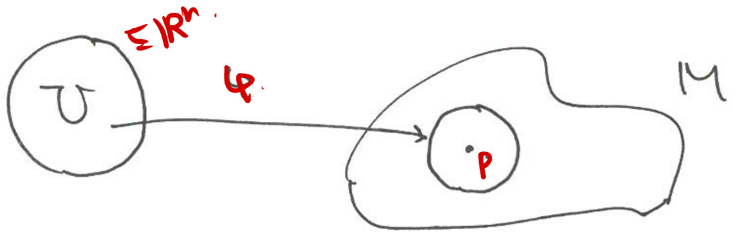
• diff. curves: $c: I \to M$, $I \subseteq \mathbb{R}$.



• Define: $p \in M^n$ $a_1 f_1 + a_2 f_2 \in D_p$, $a_1, a_2 \in \mathbb{R}$.

$$D_p := \{ f: M \to \mathbb{R} \,,\ \text{diff. at } p \in M \}$$

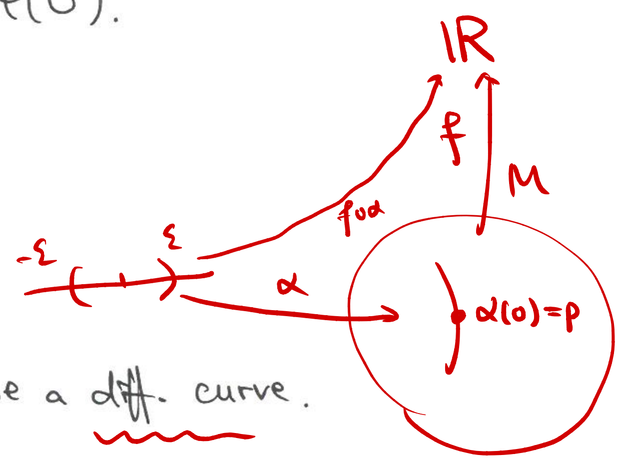
• $D_p \neq \phi$:



Take $\eta \in C_0^\infty(U)$, then: $\eta \in C^\infty(\mathbb{R}^n)$, $\eta|_{\mathbb{R}^n \setminus U} \equiv 0$.

$$f(y) := \begin{cases} \eta \circ \varphi^{-1}(y), & y \in \varphi(U) \\ 0, & y \notin \varphi(U). \end{cases}$$

then f is diff. on M.



§§4.2 Tangent Space.

Def. 10. • Let $\alpha: (-\varepsilon, \varepsilon) \to M$, $\alpha(0) = p$, be a diff. curve.

• The tangent vector at $t=0$ of α is a map:

$$\boxed{\alpha'(0)}: D_p \to \mathbb{R}, \qquad f \overset{\alpha'(0)}{\longmapsto} \frac{d}{dt}(f \circ \alpha)\Big|_{t=0}, \qquad f\circ\alpha: (-\varepsilon, \varepsilon) \to \mathbb{R}.$$

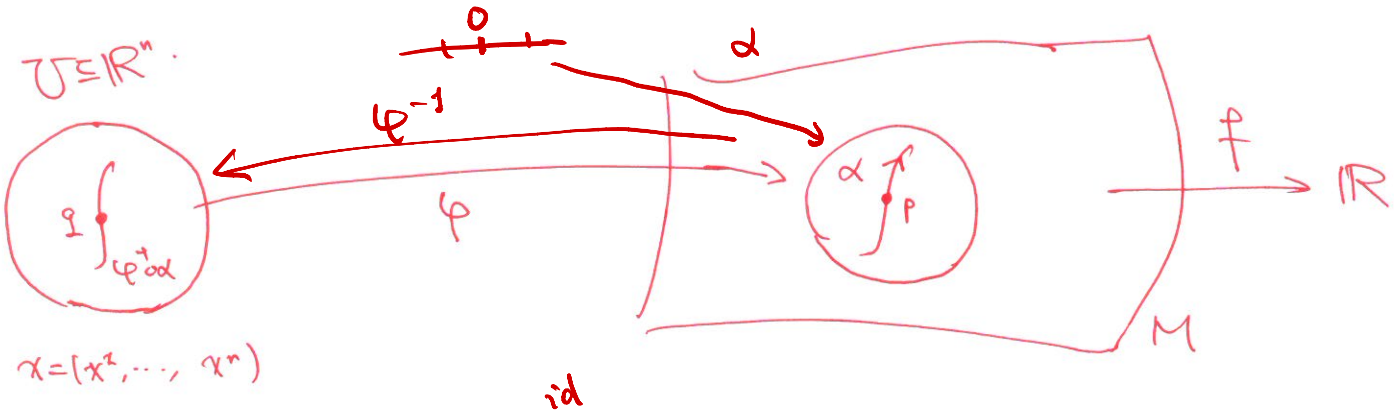
• Denote: $T_pM := \{ \alpha'(0) \mid \alpha: (-\varepsilon, \varepsilon) \to M, \text{ diff. curve}, \alpha(0) = p \}$.

