Lemma: Suppose γ has no conjugate point ~~along~~ on $[0,l]$.

Let $J \in \mathcal{J}_0^\perp$, X be a vector field, satisfying

$$X \perp \gamma', \quad X(0)=0, \quad X(l) = J(l).$$

Then, $I(X,X) \geq I(J,J)$, with "=" hold iff $X \equiv J$.

Pf: Suppose $J(l) \neq 0$, otherwise $J \equiv 0$.

Take a basis $J_1, \cdots, J_{n-1} = J \in \mathcal{J}_0^\perp$

Then $\{J_i(s)\}_{i=1}^{n-1}$ is a basis of $\gamma'(s)^\perp$.

$\exists$ functions $a^i(s)$, s.t., $X(s) = a^i(s) J_i(s)$, $s \in (0,l]$.

$a^i(l) = 0$, $i=1,\cdots,l-2$; $a^{n-1}(l) = 1$.

- $I(J,J) = \int_0^l \left(|\nabla_{\gamma'} J|^2 - R(\gamma',J,\gamma',J)\right) ds$

where $R(\gamma,J,\gamma',J) = -\langle R_{\gamma',J}\gamma', J\rangle = -\langle J'', J\rangle = -\langle J,J'\rangle' + |J'|^2$.

So, $I(J,J) = \int_0^l \langle J,J'\rangle' ds = \langle J,J'\rangle\Big|_0^l = \langle J,J'\rangle(l)$.

- $I(X,X) = \int_0^l \left(\langle \nabla_{\gamma'}(a^iJ_i), \nabla_{\gamma'}(a^jJ_j)\rangle - a^ia^j R(\gamma',J_i,\gamma',J_j)\right) ds$

$$= \int_0^l \left(\langle (a^i)'J_i + a^iJ_i', (a^j)'J_j + a^jJ_j'\rangle + a^ia^j\langle J_i'', J_j\rangle\right) ds$$

$$R(\gamma',J_i,\gamma',J_j) = -\langle J_i'', J_j\rangle$$

$$\int_0^l a^ia^j\langle J_i'', J_j\rangle ds = a^ia^j\langle J_i', J_j\rangle\Big|_0^l - \int_0^l \left[\left((a^i)'a^j + a^i(a^j)'\right)\langle J_i', J_j\rangle + a^ia^j\langle J_i', J_j'\rangle\right] ds$$

Boundary term $a^ia^j\langle J_i', J_j\rangle\Big|_0^l = \langle a^iJ_i', X\rangle\Big|_0^l = \langle a^iJ_i', X\rangle(l) = \langle J', J\rangle(l)$

So, $I(X,X) = I(J,J) + \int_0^\ell \Big(\langle (a^i)' J_i, (a^j)' J_j \rangle + (a^i)' a^j \langle J_i', J_j \rangle$

$\qquad\qquad - (a^i)' a^j \langle J_i, J_j' \rangle \Big) ds$

We claim that $\langle J_i', J_j \rangle = \langle J_i, J_j' \rangle$ along γ.

Actually, $\langle J_i', J_j \rangle (0) = \langle J_i, J_j' \rangle (0) = 0$, and

$$\big(\langle J_i', J_j \rangle - \langle J_i, J_j' \rangle \big)' = \langle J_i'', J_j \rangle - \langle J_i, J_j'' \rangle$$

$$= -R(\gamma', J_i, \gamma', J_j) + R(\gamma', J_j, \gamma', J_i) = 0.$$

Therefore, $I(X,X) = I(J,J) + \int_0^\ell |(a^i)' J_i|^2 ds \geq I(J,J)$.

"=" only if $(a^i)' \equiv 0$, i.e., $X \equiv J$ along γ. $\square$

Theorem (Rauch comparison for Jacobi fields)

Let M^n, $\underline{M}^n$ be complete Rie. mfds.

Let $\gamma / \underline{\gamma} : [0, \ell] \to M / \underline{M}$ be unit speed geodesics.

Let J, $\underline{J}$ be Jacobi fields along γ, $\underline{\gamma}$.

Assume: ① $K(\gamma', v) \geq \underline{K}(\underline{\gamma}', \underline{v})$ for any $v \in T_{\gamma(s)} M$,

$\underline{v} \in T_{\underline{\gamma}(s)} \underline{M}$, $v \perp \gamma'$, $\underline{v} \perp \underline{\gamma}'$, $\forall s \in [0, \ell]$.

