§9. Analytic Approach to Comparison Theorems.

§9.1 Distance function.

Let (M^n, g) be a complete, Rie. mfd. Fix $p \in M$.

Let $\rho : M \to \mathbb{R}$ be the distance function based at p.

$\rho(q) = d(p, q)$.

- ρ is smooth on $M \setminus (\mathcal{C}_p \cup \{p\}) = \mathcal{V}_p \setminus \{p\}$.

Lemma: ① $\forall q \in \mathcal{C}_p \setminus \{p\}$. $\nabla \rho(q) = \gamma'_{p,q}$,

where $\gamma_{p,q}$ is the unique minimal geodesic from p to q.
unit speed

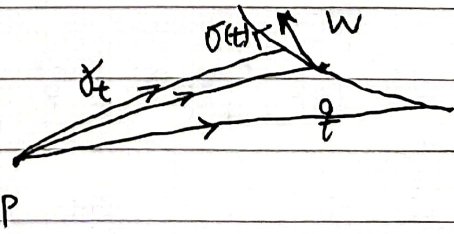
② $\forall q \in \mathcal{C}_p \setminus \{p\}$, at q,

$\text{Hess}\,\rho\,(w, w) = I(J, J)$, $\forall\, w \in T_q M$, $w \perp \gamma'_{p,q}$,

where J is the unique Jacobi field along $\gamma_{p,q}$ with

$J(p) = 0$, $J(q) = w$

Pf: ① Let $w \in T_q M$ be any vector,



$\sigma(t)$ be any curve at q, with

$\sigma(0) = q$, $\sigma'(0) = w$.

Let $\gamma_t : [0, 1] \to M$ be minimal geodesic from to $\sigma(t)$.

γ_t is smooth variation at $t = 0$, when $q \in \mathcal{C}_p \setminus \{p\}$.

1st variation formula:

$$\langle \nabla\rho, w\rangle = \frac{d}{dt}\Big|_{t=0} L(\gamma_t) = \left\langle \frac{\gamma_0'}{|\gamma_0'|}, \cancel{\alpha_0'} \right\rangle (q). = \langle \gamma_{p,q}', w\rangle (q).$$

② Let $J(s) = \frac{\partial}{\partial t}\Big|_{t=0} \gamma_t(s)$ be Jacobi field, $J(0)=0$, $J(1)=w$.

Then, by 2nd variation formula.

$$\text{Hess}\rho(w,w) = ww\rho - (\nabla_w w)\rho$$

Extend w to be a vector field on M

$$= \frac{d^2}{dt^2}\Big|_{t=0} L(\gamma_t) - \langle \nabla_w w, \nabla\rho\rangle(1)$$

$$= \int_0^1 \left(|(\nabla_{\gamma_0'} J)^\perp|^2 - R(\gamma_0', J, \gamma_0', J) \right) ds$$

Notice that when $J \in \gamma_0^\perp$,

$$\langle \nabla_{\gamma_0'} J, \gamma_0' \rangle = \gamma_0' \langle J, \gamma_0'\rangle - \langle J, \nabla_{\gamma_0'} \gamma_0' \rangle = 0$$

i.e., $(\nabla_{\gamma_0'} J)^\perp = \nabla_{\gamma_0'} J$. We get the required identity. □

Lemma: $\text{Hess}\rho(\nabla\rho, w) = 0$, $\forall w$, at $\forall q \in \mathcal{C}_p \setminus \{p\}$.

Pf: Extend w to be a vector field on M.

$$\text{Hess}\rho(\nabla\rho, w) = (\nabla_w d\rho)(\nabla\rho) = \langle \nabla_w \nabla\rho, \nabla\rho\rangle = \frac{1}{2} w |\nabla\rho|^2 = 0. \quad \square$$

- We used that

$$|\nabla\rho|^2 \equiv 1 \quad \text{on} \quad \mathcal{C}_p \setminus \{p\}.$$

Lemma $|\mathrm{Hess}\,\rho|^2 \geq \dfrac{(\Delta\rho)^2}{n-1}$ on $C_p \setminus \{p\}$.

