§6. Manifolds of Non-positive Curvature.

§6.1 Fundamental group.

Let M^n be a connected diff mfd. Define its fundamental group:

$$\pi_1(M, p) = \{ \gamma : [0,1] \to M \text{, continuous} \mid \gamma(0) = \gamma(1) = p \} / \sim$$

where $\gamma_0 \sim \gamma_1$ iff $\exists$ homotopy equivalence, based at p,

$$h : [0,1] \times [0,1] \to M, \qquad \gamma_0 = h(0, \cdot), \quad \gamma_1 = h(1, \cdot)$$

$$h(\cdot, 0) \equiv h(\cdot, 1) \equiv p.$$

- Trivial element: $\gamma(s) \equiv p$. The class is denoted $[pt]$.
- Group structure:
  - product $(\gamma_1 * \gamma_2)(s) = \begin{cases} \gamma_1(2s), & 0 \le s \le \frac{1}{2} \\ \gamma_2(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$.
  - inverse $\gamma^{-1}(s) = \gamma(1-s)$.
- The mfd M is called simply-connected, if $\pi_1(M, p) = \{[pt]\}$.

Def (Covering map) A covering map $\phi : M^n \to N^n$, is a diff map, s.t., for any $p \in N$, $\exists$ neighborhood $U \subset N$, s.t.,

$$\phi^{-1}(U) = \bigcup U_\alpha \subset M, \quad \text{with each}$$

$$\phi|_{U_\alpha} : U_\alpha \to U \quad \text{is diffeomorphic}$$

$$\& \quad U_\alpha \cap U_\beta = \phi, \quad \forall\, \alpha \neq \beta.$$

- The covering map has curve lifting property:

Any curve $\gamma: [0, \ell] \to N$, $\gamma(0) = p$, any $\tilde{p} \in M$ with

$\phi(\tilde{p}) = \gamma(0) = p$, $\exists!$ curve $\tilde{\gamma}: [0, \ell] \to M$,

s.t: $\phi \circ \tilde{\gamma} = \gamma$.

$$\begin{tikzcd} & \tilde{M} \arrow[d, "\phi"] \\ {[0,\ell]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

- The covering map also has homotopy lifting property:

for any homotopy $h: [0,1] \times [0,1]$ based at $p \in N$,

for any $\tilde{p} \in M$, with $\tilde{p} \in \phi^{-1}(p)$, $\exists!$ lifting homotopy

$\tilde{h}: [0,1] \times [0,1] \to M$, s.t.,

$\tilde{h}(\cdot, 0) \equiv \tilde{p}$, $\phi \circ \tilde{h} = h$.

$$\begin{tikzcd} & M \arrow[d, "\phi"] \\ {[0,1] \times [0,1]} \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "h"'] & N \end{tikzcd}$$

- For any mfd M, $\exists!$ universal covering space $\tilde{M}$,

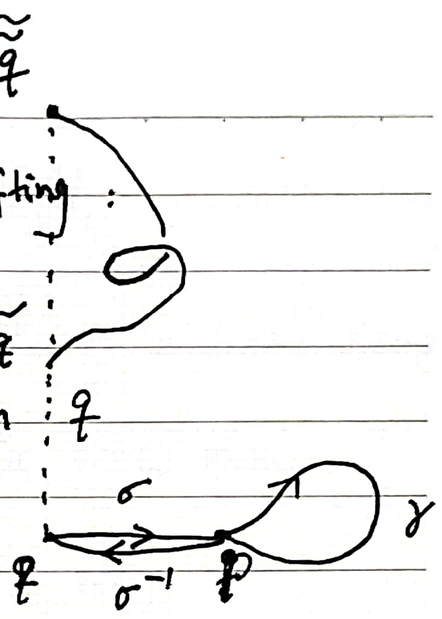
together with a covering map $\pi: \tilde{M} \to M$.

s.t., $\tilde{M}$ is simply connected.

- Any map $f: M \to N$ induce a morphism

$$f_*: \pi_1(M, p) \to \pi_1(N, f(p)).$$

by $$[\gamma] \mapsto [f \circ \gamma].$$



- $\pi_1(M,p)$ acts on $\tilde{M}$ via homotopy lifting:

  $\forall\ \tilde{q} \in \tilde{M}$, $\quad q = \pi(\tilde{q})$.

