K_0 ...

Apply cosine Law. to get comparison on $\underline{M}$ with curvature K_0. $\square$

§7 Volume. Bishop-Gromov Relative Volume Comparison.

§7.1 Volume form.

Let M^n be an orientable mfd. An orientation is a family of coor. covering $\{(\varphi_\alpha, U_\alpha)\}$ s.t.,

$$\mathrm{Jac}(\varphi_{\alpha\beta}) > 0, \quad \text{on } \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta).$$

Equivalently, ~~an orientation is~~ there is an n-form that is non-vanishing anywhere.

Let ω be a nowhere vanishing n-form, ω is called to be compatible with the orientation, if

$$\omega = f(x)\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n, \quad f(x) > 0$$

on each coor. $(\varphi_\alpha, U_\alpha)$, where

$x_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$ is coor on U_α.

- Let $\{(\varphi_\alpha, U_\alpha)\}$ be an orientation of M. A coor. (φ, U) is called compatible with the orientation, if $Jac(\varphi_\alpha^{-1} \circ \varphi) > 0$, $\forall \alpha$.

<u>Def</u>: Let $\{(\varphi_\alpha, U_\alpha)\}$ be an orientation and ω be a nowhere vanishing n-form, compatible with the orientation.

In each orientation compatible coor. (φ, U), define

$$\int_{\varphi(U)} \omega = \int_U f(x)\, dx^1 \cdots dx^n$$

where $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ on U.

- Let ω be a nowhere vanishing n-form, compatible with the orientation $\{(\varphi_\alpha, U_\alpha)\}$.

Let β be any other n-form. $\beta = h \cdot \omega$, for some $h \in C^\infty(M)$.

Let $\{\eta_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$.

Then, define integration on compact subsets

$$\int_K \beta = \sum \int_{K \cap \varphi_\alpha(U_\alpha)} \eta_\alpha \cdot \beta = \sum \int_{\varphi_\alpha^{-1}(K \cap \varphi_\alpha(U_\alpha))} (\eta_\alpha \circ \varphi_\alpha) \cdot f_\alpha\, dx_\alpha^1 \cdots dx_\alpha^n$$

where $x_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$ is coor. on U_α,

& $\omega = f_\alpha\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ on $\varphi_\alpha(U_\alpha)$.

•• The integration is indep. of choice of $\{\eta_\alpha\}$.

• Rie. volume form.

Let (M^n, g) be an oriented, Rie mfd. Define a canonical n-form

$$\omega_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n.$$

in each coor (φ, U), compatible with orientation.

•• ω_g is indep of coor.

•• Define volume, for compact set K,

$$\mathrm{vol}(K) = \int_K \omega_g$$

For open set U, with an exhaution by compact sets K_i,

with $\cup K_i = U$, $K_i \subset K_{i+1}$,

$$\mathrm{vol}(U) = \lim_{i\to\infty} \mathrm{vol}(K_i).$$

§ 7.2 Cut locus.

~~Let $\gamma: [0, \ell] \to M$ be a geodesic, unit speed.~~

Define the cut point in direction $v \in T_pM$, $|v| = 1$,

$$t_v = \sup\{ s_0 > 0 \mid \exp_p(sv) \text{ is minimal up to } s_0 \}$$

• By Gauss Lemma, $t_v > 0$, for any direction v.

• t_v may be infinity.

Let $\mathcal{U}_p = \{ t \cdot v \mid v \in T_pM, \ |v|=1, \ 0 \le t < t_v \}$.

Define the cut point along γ_v: to be the unique point $\gamma_v(t_v \cdot v)$

Let $\mathcal{V}_p = \{ \gamma_v(tv) \mid v \in T_pM, \ |v|=1, \ 0 \le t < t_v \} = \exp_p(\mathcal{U}_p)$.