• Each element in T_pM is called a tangent vector at p.

• D_p is a linear space over $\mathbb{R}$, $\alpha'(0): D_p \to \mathbb{R}$ is linear

i.e., $\alpha'(0) \in D_p^*$ - dual space of D_p.

• Representation in local coord. $\varphi: U \subseteq \mathbb{R}^n \to M$, coord. at p.



$$f\circ\alpha = (f\circ\varphi)\overset{\text{id}}{\circ}(\varphi^{-1}\circ\alpha).$$

write: $\varphi^{-1}\circ\alpha(t) = (x^1(t),\dots,x^n(t))$, $q := \varphi^{-1}(p) = (\varphi^{-1}\circ\alpha)(0)$, then:

(1). $$\alpha'(0)f = \frac{d}{dt}\Big|_{t=0}(f\circ\alpha) = \frac{d}{dt}\Big|_{t=0}(f\circ\varphi)\circ(\cancel{\varphi}\; x^1(t),\dots,x^n(t))$$

$$= \sum_{i=1}^{n} \frac{\partial(f\circ\varphi)}{\partial x^i}(q)\cdot(x^i)'(0).$$

• **Lemma II**: T_pM is a linear space of dim. n.

Proof: • Take a coord. $\varphi: U \to M$, $U \subseteq \mathbb{R}^n$, $p \in \varphi(U)$.

• Let $q = \varphi^{-1}(p)$, $x = (x^1,\dots,x^n)$ be coord. on $U \subseteq \mathbb{R}^n$.

• Define.

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[rr, bend right] & M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$\frac{\partial}{\partial x^i}: \overset{f_n}{D_p} \to \mathbb{R}, \qquad f \mapsto \frac{\partial}{\partial x^i}(f\circ\varphi)(q).$$

• Claim: $T_pM = \operatorname{Span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{n}$.

Proof: • By Eqn. (1), $T_pM \subseteq \operatorname{Span}\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^{n}$

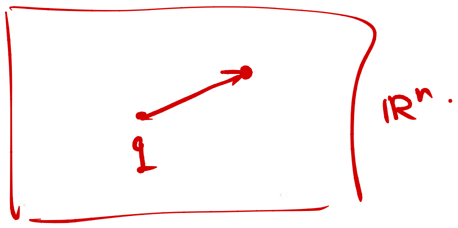
- For $T_pM \supseteq \mathrm{Span}\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{n}$.

$\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$.

- For each $v = \sum_{i=1}^{n} v^i \cdot \frac{\partial}{\partial x^i}$, $v^i \in \mathbb{R}$, define a curve on M:

  - $\alpha(t) = \varphi\left(\underset{\mathbb{R}^n}{q} + \underset{\mathbb{R}}{t} \cdot \underset{\mathbb{R}^n}{\bar{v}}\right)$, $t \in (-\varepsilon, \varepsilon)$, $\bar{v} = (v^1, \dots, v^n) \in \mathbb{R}^n$.

  - $q = (q^1, \dots, q^n) \in \mathbb{R}^n$.