  $\forall\ [\gamma] \in \pi_1(M,p)$, take a curve σ from q to p. Then take the lifting of $\sigma^{-1} * \gamma * \sigma$, starting at $\tilde{q}$. The end point $\tilde{\tilde{q}} \in \pi^{-1}(q)$ is defined to be the action: $\quad \tilde{\tilde{q}} = [\gamma](\tilde{q})$.

  σ: a curve from q to p.

$$\pi_1(M,p) \times \tilde{M} \to \tilde{M}, \quad \tilde{q} \mapsto \tilde{\tilde{q}} = [\gamma](\tilde{q})$$

$\because$ Homotopy lifting $\Rightarrow$ the action $\tilde{q} \mapsto \tilde{\tilde{q}}$ is well-defined

§6.2. Cartan-Hadamard Theorem

<u>Theorem (Cartan-Hadamard)</u>. Suppose (M^n, g) is complete, connected, with $K \leq 0$. Then

$$\exp_p : T_pM \to M \quad \text{is a covering map}, \quad \forall p \in M.$$

In particular, if M is simply connected, then $\exp_p$ is a diffeomorphism, $\forall p \in M$.

The proof uses the local isometry of Rie. mfds.

<u>Def</u> (Local isometry). A map $f: M \to N$ is a local isometry if: ① f is local diffeomorphism, in the sense that

for any $p \in M$, $\exists$ neighborhood $U \subset M$, s.t.,

$$f: U \to f(U) \subset N \quad \text{is a diffeomorphism.}$$

~~② The restriction in ① is an isometry, such that~~

② f is isometry on each T_pM, such that

$$\langle df_p(u), df_p(v)\rangle = \langle u, v\rangle, \quad \forall u, v \in T_pM.$$

- The local isometry iduces isomorphism of Levi-Civita connection

$$df(\nabla_X Y) = \nabla_{df(X)} df(Y), \quad \forall X, Y \in \Gamma(TM).$$

∵ The formula is understood pointwisely.

- Local isometry iduces same curvature

$$df(R_{X,Y} Z) = R_{df(X), df(Y)} df(Z), \quad \forall X, Y, Z \in \Gamma(TM)$$

- Local isometry maps geodesics to geodesics, with same speed:

$$\nabla_{(\gamma\circ f)'} (\gamma \circ f)' = \nabla_{df(\gamma')} df(\gamma') = df(\nabla_{\gamma'} \gamma').$$

In fact, we may assume $\sigma(s) \neq 0$, $\forall s \in [0, \ell]$, then write

$$\sigma'(s) = r(s) \cdot \sigma(s) + w(s), \qquad w(s) \perp \sigma(s),$$

So: $L(\gamma) = \int_0^\ell \left| d(\exp_p)_{\sigma(s)} \sigma'(s) \right| ds = \int_0^\ell \sqrt{r^2 \left| d(\exp_p)_\sigma \sigma \right|^2 + \left| d(\exp_p)_\sigma w \right|^2} \, ds$

$$\geq \int_0^\ell |r(s)| \cdot \left| d(\exp_p)_{\sigma(s)} \sigma(s) \right| ds = \int_0^\ell |r \cdot \sigma| \, ds.$$

Notice that: $|\sigma|' = \frac{\langle \sigma', \sigma \rangle}{|\sigma|} = r \cdot |\sigma|$

It implies: $L(\gamma) \geq \int_0^\ell |r \cdot \sigma| \, ds \geq \int_0^\ell r \cdot |\sigma| \, ds = \int_0^\ell |\sigma|' \, ds = |\sigma|(\ell)$

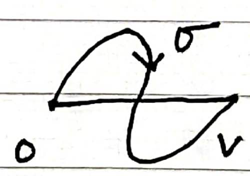
"=" holds iff $r(s) \geq 0$ & $w(s) = 0$, $\forall s \in [0, \ell]$.

i.e., $\sigma'(s) = r(s) \cdot \sigma(s)$, $r(s) \geq 0$, $\forall s \in [0, \ell]$.

So: $\sigma(s) = f(s) \cdot \sigma(\ell)$ for some function f with $f' \geq 0$. #

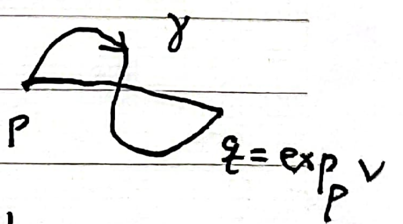
Now, Let $\gamma(s) : [0, \ell] \to M$ be any curve joining p & $q = \exp_p v$.