② $J(0) = \underline{J}(0) = 0$, $|J'(0)| = |\underline{J}'(0)|$, $\langle J', \gamma' \rangle (0) = \langle \underline{J}', \underline{\gamma}' \rangle (0)$

③ There is no conjugate point along $\underline{\gamma}$.

Then, $|J(s)| \leq |\underline{J}(s)|$, $\forall s \in [0, \ell]$

We may assume that $\underline{e_n}(s) = \underline{\gamma}'$, $e_n(s) = \gamma'$. and $\underline{e}_{n-1}(s_0) = \underline{X}(s$

$e_{n-1}(s_0) = X(s_0)$.

Suppose: $\underline{X}(s) = \eta^i(s)\, \underline{e}_i(s)$, $s \in [0, s_0]$.

$\underline{X} \perp \underline{\gamma}' \Rightarrow \eta^n \equiv 0$.

Define vector field $Y(s) = \eta^i(s) \cdot e_i(s)$ along γ.

$Y(0) = 0$, $Y(s_0) = e_{n-1}(s_0) = X(s_0)$.

Then, $$\underline{I}(\underline{X}, \underline{X}) = \int_0^{s_0} \left(|\underline{X}'|^2 - \underline{R}(\underline{\gamma}', \underline{X}, \underline{\gamma}', \underline{X}) \right) ds$$
$$= \int_0^{s_0} \left(\sum [(\eta^i)']^2 - |\underline{X}|^2 \cdot \underline{K}(\underline{\gamma}', \underline{X}) \right) ds$$
$$\geqslant \int_0^{s_0} \left(\sum [(\eta^i)']^2 - |Y|^2 K(\gamma', Y) \right) ds$$
$$= I(Y, Y) \geqslant I(X, X).$$

This proves ⊛. □

Remark: ① By construction, the Theorem remains hold when $\dim M \geqslant \dim M_0$.

② . In the proof, the geodesics γ & $\underline{\gamma}$ ~~they~~ may be not unit speed. What used is that that $|\gamma'| = |\underline{\gamma}'|$. □

Pf: By assumption ②, J^T and $\underline{J}^T$ have equal norms. So we may assume that $J \perp \gamma'$, $\underline{J} \perp \underline{\gamma}'$.

If $J'(0) = \underline{J}'(0) = 0$, then both Jacobi fields vanish.

If $J'(0) \neq 0$, then assumption ③ implies that $J(s) \neq 0$, $\forall s \in (0, \ell]$.

Put $f(s) = \dfrac{|\underline{J}(s)|^2}{|J(s)|^2}$, $s \in (0, \ell]$.

$$\lim_{s\to 0} f(s) = \lim_{s \to 0} \frac{\langle \underline{J}, \underline{J}' \rangle}{\langle J, J' \rangle} = \lim_{s\to 0} \frac{|\underline{J}'|^2}{|J'|^2} = 1.$$

So, it suffices to prove that $f'(s_0) \geq 0$, $\forall s_0 \in (0, \ell]$.

Equivalently, $\dfrac{\langle \underline{J}, \underline{J}' \rangle}{|\underline{J}|^2}(s_0) \geq \dfrac{\langle J, J' \rangle}{|J|^2}(s_0)$.

Define Jacobi fields $\underline{X}(s) = \dfrac{\underline{J}(s)}{|\underline{J}|(s_0)}$, $X(s) = \dfrac{J(s)}{|J(s_0)|}$, $s \in [0, s_0]$.

It suffices to prove that: $\langle \underline{X}, \underline{X}' \rangle (s_0) \geq \langle X, X' \rangle (s_0)$. ...... ⊛.

Notice that: $\langle \underline{X}, \underline{X}' \rangle (s_0) = \int_0^{s_0} \langle \underline{X}, \underline{X}' \rangle' ds = \int_0^{s_0} \left(|\underline{X}'|^2 + \langle \underline{X}, \underline{X}'' \rangle \right) ds$

$$= \int_0^{s_0} \left(|\underline{X}'|^2 - \underline{R}(\underline{\gamma}', \underline{X}, \underline{\gamma}', \underline{X}) \right) ds = I(\underline{X}, \underline{X}).$$

& $\langle X, X' \rangle (s_0) = I(X, X)$.

We are going to construct a comparison vector field along γ, and apply comparison of index. Let $\{e_i(s)\}$ be an orthonormal, parallel vector fields along γ; $\{\underline{e}_i(s)\}$ ......, along $\underline{\gamma}$.