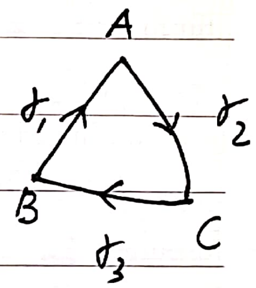
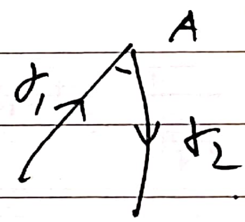
§5. Toponogov Comparison Theorem

Def: A geodesic triangle, $\triangle ABC$, is a set of three vertices, A, B, C, together with three edges $\gamma_1, \gamma_2, \gamma_3$, of unit speed minimal geodesics connecting vertices.



- Denote by $\measuredangle A$, $\measuredangle B$, $\measuredangle C$ the angles formed by the edges.

Def: A hinge is a point A, together with two minimal geodesics segments γ_1, γ_2 with $\gamma_1(l_1) = \gamma_2(0) = A$.



~~Denote by $\measuredangle A$ the angle formed by γ_1, γ_2~~ at A.

Theorem (Toponogov) (M^n, g), complete Rie. mfd. sectional curv. $K \geq K_0$

Let $(\underline{M}, \underline{g})$ be 2-dim space form of curvature K_0.

(i) (Angle version) For a geod. triang $\triangle ABC \subset M$, $\exists$ comparison triangle $\triangle \underline{A}\,\underline{B}\,\underline{C} \subset \underline{M}$ of same edge lengths, s.t.:

$$\measuredangle \underline{A} \leq \measuredangle A, \qquad \measuredangle \underline{B} \leq \measuredangle B, \qquad \measuredangle \underline{C} \leq \measuredangle C.$$

The comparison triangle is unique up to congruence.

(ii). (Hinge version) For hinge (A, γ_1, γ_2), $\exists$ comparison hinge $(\underline{A}, \underline{\gamma_1}, \underline{\gamma_2})$, s.t.:

$$d(\gamma_1(0), \gamma_2(l_2)) \leq \underline{d}(\underline{\gamma_1}(0), \underline{\gamma_2}(l_2)).$$

Pf: The proof is take from Cheeger-Ebin « Comparison theorems in Riemannian Geometry ».

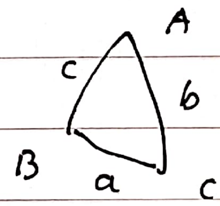
There are a number of steps. We work with $\underline{M}$ of curvature $K_0 - \varepsilon$ at first, instead of $\underline{M}$ of curvature K_0.

Step 1: Let $(\underline{A}, \underline{\gamma_1}, \underline{\gamma_2})$ be a hinge in $\underline{M}$, with $L(\gamma_i) = l_i$.

Then $\underline{d}(\underline{\gamma_1}(0), \underline{\gamma_2}(l_2))$ increase strictly in $\measuredangle A$.

& $$|l_1 - l_2| \le \underline{d}(\underline{\gamma_1}(0), \underline{\gamma_2}(l_2)) \le \min\left(l_1 + l_2, \frac{2\pi}{\sqrt{K_0 - \varepsilon}} - l_1 - l_2\right).$$

Pf: 1^{st} variation formula. Or apply cosine Law



$K_0 < 0$: $$\cosh(\sqrt{-K_0}\, a) = \cosh(\sqrt{-K_0}\, b) \cdot \cosh(\sqrt{-K_0}\, c) - \sinh(\sqrt{-K_0}\, b) \cdot \sinh(\sqrt{-K_0}\, c) \cdot \cos A.$$

a, b, c: lengths of edges.