If $\gamma \subset \exp_p B_r(0)$, then $\exists\ \sigma : [0, \ell] \to B_r(0)$,



s.t., $\exp_p \sigma(s) = \gamma(s)$.

$\circledast \Rightarrow L(\gamma) \geq |\sigma(\ell)| = |v|$, with "=" iff



$\sigma(s) = f(s) \cdot v$.

Because γ, as well as σ, is of unit speed,

$\sigma(s) = \frac{s}{\ell} v$, & $\gamma(s) = \exp_p\left(\frac{s}{\ell} v\right)$.

If, on the other hand, $\gamma \not\subset \exp_p(B_r(0))$, then $\exists\, s_0 \in (0, \ell]$,

s.t., $\gamma(s_0) = \exp_p v_0$, with $r < |v_0| < \mathrm{inj}_p$,

& $\gamma|_0^{s_0} \subset \exp_p(\overline{B_{|v_0|}(0)})$.

Apply ⊛ on $\overline{B_{|v_0|}(0)} \subset T_pM$,

$$L(\gamma) \geq L(\gamma|_0^{s_0}) \geq |v_0| > r. \qquad \square$$

We shall apply the following Lemma in Cartan-Hadamard Theorem

<u>Lemma</u>: Let $\phi: M^n \to N^n$ be a surjective, local isometry of Rie. mfds. Then ϕ is a covering map, if M is complete.

Pf: For any, $p \in N$, we show that $B_r(p) = \exp_p(B_r(0))$ satisfyies the neighborhood property in covering map, where $r < \frac{1}{2}\mathrm{inj}_p$.

Let $\phi^{-1}(p) = \{p_\alpha\}$. <u>Consider diagram</u>:

$$\begin{tikzcd} T_{p_\alpha}M \arrow[r, "d\phi"] \arrow[d, "\exp_{p_\alpha}"'] & T_pN \arrow[d, "\exp_p"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

- Commutable. $\forall\, v \in T_{p_\alpha}M$,

$$\phi(\exp_{p_\alpha} v) = \phi(\gamma_v(1))$$
$$= \gamma_{d\phi(v)}(1) = \exp_p(d\phi(v)).$$

$$\begin{tikzcd} T_{p_\alpha}M \supset B_r(0) \arrow[r, "d\phi"] \arrow[d, "\exp_{p_\alpha}"'] & B_r(0) \subset T_pM \arrow[d, "\exp_p"] \\ B_r(p_\alpha) \arrow[r, "\phi"] & B_r(p) \end{tikzcd}$$

- Restriction on $B_r(0) \subset T_{p_\alpha}M$.

Let $\exp_{p_\alpha}(B_r(0)) = B_r(p_\alpha)$.

Commutable $\Rightarrow$ $\exp_{p_\alpha}: B_r(0) \to B_r(p_\alpha)$ diffeomorphism.

Lemma (Gauss Lemma): Let (M^n, g) be any complete, Rie. mfd.

① At any $v \in T_pM$, $v \neq 0$,

$$\langle d(\exp_p)_v w, \ d(\exp_p)_v v \rangle = \langle w, v \rangle, \qquad \forall \ w \in T_pM.$$

② For any $p \in M$, $\exists \ r > 0$, s.t., $\gamma_v(s) = \exp_p(sv)$, $0 \leq s \leq 1$, is the unique minimal geodesic joining p & $\exp_p v$, whenever $|v| < r$. In particular, $d(p, \exp_p v) = |v|$, $\forall \ |v| < r$.