  - $$\begin{tikzcd} (-\varepsilon, \varepsilon) \arrow[r] \arrow[rr, bend right, "\alpha \text{ curve}"'] & U \arrow[r, "\varphi"] & M \end{tikzcd}$$
    $t \longmapsto q + t \cdot \bar{v}$

  - $(x^1(t), \dots, x^n(t)) = \varphi^{-1} \circ \alpha(t) = \boxed{q + t \cdot \bar{v}} = (q^1 + t \cdot v^1, \dots, q^n + t \cdot v^n)$.

    $\Rightarrow x^i(t) = q^i + t \cdot v^i$, $1 \le i \le n$.

  - Use (1), we have:

$$\alpha'(0) f = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(f \circ \varphi)(q) \cdot \left(q^i + t \cdot v^i\right)'(0)$$

$$= \sum_{i=1}^{n} v^i \cdot \frac{\partial}{\partial x^i}(f \circ \varphi)(q)$$

Def. of $\frac{\partial}{\partial x^i}$

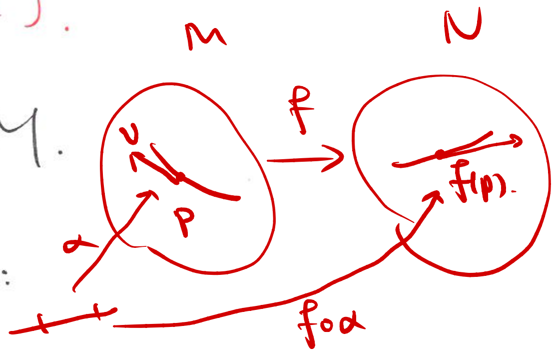
$$= \left(\sum_{i=1}^{n} v^i \cdot \frac{\partial}{\partial x^i}\right) f . = v(f).$$

  - Hence $v = \alpha'(0) \in T_pM$.

- In conclusion, $T_pM = \mathrm{Span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{n}$.

• Prop. 12. (Def. of tangent map / differential).

• Suppose $f : M^m \to N^n$ is diff. at $p \in M$.



• Define $df_p : T_pM \to T_{f(p)}N$ as follows:

$\forall\, v \in T_pM$, choose a curve $\alpha : (-\varepsilon, \varepsilon) \to M$, with

$\alpha(0) = p$, $\alpha'(0) = v$, then put:

$$df_p(v) := (f \circ \alpha)'(0) \in T_{f(p)}N.$$

• Note that

$h \in D_{f(p)}$

$$df_p(v) : D_{f(p)} \to \mathbb{R}, \qquad df_p(v)(h) = \frac{d}{dt}\Big|_{t=0} h \circ f \circ \alpha(t).$$

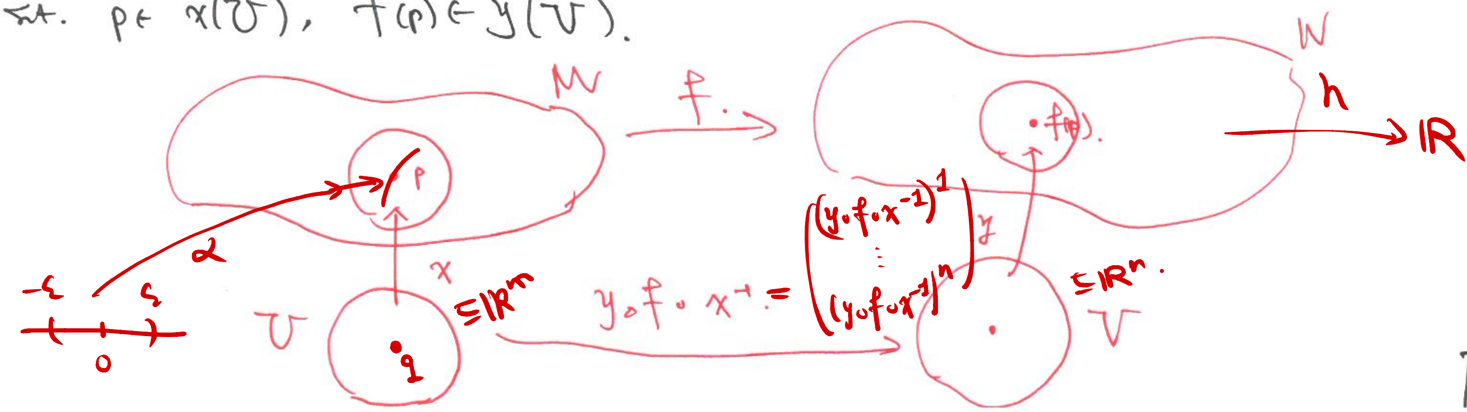
$$\begin{tikzcd} (-\varepsilon,\varepsilon) \arrow[r, "\alpha"] \arrow[rrr, bend right, "h \circ f \circ \alpha"'] & M \arrow[r, "f"] & N \arrow[r, "h"] & \mathbb{R} \end{tikzcd}$$