$K_0 > 0$:

$$\cos(\sqrt{K_0}\, a) = \cos(\sqrt{K_0}\, b) \cdot \cos(\sqrt{K_0}\, c) + \sin(\sqrt{K_0}\, b)\, \sin(\sqrt{K_0}\, c) \cdot \cos A.$$

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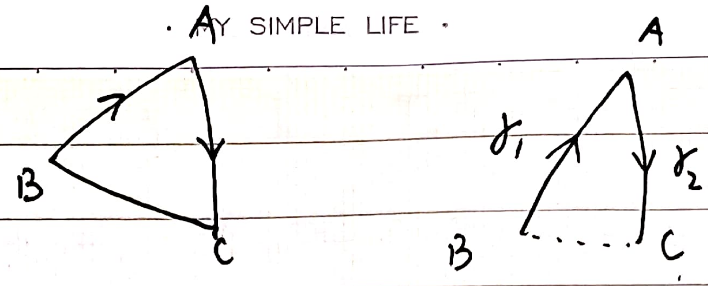
Step 2: The triangle $\Delta \underline{A}\,\underline{B}\,\underline{C} \subset \underline{M}$ of edge length $\le \frac{\pi}{\sqrt{K_0}}$ is unique, up to congrunce.

Pf: Step 1 $\Rightarrow$ The angles $\measuredangle A, \measuredangle B, \measuredangle C$ is determined by edge length. Then apply homogeneous property of $\underline{M}$. #

Step 3.



In comparison triangle $\triangle \underline{A}\underline{B}\underline{C}$, comparison hinge $(\underline{A}, \underline{\gamma_1}, \underline{\gamma_2})$.

$$\measuredangle \underline{A} \le \measuredangle A \iff \underline{d}(\underline{\gamma_1}(0), \underline{\gamma_2}(\ell_2)).$$

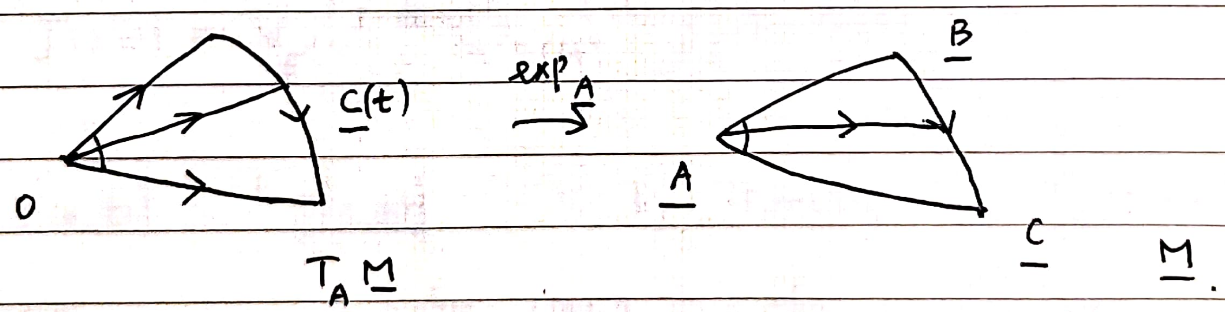
Pf: By monotonicity, Step 1. #

Step 4 (Local version). (i) & (ii) hold for small triangles.

- Small triangle:

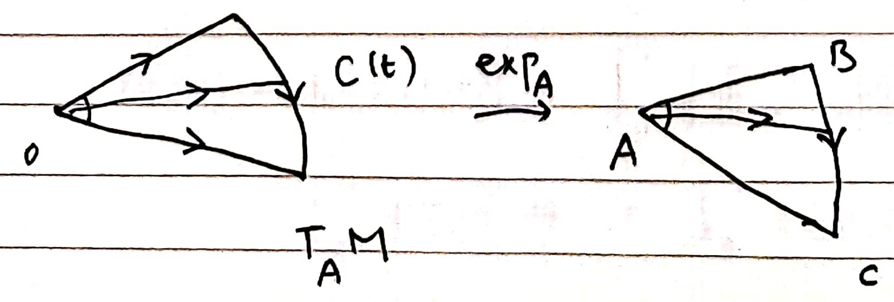
$$d(A,B),\ d(B,C),\ d(A,C) < \tfrac{1}{2}\min(\mathrm{inj}_A, \mathrm{inj}_B, \mathrm{inj}_C).$$

Pf: Apply Rauch comparison theorem. Hinge version.