Pf: ① Let $\gamma(s,t) = \exp_p(s(v + tw))$, $(s,t) \in [0,1] \times [0,1]$, be a variation. The Jacobi field $J(s) = \frac{\partial}{\partial t}\Big|_{t=0} \gamma(s,t)$, satisfies:

$$J(1) = d(\exp_p)_v w, \quad J(0) = 0, \quad J'(0) = w.$$

$$\langle J, \gamma' \rangle(0) = 0, \quad \langle J', \gamma' \rangle(0) = \langle w, v \rangle, \quad \langle J'', \gamma' \rangle(s) \equiv 0.$$

$$\Rightarrow \quad \langle J, \gamma' \rangle(1) = \int_0^1 \langle J', \gamma' \rangle(s)\, ds = \int_0^1 \langle J', \gamma' \rangle(0)\, ds = \langle w, v \rangle.$$

② Let r be any radius such that $r < inj_p$, where

$$inj_p = \sup\{ s > 0 \mid \exp_p \text{ defines diffeomorphism from } B_s(0) \subset T_pM \text{ onto its image} \}.$$

We claim that, for any curve on tange space

$$\sigma : [0, \ell] \to B_r(0) \subset T_pM, \quad \text{with } \sigma(0) = 0,$$

the curve $\gamma(s) = \exp_p \sigma(s)$ has length

$$L(\gamma) \geq |\sigma(\ell)|, \quad \text{with "=" iff } \sigma(s) = f(s) \cdot \sigma(\ell) \qquad (*)$$

for some monotone function f.

- $\bigcup B_r(p_\alpha) = \phi^{-1}(B_r(p))$: $\quad \bigcup B_r(p_\alpha) \subset \phi^{-1}(B_r(p))$ ✓

  $\bigcup B_r(p_\alpha) \supset \phi^{-1}(B_r(p))$: curve lifting. ✓.

- $B_r(p_\alpha) \cap B_r(p_\beta) = \phi, \quad \forall \alpha \neq \beta.$

Only need to show $d(p_\alpha, p_\beta) > 2r, \quad \forall \alpha \neq \beta.$

If, so, then, $\forall x \in B_r(p_\alpha), \quad y \in B_r(p_\beta),$

$$d(x,y) \geq d(p_\alpha, p_\beta) - d(p_\alpha, x) - d(p_\beta, y) > 2r - r - r = 0.$$

Finally, suppose $d(p_\alpha, p_\beta) \leq 2r$, then choose a minimal geodesic $\tilde{\gamma}$ connecting p_α, p_β. Then $\gamma = \phi \circ \tilde{\gamma}$ is a geodesic with endpoints p. The length $L(\gamma) = L(\tilde{\gamma}) \leq 2r$.

It's impossible because $r < \frac{1}{2} \mathrm{inj}_p$. □

Proof of Cartan-Hadamard Theorem:

Let $\gamma: [0, \ell] \to M$ be any unit speed geodesic. Let $J \in \mathcal{J}_0^\perp$ be any Jacobi field, with $a = |J'(0)| > 0$. Then Rauch comparison gives:

$$|J(s)| \geq a \cdot s \qquad \forall \ s \in [0, \ell].$$

Recall the Jacobi fields on Euclidean space are

$$\underline{J}(s) = a \cdot s \cdot e, \qquad \text{for a unit vector } e.$$

Thus, Any geodesic on M is conjugate point free.

- $\phi = \exp_p : T_pM \to M$ is nondegenerate everywhere.

Let $\tilde{g} = \phi^* g$ be the pull-back metric, defined by

$$\tilde{g}(u,v) = g(d\phi(u), d\phi(v)), \qquad \forall\ u,v \in T_wT_oM \cong T_oM,$$

Then $\phi : (T_pM, \tilde{g}) \to (M, g)$ is a local isometry.

- $(T_pM, \tilde{g})$ is complete, so, by previous lemma, ϕ is a covering.

$\because$ For any $v \in T_pM$, $|v| = 1$, $\sigma(s) = sv$ is a geodesic on $(T_pM, \tilde{g})$. Actually, since $\gamma(s) = \exp_p(\sigma(s)) = \phi(\sigma(s))$ is a geodesic on M, the lifting σ is a unit speed geodesic.

$\therefore$ Hopf - Rinow $\Rightarrow$ $(T_pM, \tilde{g})$ is complete. $\square$

§6.3 Preissman Theorem

Theorem (Preissman): Let (M^n, g) be a compact Rie. mfd. The sectional curvature $K < 0$ anywhere. Then, any nontrivial, Abelian subgroup $H \subset \pi_1(M, p)$ is infinite cyclic.

The proof depends on the action $\pi_1(M,p)$ on $\tilde{M}$. We start with a definition.

Def: Let $\gamma : \mathbb{R} \to \tilde{M}$ be a geodesic. An isometry $f : \tilde{M} \to \tilde{M}$, without fixed points, is a translation along γ, if $f(\gamma) = \gamma$.