• The Def. does not depend on the choice of α.

Proof: • Choose local coords:

$$x : U^{\subseteq \mathbb{R}^m} \to M, \qquad y : V^{\subseteq \mathbb{R}^n} \to N.$$

s.t. $p \in x(U)$, $f(p) \in y(V)$.



- Let $x=(x^1,\dots,x^m)$, $y=(y^1,\dots,y^n)$ be coords on U, V.

- Let $q = x^{-1}(p)$.

- $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^{m}$: basis of T_pM.

  $\left\{\frac{\partial}{\partial y^j}\right\}_{j=1}^{n}$: basis of $T_{f(p)}N$.

- **Claim**: $\forall\ v=\sum_{i=1}^{m} v^i \cdot \frac{\partial}{\partial x^i} \in T_pM$, we have:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}$$

$$df_p(v) = \sum_{i,j} v^i \frac{\partial (y^{-1}\circ f\circ x)^j}{\partial x^i}(q)\cdot \frac{\partial}{\partial y^j}.$$

Hence df_p is a linear map.

Proof: • Take a curve α on M, s.t. $\alpha(0)=p$, $\alpha'(0)=v$.

- Then, $\forall\ h \in D_{f(p)}$, by chain rule:

$$(f\circ\alpha)'(0)\, h \overset{\text{Def.}}{=} \frac{d}{dt}\Big|_{t=0} h\circ f\circ\alpha(t).$$

• $(-\varepsilon,\varepsilon) \xrightarrow{x^{-1}\circ\alpha} U \xrightarrow{y^{-1}\circ f\circ x} V$

U in $\mathbb{R}^m$, V in $\mathbb{R}^n$

$V \to \mathbb{R}$ $\qquad \mathbb{R}^m \to \mathbb{R}^n \qquad \mathbb{R} \to \mathbb{R}^m$

$$= \frac{d}{dt}\Big|_{t=0} (h\circ y)\circ \underbrace{(y^{-1}\circ f\circ x)\circ (x^{-1}\circ\alpha)(t)}_{\text{curve on } V \subseteq \mathbb{R}^n}.$$

$$= \sum_{j=1}^{n} \Big[(y^{-1}\circ f\circ x)^j \circ (\underbrace{x^{-1}\circ\alpha}_{x^1,\dots,x^m}) \Big]'(0) \cdot \frac{\partial}{\partial y^j}(h\circ y).$$

$(x^{-1}\circ\alpha)'(0) = (v^1,\dots,v^m)$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} \frac{\partial (y^{-1}\circ f\circ x)^j}{\partial x^i}(q)\cdot v^i \cdot \frac{\partial}{\partial y^j}(h\circ y)(f(p)).$$

$$= \left(\sum_{j=1}^{n}\sum_{i=1}^{m} v^i \frac{\partial (y^{-1}\circ f\circ x)^j}{\partial x^i}(q) \cdot \frac{\partial}{\partial y^j} \right) h.$$

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- The associated mapping matrix is :

$$\left(\frac{\partial (y^{-1} \circ f \circ x)^j}{\partial x^i}(q) \right)_{m \times n}.$$

- If $f: M^m \to N^n$ is diff. everywhere, we call f ~~is~~ a diff. map.