$\underline{c}(t) \subset T_{\underline{A}}\underline{M}$, $\exp_{\underline{A}}(\underline{c}(t)) = \underline{\gamma}(t)$

minimal geodesic from $\underline{B}$ to $\underline{C}$.



$\gamma(t) = \exp_A c(t)$.

$\exp_A c(0) = B$, $\exp_A c(1) = C$. $c(t) \subset P(0, c(0), c(1))$,

$\exists$ isometric embedding $I : T_{\underline{A}} \underline{M} \to T_A M$. of linear space

$$I(0) = 0, \quad I(\underline{c}(0)) = c(0), \quad I(\underline{c}(1)) = c(1).$$

Let $c(t) = I(\underline{c}(t))$.

Let $\gamma(s,t) = \exp_A(s \cdot c(t))$, $\underline{\gamma}(s,t) = \exp_{\underline{A}}(s \cdot \underline{c}(t))$.

$$s, t \in [0,1].$$

Let $J(s,t) = \frac{\partial}{\partial t}\gamma(s,t)$, $\underline{J}(s,t) = \frac{\partial}{\partial t}\underline{\gamma}(s,t)$ be Jacobi fields.

They satisfy common initials:

$$J(0,t) = \underline{J}(0,t) = 0, \quad \forall\, t \in [0,1].$$

$$J(s,t) = d(\exp_A)_{s\cdot c(t)}(s \cdot c'(t)) = s \cdot d(\exp)_{A\, s\cdot c(t)}(c'(t)).$$

$$\Rightarrow \quad J'(0,t) = \nabla_{\gamma'}\left(s \cdot d(\exp_A)_{s\cdot c(t)}\, c'(t)\right)\Big|_{s=0} = c'(t).$$

$$|J'(0,t)| = |\underline{J}'(0,t)| \quad \text{by isometry } I : T_{\underline{A}} \underline{M} \to T_A M.$$

$$\& \quad \langle J', \gamma' \rangle(0,t) = \langle c'(t), c(t) \rangle = \langle \underline{c}'(t), \underline{c}(t) \rangle = \langle \underline{J}', \underline{\gamma}' \rangle(0).$$

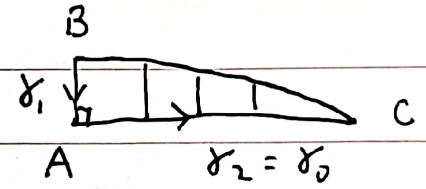
Rauch comparison Thm $\Rightarrow$ $|J(1,t)| \le |\underline{J}(1,t)|$, $\forall\, t \in [0,1]$.

<u>Now</u>, $L(\sigma(t)) = L(\gamma(1,t)) = \int_0^1 \left|\frac{\partial}{\partial t}\gamma(1,t)\right| dt$

$$= \int_0^1 |J(t)|\, dt \le \int_0^1 |\underline{J}(t)|\, dt = L(\underline{\sigma}(t)).$$

Therefore: $d(B,C) \le L(\sigma(t)) \le L(\underline{\sigma}(t)) = \underline{d}(\underline{B}, \underline{C})$. #

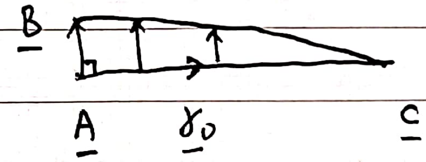
Step 5. (Thin right hinge)



$\gamma(s,t) = \exp_{\gamma_0(s)}(t \cdot f(s)\, e(s))$.

$e(s)$: unit, parallel, orthogonal to γ_0'.

s.t. $B = \exp_{\gamma_0(0)}(f(0)\, e(0))$.