- The translation satisfies that $f(\gamma(s)) = \gamma(s+\theta)$, $\forall s \in \mathbb{R}$, for some $\theta \in \mathbb{R}$, $\theta \neq 0$.

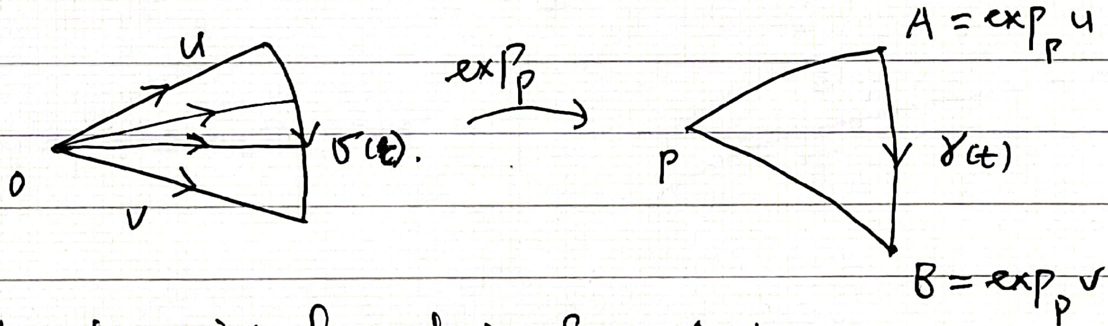
Lemma: $\exp_p : T_p\tilde{M} \to \tilde{M}$ is an expanding map, i.e.,

$$d(\exp_p u, \exp_p v) \geq |u-v|, \quad \forall\, u, v \in T_p\tilde{M}.$$

In particular, any geodesic $\gamma : \mathbb{R} \to \tilde{M}$ is a line, i.e.,

$$d(\gamma(s), \gamma(t)) = |s-t|, \quad \forall\, t, s \in \mathbb{R}.$$

Pf: The proof is totally same as the local version of Toponogov Theorem.



Let $\gamma(t)$ be the minimal geodesic from A to B.

Since $\exp_p$ is a diffeomorphism, $\sigma(t) = \exp_p^{-1}(\gamma(t))$, is well defined. Then, consider the variation

~~$\tilde{\gamma}(s,t) = \exp_p(s(\sigma(t)))$~~ $\tilde{\gamma}(s,t) = \exp_p(s \cdot \sigma(t))$.

Jacobi field $J(s,t) = \frac{\partial}{\partial t}\tilde{\gamma}(s,t) = s \cdot d(\exp_p)_{s\cdot\sigma(t)}\sigma'(t)$.

Rauch comparison $\Rightarrow$ $|J(1,t)| \geq |\sigma'(t)|$.

$$L(\gamma) = d(A,B) = \int_0^1 |J(1,t)|\,dt \geq \int_0^1 |\sigma'(t)|\,dt \geq |u-v|.$$

If γ is a unit speed geodesic on $\tilde{M}$, then, by exponential map

at $\gamma(s)$, $\quad d(\gamma(s), \exp_{\gamma(s)}((t-s)\gamma'(s))) = d(\gamma(s), \gamma(t)) \geq |t-s|$.

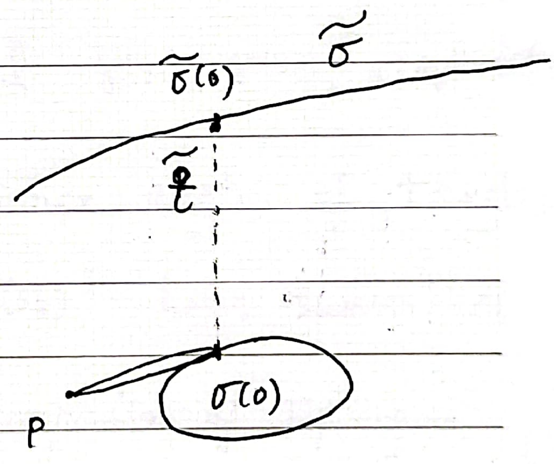
So: $\quad d(\gamma(s), \gamma(t)) = |t-s|$. $\square$

- Any element $[\gamma] \in \pi_1(M, p)$ defines a translation on $\tilde{M}$.