- If $m = n$, $df_p: T_pM \to T_{f(p)}N$ is an isomorphism, then $\exists$ nbhds

$W_1 \subset M$ of p, $W_2 \subseteq N$, s.t. $f|_{W_1}: W_1 \to W_2$ is a diffeomorphism.

$(f|_{W_1})^{-1}: W_2 \to W_1$ diff. map.

- Def. 13. (Diffeomorphism).

  - A diff. map $\varphi: M^n \to N^n$ is a diffeomorphism, if:

    φ is bijective and $\varphi^{-1}: N \to M$ is diff.

    $d\varphi_p: T_pM \to T_{\varphi(p)}N$ isomorphism

  - Denote $M \overset{\text{diff.}}{\cong} N$.

- Chain rule: If $f_1: M_1 \to M_2$, $f_2: M_2 \to M_3$,

f_1 diff. at $p \in M_1$, f_2 diff. at $f(p) \in M_2$.

then $f_2 \circ f_1: M_1 \to M_3$ is diff. at p, and:

$$d(f_2 \circ f_1)_p = (df_2)_{f_1(p)} \circ (df_1)_p : T_pM_1 \xrightarrow{(df_1)_p} T_{f_1(p)}M_2 \xrightarrow{(df_2)_{f_1(p)}} T_{f_2 \circ f_1(p)}M_3.$$

- If $\varphi: M \to N$ is diffeomorphism, then:

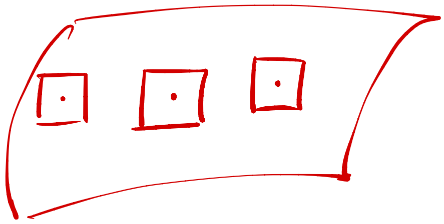
$$Id = d(\underbrace{\varphi^{-1} \circ \varphi}_{= Id})_p = d\varphi^{-1}_{\varphi(p)} \circ d\varphi_p : T_pM \xrightarrow{d\varphi_p} T_{\varphi(p)}N \xrightarrow{d\varphi^{-1}_{\varphi(p)}} T_pM \qquad (d\varphi_p)^{-1} = d\varphi^{-1}_{\varphi(p)}$$

$\Rightarrow d\varphi_p$ is isomorphic at each $p \in M$.

• Def 1.15 (Tangent bundle).

• Let M^n be an n-dim. diff. mfd, with diff. structure $\{(U_\alpha, \varphi_\alpha)\}$.

• Put $TM = \coprod_{p\in M} T_pM$, $\forall \alpha$, define coord.



$$\Phi_\alpha : \overset{\subseteq \mathbb{R}^n}{U_\alpha} \times \mathbb{R}^n \longrightarrow TM$$

$$(x_\alpha, v) \longmapsto \xi = \sum_{i=1}^{n} v^i \cdot \frac{\partial}{\partial x_\alpha^i} \in T_{\varphi_\alpha(x_\alpha)}M.$$