$\underline{\gamma}(s,t) = \exp_{\underline{\gamma}(s)}(t f(s) \cdot \underline{e}(s))$

$\underline{e}(s)$ = parallel, unit, orthogonal to $\underline{\gamma_0}'$.

$(s,t) \in [0,1] \times [0,1]$.

Jacobi fields: $J(s,t) = \frac{\partial}{\partial t}\gamma(s,t)$, $\underline{J}(s,t) = \frac{\partial}{\partial t}\underline{\gamma}(s,t)$.

Rauch comparison: (when $\sup|f(s)| \leq \frac{1}{2}\,\mathrm{inj}_{\underline{\gamma_0}}$

$\Uparrow$

Another version

NOT PROVED, YET.

$\mathrm{inj}_{\gamma_0} = \sup\{ r > 0 \mid \exp : \underline{\nu_{\gamma_0}} \to \underline{M}$ defines diff from $\{ |v| < r \mid v \in \nu_{\gamma_0} \} \}$)

where we require "thin" condition & $K_0 - \varepsilon$.

$$|J(s,1)| \leq |\underline{J}(s,1)|, \quad \forall s \in [0,1].$$

Similarly, as in Step 4,

$$d(B,C) \leq L(\gamma(s,1)) = \int_0^1 |J(s,1)|\, ds \leq \int_0^1 |\underline{J}(s,1)|\, ds$$

$$= d(\underline{B}, \underline{C}).$$

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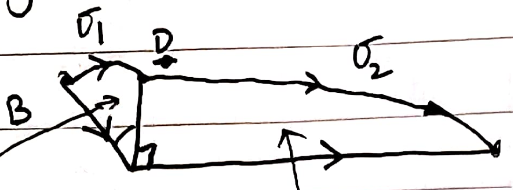
Step 6 Obtuse hinge for (ii).

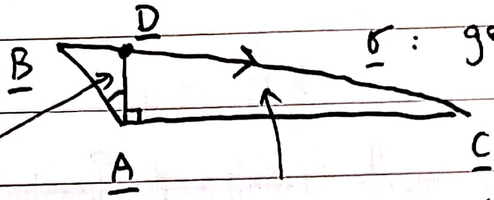


small hinge

right, thin hinge.

all angles at A in a same plane.

$d(A,D) = \underline{d}(\underline{A},\underline{D})$.



γ : geodesic from $\underline{B}$ to $\underline{C}$

Comparison small hinge

Comparison right, thin hinge.

$d(B,D) \leq \underline{d}(\underline{B},\underline{D})$, & $d(C,D) \leq \underline{d}(\underline{C},\underline{D})$

$d(B,C) \leq d(B,D) + d(C,D) \leq \underline{d}(\underline{B},\underline{D}) + \underline{d}(\underline{C},\underline{D}) = \underline{d}(\underline{B},\underline{C})$

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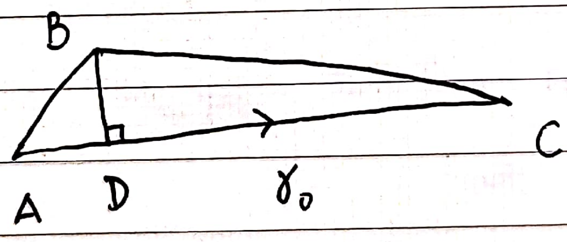
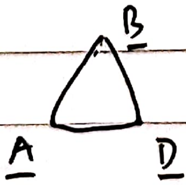
Step 7. Acute hinge for (ii)



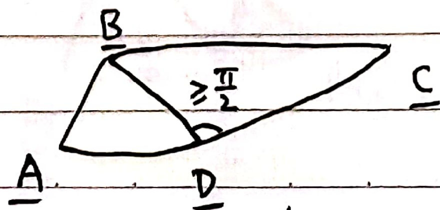
D : closest pt on γ_0 from B.