- $t_v : S_pM \to \mathbb{R}$ is continuous,

  $S_pM = \{ v \in T_pM \mid, |v| = 1 \}$ is the unit sphere of T_pM.

- The cut point along a geodesic γ has a characterization:

  Either ① a conjugate point of $\gamma(0)$, or ② $\exists$ at least two minimal geodesic from $\gamma(0)$ to the point.

- $\mathcal{U}_p$ is star-shaped, open, domain of T_pM;

  $\exp_p : \mathcal{U}_p \to \mathcal{V}_p$ is diffeomorphism.

- Let $\mathcal{C}_p = \{ \gamma_v(t_v \cdot v) \mid v \in S_pM \}$ be the set of cut locus.

  $\mathrm{vol}(\mathcal{C}_p) = 0$, $\mathcal{V}_p = M \setminus \mathcal{C}_p$.

- For any compact / open set $A \subset M$, the volume

  $\mathrm{vol}(A) = \mathrm{vol}(A \setminus \mathcal{C}_p)$.

  It can be calculated in the normal coor $\exp_p : \mathcal{U}_p \to \mathcal{V}_p \subset M$.

How to compute the volume form in coordinate $(\mathcal{U}_p, \exp_p)$?

Let $\{e_i\}$ be an orthonormal frame at p, compatible with orientation.

Let $x = x^i e_i$ be decomposition of $x \in T_pM$.

Let $(x^1, \cdots, x^n)$ be Descartes coordinate on T_pM.

Write

$$\omega_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

- For any $x_0 \in T_pM$, let γ_{x_0} be the geodesic

and define variations

$$\gamma_i(s,t) = \exp_p(s(x_0 + te_i)). \qquad i = 1, \cdots, n.$$

The Jacobi field

$$J_i(s,t) = \frac{\partial}{\partial t}\gamma_i = s \cdot d(\exp_p)_{s(x_0+te_i)}(e_i) = s \cdot \frac{\partial}{\partial x^i}.$$

- Along the geodesic γ_{x_0}:

$$g_{ij} = s^{-2}\langle J_i, J_j \rangle$$

After orthogonal transformation on T_pM, we may assume $x_0 = |x_0|\, e_n$

Then, by Gauss lemma:

$$g_{in} = g_{ni} = 0, \qquad i = 1, \cdots, n-1,$$

& $$g_{nn} = 1.$$

- Along γ_{x_0}, $x_0 = |x_0| e_n$,

$J_i(0) = 0$, $J_i'(0) = e_i$, $J_i(s) = s\frac{\partial}{\partial x^i}$

At $s=0$: $\lim\limits_{s\to 0} g_{ij}(s) = \lim\limits_{s\to 0} \frac{\langle J_i(s), J_j(s)\rangle}{s^2} = \delta_{ij}$.

- Along γ_{x_0}, $\frac{d}{ds}\log\sqrt{\det(g_{ij})} = \frac{1}{2} g^{ij}\frac{\partial}{\partial s} g_{ij}$

$$= \frac{1}{2} g^{ij}\left(s^{-2}\langle J_i', J_j\rangle + s^{-2}\langle J_i, J_j'\rangle - 2s^{-3}\langle J_i, J_j\rangle\right)$$

$$= s^{-2} g^{ij}\langle J_i', J_j\rangle - n s^{-1}.$$

Apply $g_{in} = g_{ni} = \delta_{in}$, we get $g^{in} = g^{ni} = \delta_{in}$, and,

$$g^{ij}\langle J_i', J_j\rangle = \sum_{i,j\le n-1} g^{ij}\langle J_i', J_j\rangle + \langle J_n', J_n\rangle$$

$$= \sum_{i,j\le n-1} g^{ij}\langle J_i', J_j\rangle + s.$$

So:

$$\frac{d}{ds}\log\sqrt{\det(g_{ij})} = s^{-2}\sum_{i,j\le n-1} g^{ij}\langle J_i', J_j\rangle - (n-1)s^{-1}. \quad (*)$$

- At any $0 < s_0 \le 1$, we estimate $\sum g^{ij}\langle J_i', J_j\rangle$ as follows.