Pf: In any orthonormal frame $\{e_i\}$.

$$\mathrm{Hess}\,\rho = \nabla_i\nabla_j\rho \; e^i \otimes e^j, \qquad \nabla_i\nabla_j\rho = \mathrm{Hess}\,\rho\,(e_i, e_j)$$

$$|\mathrm{Hess}\,\rho|^2 = \sum_{i,j}(\nabla_i\nabla_j\rho)^2.$$

Let $(\nabla_i\nabla_j\rho)$ be the matrix. After an orthogonal transformation, we may assume that $(\nabla_i\nabla_j\rho) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ at a point.

The previous lemma shows that some $\lambda_i = 0$ is an eigenvalue of $\mathrm{Hess}\,\rho$, with eigenvector $\nabla\rho$.

In fact, suppose $\nabla\rho = v^i e_i$, $v^1 \neq 0$, then we claim that $\lambda_1 = 0$.

For any $w = w^i e_i$,

$$0 = \mathrm{Hess}\,\rho(v^i e_i, w^j e_j) = \nabla_i\nabla_j\rho\, v^i w^j = \sum_{i=1}^{n} \lambda_i v^i w^i$$

$\Rightarrow$ The vector $(\lambda_1 v^1, \lambda_2 v^2, \cdots, \lambda_n v^n) = 0$.

$v^1 \neq 0 \Rightarrow \lambda_1 = 0$.

Then,
$$|\mathrm{Hess}\,\rho|^2 = \sum_{i=2}^{n} \lambda_i^2 \geq \frac{(\sum \lambda_i)^2}{n-1} = \frac{(\Delta\rho)^2}{n-1}. \qquad \square$$

§9.2. Comparison Theorems.

- We first apply Bochner formula to distance function; $\rho = d(p,\cdot)$.

$$0 = \frac{1}{2}\Delta|\nabla\rho|^2 = |\mathrm{Hess}\,\rho|^2 + \langle\nabla\Delta\rho, \nabla\rho\rangle + \mathrm{Ric}(\nabla\rho, \nabla\rho).$$

Along a minimal geodesic $\gamma : [0,\ell] \to M$, $\gamma(0) = p$.

$$\langle\nabla\Delta\rho, \nabla\rho\rangle = \frac{d}{ds}\Delta\rho.$$

$$\Rightarrow \quad \frac{d}{ds}\Delta\rho \le -\frac{(\Delta\rho)^2}{n-1} - \mathrm{Ric}(\nabla\rho, \nabla\rho), \quad \forall\, 0 < s < \ell.$$

- On space form $\underline{M}$ of curvature K_0,

$$\frac{d}{ds}\Delta\underline{\rho} = \frac{(\Delta\underline{\rho})^2}{n-1} - (n-1)K_0, \quad \forall\ 0 < s < \frac{\pi}{\sqrt{K_0}}.$$

Recall that, by Bonnet–Myers Theorem, $\ell \le \frac{\pi}{\sqrt{K_0}}$ once $\mathrm{Ric} \ge (n-1)K_0$.

An easy calculation shows that

$$\Delta\rho - \Delta\underline{\rho} = o(1) \quad \text{as } \rho \to 0.$$

So, the comparison of ODE gives

<u>Prop</u> (Laplacian comparison) Suppose (M^n, g) is complete, $\mathrm{Ric} \ge (n-1)K_0\, g$. Then along any minimal geodesic,

$$\Delta\rho \le \Delta\underline{\rho}$$

evaluated at same distance point. □

Let $I_p : S^{n-1} \to S_pM$ be an isometry

Let $(\theta^1, \cdots, \theta^{n-1})$ be a coor. on unit sphere S^{n-1}.