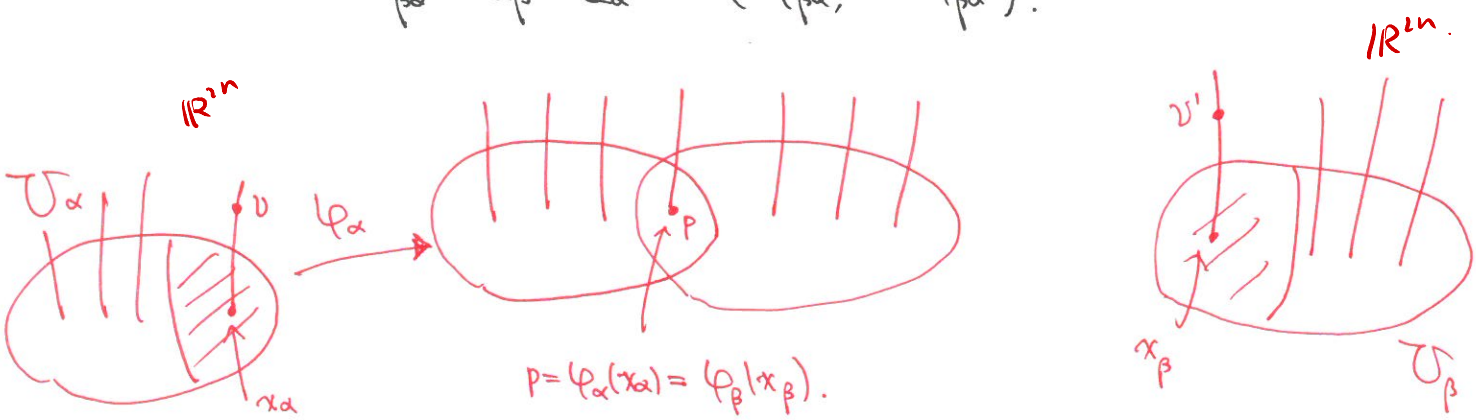
where $x_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$ is coord. on U_α, $v = (v^1, \cdots, v^n)$.

• The diff. mfd TM (of dim. $2n$), with diff. structure determined by $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$, is called tangent bundle of M.

• T_pM: the fiber (纤维) over p of the bundle.

• $\pi : TM \to M$, $v \mapsto p$, $\forall v \in T_pM$, is called the projection map.

• Transition: $\Phi_{\beta\alpha} = \Phi_\beta^{-1} \circ \Phi_\alpha = (\varphi_{\beta\alpha}, d\varphi_{\beta\alpha})$.



$$(x_\alpha, v) \mapsto \xi = v^i \cdot \frac{\partial}{\partial x_\alpha^i} = v^i \cdot \frac{\partial(\varphi_\beta^{-1} \circ \varphi_\alpha)^k}{\partial x_\alpha^i} \cdot \frac{\partial}{\partial x_\beta^k} \mapsto (x_\beta, \overset{v'}{\overset{\shortparallel}{d\varphi_{\beta\alpha}(v)}})$$

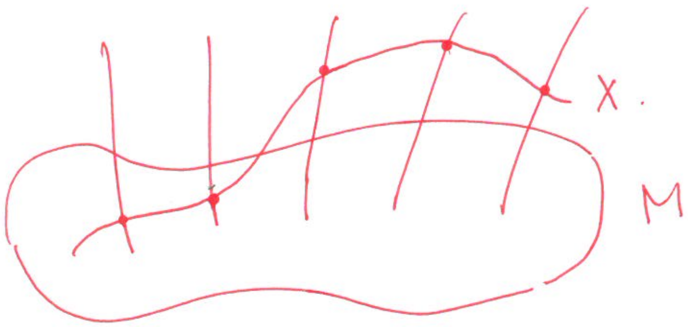
§§3. Vector field.

· **Def. 16**. · A vector field on a diff. mfd M^n, is a diff. map:

$$X : M \to TM, \quad \text{s.t. } X(p) \in T_pM, \ \forall p \in M.$$

· In local coord. $\varphi : \underset{(x^1,\cdots,x^n)}{U} \to M$,

$$X = X^i \cdot \frac{\partial}{\partial x^i}, \quad \text{each } X^i \text{ diff. on } U.$$



· $\varphi : M \to N$ is a diffeomorphism, then $\forall$ vector field X on M, $d\varphi(X)$ is a vector field on N.

· **Lemma 17**. · For any vector fields X, Y, $\exists!$ vector field Z, s.t.

$$Zf = X(Yf) - Y(Xf), \quad \forall \text{ diff. function } f.$$

$f : M \to \mathbb{R}$

$Y(f) : M \to \mathbb{R}$

$X(f) : M \to \mathbb{R}$

· **Proof**: · Uniqueness: trivial.