Comparison triangle $\Delta\underline{A}\,\underline{B}\,\underline{D}$



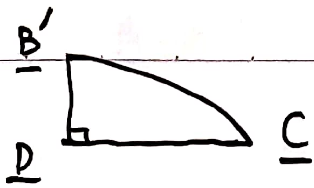
$\measuredangle\underline{A} \leq \measuredangle A$, $\measuredangle\underline{D} \leq \measuredangle D = \frac{\pi}{2}$.

Extend $\underline{A}\,\underline{D}$ to $\underline{C}$:



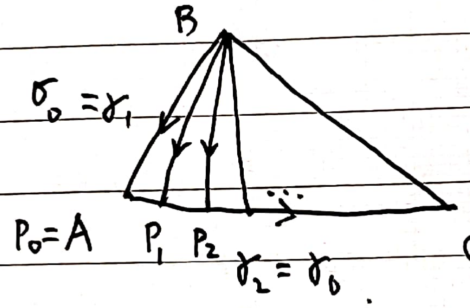
$\Delta\underline{B}\,\underline{C}\,\underline{D}$:

$\measuredangle\underline{D} \geq \frac{\pi}{2}$



$\underline{d}(\underline{B},\underline{C}) \underset{\text{monotone}}{\geq} \underline{d}(\underline{B}',\underline{C}) \underset{\text{thin right}}{\geq} d(B,C)$

Step 8. Decompose a general triangle into "thin" triangles.



$\gamma_0 : [0, \ell_2] \to M.$

Let $P_i = \gamma_0\left(\frac{i}{N}\ell\right)$. $i = 0, \cdots, N.$

$N \gg 1$.

$P_0 = A$, $P_N = C$. $t_i = \frac{i}{N}\ell$

Let σ_i be minimal geodesic from B to P_i, $i = 0, \cdots, N$.

Let $\tau_{i,j} = \gamma_0 \big|_{t_i}^{t_{i+j}}$, $0 \le i \le N$, $0 \le j \le N$

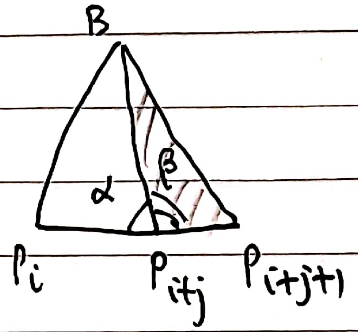
$0 \le i+j \le N$.

$\Delta_{i,j} = \Delta B P_i P_{i+j}$.

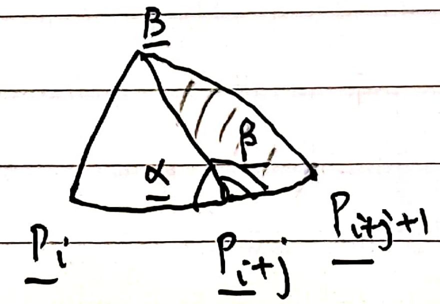
- Each $\Delta_{i,i+1}$ is thin, if N is sufficently large.
- Suppose $\Delta_{i,j}$ satisfies (ii) for each i, j, at vertex P_i.

Then, claim: $\Delta_{i,j+1}$ satisfies (ii) for each i, j at vertex. P_i.

Same argument as Step 7:

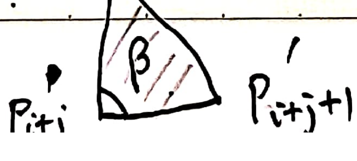


Comparison triangle $\Delta \underline{B} \underline{P_i} \underline{P_{i+j}}$.



Extend to $\underline{P_{i+j+1}}$, with $d(\underline{P_{i+j}}, \underline{P_{i+j+1}}) = d(P_{i+j}, P_{i+j+1})$

$\underline{\beta} \ge \beta$.



Step 9. Approximation of space form $\underline{M}$, as curvature

$$K_0 - \varepsilon \to K_0 .$$

Apply cosine law to get comparison on $\underline{M}$ with curvature K_0. $\square$