§8. Bochner Formula

We first recall some notion in Rie. Geometry.

§8.1 Connection on Tensor Bundles.

Recall Levi-Civita Conn: $\nabla: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, $(X,Y) \mapsto \nabla_X Y$.

Define <u>dual connection</u> on cotangent bundle:

$$\nabla: \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M), \quad (X, \omega) \mapsto \nabla_X \omega$$

via: $(\nabla_X \omega)(Y) = X[\omega(Y)] - \omega(\nabla_X Y)$

or, $X[\omega(Y)] = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$

<u>Tensor bundle</u>: $\nabla: \Gamma(TM) \times \Gamma(T^{r,s}M) \to \Gamma(T^{r,s}M)$, $(X,T) \mapsto \nabla_X T$

via:
$$\begin{aligned}(\nabla_X T)(Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s) &= X\left[T(Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s)\right] \\ &- \sum_{1 \le i \le r} T(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_r, \omega^1, \cdots, \omega^s) \\ &- \sum_{1 \le j \le s} T(Y_1, \cdots, Y_r, \omega^1, \cdots, \nabla_X \omega^j, \cdots, \omega^s)\end{aligned}$$

Or, $\nabla: \Gamma(T^{r,s}M) \to \Gamma(T^{r+1,s}M)$, $T \mapsto \nabla T$

via. $\nabla T(X, Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s) = (\nabla_X T)(Y_1, \cdots, Y_r, \omega^1, \cdots, \omega^s)$.

<u>Locally</u>, in coor., write $T = T_{i_1 \cdots i_r}^{j_1 \cdots j_s} dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}$

Denote $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}} : \Gamma(T^{r,s}M) \to \Gamma(T^{r,s}M)$.

<u>Denote by</u> $\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r}$ the coefficients of ∇T, namely,

$$\nabla T = \underline{\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r}}\, dx^k \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}.$$

- Examples: $\nabla_k X^i = \frac{\partial X^i}{\partial x^k} + \Gamma^i_{kl} X^l$ for $X \in \Gamma(TM)$.

  $\nabla_k \omega_i = \frac{\partial \omega_i}{\partial x^k} - \Gamma^j_{ki}\, \omega_j$ for $\omega \in \Gamma(T^*M)$

$$\nabla_k T^{j_1\cdots j_s}_{i_1\cdots i_r} = \frac{\partial T^{j_1\cdots j_s}_{i_1\cdots i_r}}{\partial x^k} - \sum_{1\le l\le r} \Gamma^p_{k i_l} T^{j_1\cdots j_s}_{i_1\cdots \underset{\uparrow i_l}{p} \cdots i_r} + \sum_{1\le l\le s} \Gamma^{j_l}_{kq} T^{j_1\cdots \overset{j_l \downarrow}{q} \cdots j_s}_{i_1\cdots i_r}.$$

- $\nabla_k g_{ij} = \nabla_k g^{ij} = 0$. g_{ij}, g^{ij} are metric coeff of g on TM & T^*M <u>respectively</u>. (cf. §8.2)

- <u>Ricci identity</u>: Recall, in coor.,

$$R_{X,Y} Z = X^i Y^j Z^l\, R_{ij}{}^k{}_l \frac{\partial}{\partial x^k}$$

$$R_{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} = R_{ij}{}^k{}_l \frac{\partial}{\partial x^k}.$$

<u>Prop</u>: ① $\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = R_{ij}{}^k{}_l X^l$.

② $\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = - R_{ij}{}^l{}_k\, \omega_l$ □

- There are same formula for Tensors of type (r,s).

Finally, some operation on smooth functions:

- df, differential of f, $df \in \Gamma(T^*M)$
- ∇f, gradient field of f, dual of df, via

$$\langle \nabla f, X \rangle = df(X) = Xf, \quad \forall X \in \Gamma(TM)$$

- $\mathrm{Hess} f = \nabla df$, (2,0) tensor,

$$\mathrm{Hess} f(X,Y) = (\nabla_X df)(Y) = X[df(Y)] - df(\nabla_X Y)$$
$$= XYf - (\nabla_X Y) f$$

Symmetric in X, Y:

$$\mathrm{Hess} f(X,Y) - \mathrm{Hess} f(Y,X) = XYf - YXf - (\nabla_X Y - \nabla_Y X) f = 0.$$

Locally, $\mathrm{Hess} f = \nabla_i \nabla_j f \, dx^i \otimes dx^j$,

where $\nabla_i \nabla_j f = \dfrac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \dfrac{\partial f}{\partial x^k}$.

- Laplacian $\Delta f = \sum_{i=1}^n \mathrm{Hess} f(e_i, e_i) = g^{ij} \nabla_i \nabla_j f$,

where $\{e_i\}$ is orthonormal frame.

Lemma: In coor.,

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^k} \left(\sqrt{\det(g_{ij})} \, g^{kl} \frac{\partial f}{\partial x^l} \right).$$

Pf: ... ... Direct Calculation $\square$

§8.2. Norms on tensors

Choose orthonormal basis $\{e_i\}$ on TM, with dual basis $\{e^i\}$ on T^*M.