· Existence: · for a local coord. chart $\varphi : U \overset{\subseteq \mathbb{R}^n}{\longrightarrow} M$, $x = (x^1, \cdots, x^n)$.

$$X = X^i \cdot \frac{\partial}{\partial x^i}, \quad Y = Y^j \cdot \frac{\partial}{\partial x^j}.$$

we compute:

$$X(Yf) - Y(Xf) = X\left(Y^j \cdot \frac{\partial(f\circ\varphi)}{\partial x^j}\right) - Y\left(X^j \cdot \frac{\partial(f\circ\varphi)}{\partial x^j}\right)$$

$$= X^i \cdot \frac{\partial}{\partial x^i}\left(Y^j \cdot \frac{\partial(f\circ\varphi)}{\partial x^j}\right) - Y^i \cdot \frac{\partial}{\partial x^i}\left(X^j \cdot \frac{\partial(f\circ\varphi)}{\partial x^j}\right)$$

$$X^iY^j \cdot \frac{\partial^2(f\circ\varphi)}{\partial x^i \partial x^j} \qquad Y^iX^j \cdot \frac{\partial^2(f\circ\varphi)}{\partial x^j \partial x^i}$$

$$= \left(X^i \cdot \frac{\partial Y^j}{\partial x^i} - Y^i \cdot \frac{\partial X^j}{\partial x^i}\right) \cdot \frac{\partial}{\partial x^j}(f\circ\varphi)$$

$$=: \left(Z^j \cdot \frac{\partial}{\partial x^j}\right)(f\circ\varphi).$$

hence we can set:

$$Z := \sum_{i=1}^{n} Z^i \cdot \frac{\partial}{\partial x^i}.$$

• Def. 18. (Lie bracket). $[X, Y] := XY - YX$, $\forall$ vector fields X, Y on M.

• Prop. 19. ①. $[X,Y] = -[Y,X]$.

$X \leftarrow Y$

$Y \leftarrow Z$

$Z \leftarrow X$

②. $[aX+bY, Z] = a\cdot[X,Z] + b\cdot[Y,Z]$, $\forall\, a,b \in \mathbb{R}$.

③ $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. Jacobi identity.

④. $[fX, gY] = f\cdot g[X,Y] + f\cdot X(g)\cdot Y - g\cdot Y(f)\cdot X$, $\forall$ diff. functions f, g.

Proof: • For ③:

$$[[X,Y],Z] = [XY-YX, Z] = \cancel{XYZ} - \cancel{YXZ} - \cancel{ZXY} + \cancel{ZYX}.$$

$$[[Y,Z]X] = \quad \cdots \quad = \cancel{YZX} - \cancel{ZYX} - \cancel{XYZ} + \cancel{XZY}$$

$$[[Z,X],Y] = \quad \cdots \quad = \cancel{ZXY} - \cancel{XZY} - \cancel{YZX} + \cancel{YXZ}.$$

- For ④: $\forall$ diff. function h,

$$[fX, gY]h = (fX)(\underbrace{(gY)h}_{g\cdot Y(h)}) - gY((fX)h)$$

$$= f\cdot[X(g)\cdot Y(h) + g\cdot(XYh)] - g\cdot[Y(f)\cdot X(h) + f\cdot(YXh)]$$

$$= f\cdot g\cdot(XYh - YXh) + (f\cdot X(g)\cdot Y - g\cdot Y(f)\cdot X)\,h.$$

§§4. Partion of unity.

- In the course, mfds satisfy the countable axiom:

  $\exists$ countable coord. covering. $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathbb{N}}$

- A covering $\{U_\alpha\}$ of M, is said to be locally finite, if $\forall\, p\in M$, has a nbhd W which intersects finite U_α's.

- Let $\{U_\alpha\}$ be an open, locally finite covering of M.

  A partition of unity subordinated to $\{U_\alpha\}$, is a family of diff. functions, $\{f_\alpha\}$, s.t.

  ①. $\operatorname{supp} f_\alpha \subset U_\alpha$. $\quad f_\alpha|_{M\setminus U_\alpha} = 0$

  ②. $\sum_\alpha f_\alpha \equiv 1$ on M.

- Prop. 20. Any diff. mfd satisfies partition of unity property.