Notice that the formula $g^{ij}\langle J_i', J_j\rangle$ is indep. of linear transformation of the Jacobi fields $\{J_i\}$. we may assume that $\left\{\frac{J_i(s_0)}{|J_i(s_0)|}\right\}$ is an orthonormal basis of $T_{\gamma_{x_0}(s_0)}M$.

and $\frac{J_n}{|J_n|}(s_0) = \frac{\gamma_{x_0}'(s_0)}{|\gamma_{x_0}'|}$. Then, at s_0,

$$g_{ij}(s_0) = s_0^{-2} \cdot \langle J_i, J_j \rangle (s_0) = s_0^{-2} |J_i(s_0)| \cdot |J_j(s_0)| \cdot \delta_{ij}$$

$$\sum_{i,j \le n-1} g^{ij} \langle J_i', J_j \rangle (s_0) = \sum_{i \le n-1} s_0^2 \cdot |J_i(s_0)|^{-2} \cdot \langle J_i', J_i \rangle (s_0)$$

$$= \sum_{i \le n-1} s_0^2 \cdot \langle \tilde{J}_i', \tilde{J}_i' \rangle (s_0)$$

where $\tilde{J}_i = \frac{J_i}{|J_i(s_0)|}$, is a normalized Jacobi field.

Recall that $J_i(s_0) \neq 0$, because $\gamma_{x_0}(s_0)$ is not conjugate to p.

Therefore, Along γ_{x_0}:

$$\frac{d}{ds} \log \sqrt{\det(g_{ij})} = \sum_{i \le n-1} \langle \tilde{J}_i', \tilde{J}_i \rangle (s) - (n-1) s^{-1}. \qquad (**)$$

where $\{\tilde{J}_i\}_{i=1}^{n-1} \subset \mathcal{J}_0^{\perp}$ is a family of Jacobi fields

Such that $\{\tilde{J}_i(s)\}_{i=1}^{n-1}$ is an orthonormal basis of

$T_{\gamma_{x_0}(s_0)} M$, orthogonal to $\gamma_{x_0}'(s_0)$.

Now, we can apply Jacobi field estimate as in Rauch comparison Theorem.

§7.3

In order to prove Bishop-Gromov Relative volume comparison, we introduce the polar coordinate:

$$\Phi: \mathcal{U}_p \to \mathcal{V}_p \subset M, \qquad (s, v) \longmapsto \exp_p(sv)$$

where $(s, v) \in \mathbb{R}_+ \times S_pM$, $0 \le s \le t_v$.

Let $\omega_g = \mathcal{A}(s, v)\, ds \wedge \omega_{S^{n-1}}$ be the volume form,

where $\omega_{S^{n-1}}$ is the volume form on unit sphere.

Then, for any subset $A \subset M$, by Fubini Theorem

$$\mathrm{vol}(A) = \mathrm{vol}(A \setminus \mathcal{C}_p) = \int_{S^{n-1}} \left(\int_{A_v} \mathcal{A}(s,v)\, ds \right) \omega_{S^{n-1}}.$$

where $A_v = \{ 0 < s < t_v \mid s \cdot v \in A \}$, for any $v \in S_pM$.

Theorem (Bishop-Gromov)

Let (M^n, g) be a connected, complete Rie. mfld, $\mathrm{Ric} \geq (n-1) K_0 g$.

(i). Along any minimal geodesic $\gamma: [0, l] \to M$, unit speed,

suppose $\omega_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$ in normal coor.

Then, $\dfrac{d}{ds} \log \dfrac{\sqrt{\det(g_{ij})}}{\sqrt{\det(\underline{g}_{ij})}} \le 0$, along γ.

where $\underline{g}$ is the corresponding metric on space form of const curvature K_0, computed along a geodesic $\underline{\gamma}: [0, l] \to \underline{M}$.