The polar coordinate

$$\Phi(s, \theta^1, \cdots, \theta^{n-1}) = \exp_p\left(s \cdot I_p(\theta^1, \cdots, \theta^{n-1})\right)$$

we write

$$\omega_g = \mathcal{B}(s, \theta^1, \cdots, \theta^{n-1})\, ds \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1}$$

- Lemma :

$$\Delta \rho = \frac{\partial}{\partial s} \log \mathcal{B}.$$

pf: Direct calculation, for any coor. $(x^1, \cdots, x^n)$.

$$\Delta \rho = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial \rho}{\partial x^j}\right)$$

Let $x^1 = s$, $x^i = \theta^{i-1}$, $i = 2, \cdots, n$, then, $\rho(s, \theta^1, \cdots, \theta^{n-1}) = s$,

$$\text{So: } \Delta \rho = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial s}\left(\sqrt{\det g}\, \cancel{\frac{\partial \rho}{\partial s}}\right) + \sum_{i,j \geq 2} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial \theta^i}\left(\cancel{\sqrt{\det g}\, g^{ij} \frac{\partial \rho}{\partial \theta^j}}\right)$$

$$= \frac{\partial}{\partial s} \log \sqrt{\det g} = \frac{\partial}{\partial s} \log \mathcal{B}.$$

□

Thus, $\Delta \rho$ comparison implies volume form comparison,

finially gives Bishop-Gromov comparison Theorem.

Relative

- We finally consider Rauch comparison Theorem.

Let $\gamma : [0, l] \to M$ be a minimal geodesic, unit speed, $p = \gamma(0)$

Let $J \in \mathcal{J}_0^{\perp}$ be a Jacobi field,

By minimality, $\rho = d(p, \cdot)$ is smooth at $\gamma(s)$, $0 < s < l$.

Lemma: $\operatorname{Hess}\rho\,(J, J) = \frac{1}{2}\frac{d}{ds}|J|^2$, along the geodesic γ.

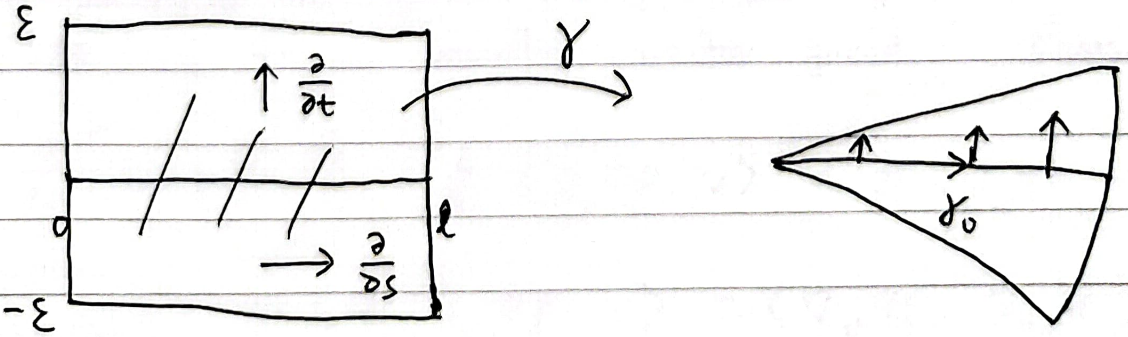
Pf: Suppose J is given by variation

$$\gamma(s,t) = \exp_p\left(s(v + tw)\right), \qquad v \perp w, \quad |v| = |w| = 1.$$

$$0 \le s \le l.$$

Let $J(s,t) = \frac{\partial}{\partial t}\gamma = d\gamma\left(\frac{\partial}{\partial t}\right)$

$$\nabla \rho = \frac{1}{|v + tw|}\frac{\partial}{\partial s}\gamma = \frac{1}{|v+tw|} d\gamma\left(\frac{\partial}{\partial s}\right)$$