- Introduce metric on T^*M, also denoted by g, so that $\{e^i\}$ is orthonormal; similarly, introduce metric on $T^{r,s}M$ such that the basis $\{e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e_{j_1}\otimes\cdots\otimes e_{j_s}\}$ is orthonormal.

- In general frame, for example, coor. fields $\{\frac{\partial}{\partial x^i}\}$ with dual basis $\{dx^i\}$ on T^*M, we have

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle,$$

$$g^{ij} = \langle dx^i, dx^j\rangle, \qquad (g^{ij}) = (g_{ij})^{-1}, \text{ inverse matrix.}$$

&: $\langle dx^{i_1}\otimes\cdots\otimes dx^{i_r}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}},\ dx^{k_1}\otimes\cdots\otimes dx^{k_r}\otimes\frac{\partial}{\partial x^{l_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{l_s}}\rangle$

$$= g^{i_1k_1}\cdots g^{i_rk_r}g_{j_1l_1}\cdots g_{j_sl_s}.$$

- $|df|^2 = |\nabla f|^2 = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$

$|\nabla\nabla f|^2 = g^{ik}g^{jl}\nabla_i\nabla_j f\,\nabla_k\nabla_l f.$

<u>Lemma</u>: (∇ compatible with inner product on Tensor bundles)

$$\nabla_X\langle T_1, T_2\rangle = \langle\nabla_X T_1, T_2\rangle + \langle T_1, \nabla_X T_2\rangle. \qquad \square$$

§ 8.3 Bochner formula

<u>Prop</u>: $\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + \langle \nabla\Delta f, \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f)$

<u>Pf</u>: By direct computation,

$$\frac{1}{2} g^{ij}\nabla_i\nabla_j|\nabla f|^2 = \frac{1}{2} g^{ij}\nabla_i\nabla_j\left(g^{kl}\nabla_k f\,\nabla_l f\right)$$

$$= \frac{1}{2} g^{ij}g^{kl}\nabla_i\left(\nabla_j\nabla_k f\,\nabla_l f + \nabla_k f\,\nabla_j\nabla_l f\right)$$

$$= \frac{1}{2} g^{ij}g^{kl}\left(\nabla_i\nabla_j\nabla_k f\,\nabla_l f + \nabla_j\nabla_k f\,\nabla_i\nabla_l f + \nabla_i\nabla_k f\,\nabla_j\nabla_l f + \nabla_k f\,\nabla_i\nabla_j\nabla_l f\right)$$

$$= g^{ij}g^{kl}\left(\nabla_i\nabla_j\nabla_k f\,\nabla_l f + \nabla_j\nabla_k f\,\nabla_i\nabla_l f\right)$$

$$= |\mathrm{Hess} f|^2 + g^{ij}g^{kl}\nabla_i\nabla_j\nabla_k f\,\nabla_l f$$

<u>where</u>,

$$\nabla_i\nabla_j\nabla_k f = \nabla_i\nabla_k\nabla_j f \quad \leftarrow \quad \nabla_j\nabla_k f = \nabla_k\nabla_j f$$

$$= \nabla_k\nabla_i\nabla_j f - R_{ik}{}^{p}{}_{j}\nabla_p f \quad \leftarrow \quad w = df$$

$$= \nabla_k\nabla_i\nabla_j f - g^{pq}R_{ikqj}\nabla_p f.$$

$$g^{ij}g^{kl}\nabla_i\nabla_j\nabla_k f\,\nabla_l f = g^{ij}g^{kl}\nabla_k\nabla_i\nabla_j f\,\nabla_l f - g^{ij}g^{kl}g^{pq}R_{ikqj}\nabla_p f\,\nabla_l f$$

$$= g^{kl}\nabla_k\Delta f\,\nabla_l f + g^{kl}g^{pq}R_{kq}\nabla_p f\,\nabla_l f$$

$$= \langle\nabla\Delta f, \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f).$$

□

- Harmonic forms.

Prop: (M^n, g), compact. $Ric > 0$. Then there is no nontrivial harmonic 1-form. In particular, $b_1 = 0$.

pf: ω, harmonic 1-forms $\Leftrightarrow$ $d\omega = d^*\omega = 0$

$\Leftrightarrow$ $d\omega = \mathrm{div}\,\omega = 0$.

where $\mathrm{div}\,\omega = g^{ij}\nabla_i\omega_j$ locally.

Poincaré lemma $\Rightarrow$, $d\omega = 0$, then $\omega = df$ locally.

$$\frac{1}{2}\Delta|df|^2 = |\mathrm{Hess} f|^2 + \langle\nabla\Delta f, \nabla f\rangle + Ric(\nabla f, \nabla f)$$

$$\Delta f = g^{ij}\nabla_i\nabla_j f = g^{ij}\nabla_i\omega_j = \mathrm{div}\,\omega = 0.$$

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + Ric(\nabla f, \nabla f).$$

$\Delta|\nabla f|^2 \geq 0$, maximum principle $\Rightarrow$ $|\nabla f| = $ constant.

- $\nabla f = 0$: done.
- $\nabla f \neq 0$, anywhere. $\Rightarrow$ $Ric(\nabla f, \nabla f) > 0$ impossible

because $0 = \frac{1}{2}\Delta|\nabla f|^2 \geq Ric(\nabla f, \nabla f)$. $\square$