(ii). Define annulus

$$A_{s,r}(p) = \{ \exp_p(tv) \mid s < t < r, \quad t < t_v, \quad v \in S_pM \}.$$

Then, $\dfrac{vol(A_{r_3,r_4}(p))}{vol(\underline{A}_{r_3,r_4}(\underline{p}))} \le \dfrac{vol(A_{r_1,r_2}(p))}{vol(\underline{A}_{r_1,r_2}(\underline{p}))}$, $\forall\ 0 < r_1 \le \min(r_2, r_3) \le \max(r_2, r_4) \le r_4$.

In particular, $\dfrac{vol(B_{r_2}(p))}{\underline{vol}(\underline{B}_{r_2}(\underline{p}))} \le \dfrac{vol(B_{r_1}(p))}{\underline{vol}(\underline{B}_{r_1}(\underline{p}))}$, $\forall\ r_1 \le r_2$

Pf: (i). According calculation in §7.2, along γ,

$$\frac{d}{ds} \log\sqrt{\det(g)}(s_0) = \sum_{i=1}^{n-1} \langle \tilde{J}_i', \tilde{J}_i \rangle(s_0) - (n-1)s_0^{-1}, \quad \forall\ 0 < s_0 \le 1,$$

where $\{\tilde{J}_i\}_{i=1}^{n-1} \in \mathcal{J}_0^1$ satisfies $\{\tilde{J}_i(s_0)\}$ is orthonormal basis of $\gamma'(s_0)^{\perp} \subset T_{\gamma(s)}M$. For each i,

$$\langle \tilde{J}_i', \tilde{J}_i \rangle(s_0) = \int_0^{s_0} (|\tilde{J}_i'|^2 + \langle \tilde{J}_i'', \tilde{J}_i \rangle)\, ds = I(\tilde{J}_i, \tilde{J}_i).$$

Similar computation on $\underline{M}$ gives

$$\langle \underline{\tilde{J}_i}', \underline{\tilde{J}_i} \rangle(s_0) = I(\underline{\tilde{J}_i}, \underline{\tilde{J}_i}).$$

where $\{\underline{\tilde{J}_i}(s_0)\}$ is orthonormal at $\underline{\gamma}(s_0)$.

Let $\{\underline{e_i}\}$ be parallel fields with $\underline{e_i}(s_0) = \underline{\tilde{J}_i}(s_0)$.

Then $\tilde{\underline{J}}_i(s) = f(s) \cdot \underline{e}_i(s)$, $i = 1, \cdots, n-1$,

for a common function f. Define comparison fields

$$X_i(s) = f(s)\, e_i(s),$$

where $\{e_i\}$ are parallel fields along γ, with $e_i(s_0) = \tilde{J}_i(s_0)$.

Then, $X_i(0) = \tilde{J}_i(0) = 0$, $X_i(s_0) = \tilde{J}_i(s_0)$, $i = 1, \cdots, n-1$.

So, $I(X_i, X_i) \geq I(\tilde{J}_i, \tilde{J}_i)$,

$$\sum_{i \le n-1} I(\underline{\tilde{J}}_i, \underline{\tilde{J}}_i) = \sum_{i \le n-1} \int_0^{s_0} \left(|f'|^2 - f^2 \underline{R}(\underline{\gamma}', \underline{e}_i, \underline{\gamma}', \underline{e}_i) \right) ds$$

$$= (n-1) \int_0^{s_0} |f'|^2 ds - \int_0^{s_0} f^2 \cdot \underline{Ric}(\underline{\gamma}', \underline{\gamma}') ds$$

$$\geq (n-1) \int_0^{s_0} |f'|^2 ds - \int_0^{s_0} f^2 \cdot Ric(\gamma', \gamma') ds$$

$$= \sum_{i \le n-1} I(X_i, X_i)$$

$$\geq \sum_{i \le n-1} I(\tilde{J}_i, \tilde{J}_i).$$

$$\Rightarrow \quad \frac{d}{ds} \log \sqrt{\det g} \le \frac{d}{ds} \log \sqrt{\det \underline{g}}.$$