~~Then, when t=0~~,

Then, $[J, \nabla \rho] = \left[d\gamma\left(\frac{\partial}{\partial t}\right), d\gamma\left(\frac{1}{|v+tw|}\frac{\partial}{\partial s}\right)\right] = d\gamma\left[\frac{\partial}{\partial t}, |v+tw|^{-1}\cdot\frac{\partial}{\partial s}\right]$

$$= -d\gamma\cdot\left(|v+tw|^{-3}\cdot\langle v+tw, w\rangle\cdot\frac{\partial}{\partial s}\right).$$

Along γ_0, i.e., $t=0$,

$$[J, \nabla\rho] = 0.$$

Thus, $\mathrm{Hess}\,\rho(J,J) = \langle \nabla_J \nabla\rho, J\rangle = \langle \nabla_{\nabla\rho} J, J\rangle + \langle [J,\nabla\rho], J\rangle$

Along γ_0: $\mathrm{Hess}\,\rho(J,J) = \langle \nabla_{\nabla\rho} J, J\rangle = \frac{1}{2}\nabla\rho(|J|^2) = \frac{1}{2}\frac{d}{ds}|J|^2$. □

Therefore the Rauch comparison Theorem reduces a Hessian estimate of distance function.

Prop: We assume that along a minimal geodesic $\gamma : [0, l] \to M$,

$$K_1 \le K(\gamma', v) \le K_2, \qquad \forall\ v \perp \gamma',\ v \ne 0.$$

Then, along γ,

$$\mathrm{Hess}_{K_2}\,\rho(v,v) \le \mathrm{Hess}\,\rho(v,v) \le \mathrm{Hess}_{K_1}\,\rho(v,v), \quad \forall\ |v| = 1,$$

where Hess_{K_i} is the Hessian on space form of constant curvature K_i.

Pf: Let V be a parallel vector field. Then, along γ,

$$\frac{d}{ds}[\mathrm{Hess}\,\rho(v,v)] = \frac{d}{ds}\langle \nabla_v\nabla\rho, v\rangle = \nabla\rho\langle \nabla_v\nabla\rho, v\rangle$$

V parallel $\quad = \langle \nabla_{\nabla\rho}\nabla_v\nabla\rho, v\rangle + \langle \nabla_v\nabla\rho, \cancel{\nabla_{\nabla\rho} v}\rangle$

$\mathrm{Hess}\,\rho(\nabla\rho) = 0 \quad = \langle R_{\nabla\rho, v}\nabla\rho, v\rangle + \langle \nabla_v \nabla_{\nabla\rho}\cancel{\nabla\rho}, v\rangle + \langle \nabla_{[\nabla\rho, v]}\nabla\rho, v\rangle$

$$= -R(\nabla\rho, v, \nabla\rho, v) - \langle \nabla_{\nabla_v\nabla\rho}\nabla\rho, v\rangle$$

where $\langle \nabla_{\nabla_v \nabla \rho} \nabla \rho, v \rangle = \langle \nabla_v \nabla \rho, \nabla_v \nabla \rho \rangle = \mathrm{Hess}^2 \rho \, (v,v)$

where $\mathrm{Hess}^2 \rho$ is a $(2,0)$ tensor defined by

$$\mathrm{Hess}^2 \rho \, (v,v) = \mathrm{Hess}\, \rho \, (\mathrm{Hess}\, \rho \, (v, \cdot), v)$$

Locally, if $\mathrm{Hess}\, \rho = \nabla_i \nabla_j \rho \, dx^i \otimes dx^j$, then

$$\mathrm{Hess}^2 \rho = g^{jk} \cdot \nabla_i \nabla_j \rho \, \nabla_k \nabla_l \rho \cdot dx^i \otimes dx^l$$

is a contraction of $\mathrm{Hess}\, \rho \otimes \mathrm{Hess}\, \rho$.

Thus:
$$\frac{d}{ds} \mathrm{Hess}\, \rho \, (v,v) = - R(\gamma', v, \gamma', v) - \mathrm{Hess}^2 \rho \, (v,v).$$

Apply comparison of ODE. $\square$