The main ingredient is to construct the line preserved by $[\gamma]$.

Recall that, $\exists$ closed geodesic σ in free homotopy class $[\gamma]$.

Let $q = \sigma(0)$, pick $\tilde{q} \in \pi^{-1}(q) \in \tilde{M}$, and let $\tilde{\sigma}$ be the geodesic, lifting of σ starting at $\tilde{q}$.



By construction, the action of $[\gamma]$ on $\tilde{\sigma}$ is:

$$[\gamma] \cdot \tilde{\sigma}(s) = \tilde{\sigma}(s+\theta)$$

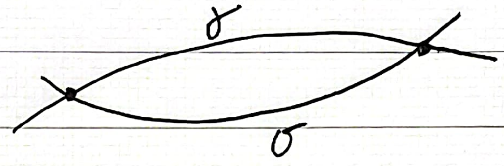
where $\theta = L(\sigma)$.

<u>Lemma</u>: Suppose f: is a translation along a geodesic $\sigma \subset \tilde{M}$. Then σ is the unique geodesic left invariant by f.

<u>pf</u>: Suppose γ is another geodesic such that $f(\gamma) = \gamma$.

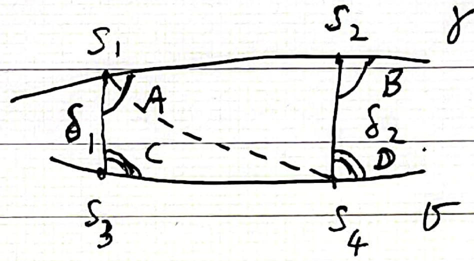
- $\gamma \cap \sigma = \phi$. Suppose otherwise $\gamma(s_1) = \sigma(s_2)$.

Then $f(\gamma(s_1)) = f(\sigma(s_2))$, i.e., $\exists$ another intersection pt.



Contradicts with diffeomorphism

exp at $\gamma(s_1) = \sigma(s_2)$.



- Suppose $f(\gamma(s_1)) = \gamma(s_2)$,

  $f(\sigma(s_3)) = \sigma(s_4)$.

Let δ_1, δ_2 be minimal geodesics joining $\gamma(s_1)$, $\sigma(s_3)$ and $\gamma(s_2)$, $\sigma(s_4)$ respectively.

As in the diagram, $\measuredangle A = \measuredangle B$, $\measuredangle C = \measuredangle D$

The inner angles of the quadrilateral $\gamma(s_1), \gamma(s_2), \sigma(s_4), \sigma(s_3)$ have a summation 2π. So, the inner angles of triangles $\Delta\gamma(s_1)\gamma(s_2)\sigma(s_4)$ and $\Delta\gamma(s_1)\sigma(s_4)\sigma(s_3)$ have summations $\geq 2\pi$. It contradicts with angle comparison of Toponogov. $\square$

<u>Lemma</u>: Let f, h be isometries without fixed points on $\tilde{M}$.

If f is a translation along γ, h commutes with f, then h is also a translation along γ.

<u>Pf</u>: $f \circ h(\gamma) = h \circ f(\gamma) = h(\gamma)$.

Then apply uniqueness of geodesics left invariant by f. $\square$

<u>Proof of Preissman Theorem</u>: Let $H \subset \pi_1(M, p)$ be a nontrivial Abelian subgroup. Take an element $g \in H$, it defines a translation along a geodesic γ on $\tilde{M}$.

By the previous lemma, each element $h \in H$ is a translation along γ. Define a map

$$\theta: H \longrightarrow \mathbb{R}, \qquad \theta(h) \in \mathbb{R} \text{ satisfies}$$
$$h(\gamma(0)) = \gamma(\theta(h)).$$

- θ is a group homomorphism.

It follows from the trivial fact $h(\gamma(s)) = \gamma(s + \theta(h))$, $\forall s \in \mathbb{R}$.

- $\inf \{ |\theta(h)| \mid h \in H,\ h \neq id \} > 0$.

By construction of translation,

$$\theta(h) = \pm \text{ length of closed geodesic in free homotopy class } h.$$

The length of closed geodesics in a compact mfd M is uniformly bounded away from 0, say bounded below by injectivity radius of the mfd.

So: θ is injective and $\theta(H)$ is discrete. H is isomorphic to a discrete subgroup of $\mathbb{R}$. $\square$