(ii). Equivalent to show:

$$\frac{\text{vol}(A_{s,r}(p))}{\text{vol}(\underline{A}_{s,r}(p))} \text{ is decreasing in both } s \;\&\; r.$$

$$\frac{\int_{S^{n-1}}\left(\int_s^r \mathscr{A}(t,v)\,dt\right)\omega_{S^{n-1}}}{\int_{S^{n-1}}\left(\int_s^r \underline{\mathscr{A}}(t,\underline{v})\,dt\right)\omega_{S^{n-1}}} = \frac{1}{\mathrm{Vol}(S^{n-1})}\cdot\int_{S^{n-1}} \frac{\int_s^r \mathscr{A}(t,v)\,dt}{\int_s^r \underline{\mathscr{A}}(t,\underline{v})\,dt}\cdot\omega_{S^{n-1}}.$$

$\uparrow$

$\underline{\mathscr{A}}(t,\underline{v})$ is indep. of $\underline{v}\in S^{n-1}_{\underline{p}}\underline{M}$.

It suffices to show that, for each $v\in S_pM$,

$$\frac{\int_s^r \mathscr{A}(t,v)\,dt}{\int_s^r \underline{\mathscr{A}}(t,\underline{v})\,dt} \quad \text{is decreasing in } s \text{ \& } r.$$

- Suppose $s<r_1<r_2$, then, by putting $\eta(t)=\dfrac{\mathscr{A}(t,v)}{\underline{\mathscr{A}}(t,\underline{v})}$,

$$\begin{aligned}
\int_s^{r_1}\mathscr{A}(t,v)\,dt\int_{r_1}^{r_2}\underline{\mathscr{A}}\,dt &= \int_s^{r_1}\eta\cdot\underline{\mathscr{A}}\,dt\int_{r_1}^{r_2}\underline{\mathscr{A}}\,dt\\
&\geq \eta(r_1)\cdot\int_s^{r_1}\underline{\mathscr{A}}\,dt\int_{r_1}^{r_2}\underline{\mathscr{A}}\,dt \geq \int_s^{r_1}\underline{\mathscr{A}}\,dt\int_{r_1}^{r_2}\eta\cdot\underline{\mathscr{A}}\,dt\\
&= \int_s^{r_1}\underline{\mathscr{A}}\,dt\int_{r_1}^{r_2}\mathscr{A}\,dt
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_s^{r_1}\mathscr{A}\,dt\cdot\int_s^{r_2}\underline{\mathscr{A}}\,dt &= \int_s^{r_1}\mathscr{A}\,dt\cdot\left(\int_s^{r_1}\underline{\mathscr{A}}\,dt+\int_{r_1}^{r_2}\underline{\mathscr{A}}\,dt\right)\\
&\geq \int_s^{r_1}\mathscr{A}\,dt\cdot\int_s^{r_1}\underline{\mathscr{A}}\,dt+\int_s^{r_1}\underline{\mathscr{A}}\,dt\cdot\int_{r_1}^{r_2}\mathscr{A}\,dt\\
&= \int_s^{r_1}\underline{\mathscr{A}}\,dt\cdot\int_{r_1}^{2}\mathscr{A}\,dt.
\end{aligned}$$

$$\frac{\int_s^{r_1}\mathscr{A}\,dt}{\int_s^{r_1}\underline{\mathscr{A}}\,dt} \geq \frac{\int_s^{r_2}\mathscr{A}\,dt}{\int_s^{r_2}\underline{\mathscr{A}}\,dt}. \quad \checkmark$$

Similarly, one can get monotonicity in s. $\square$