§3.2 Completeness, Hopf-Rinow theorem

Let (M^n, g), be a connected Rie. mfd. For any $p, q \in M$, define

$$d(p,q) = \inf \left\{ L(\gamma) \;\middle|\; \gamma : [a,b] \to M \text{ curve with } \begin{array}{l} \gamma(a) = p, \\ \gamma(b) = q \end{array} \right\}$$

- d defines a metric on M:

$$\begin{cases} d(p,q) \geq 0, \quad d(p,q) = 0 \text{ iff } p = q. \\ d(p,q) = d(q,p) \\ d(p_1, p_2) \leq d(p_1, p_3) + d(p_3, p_2), \quad \forall\, p_i \in M \end{cases}$$

- The metric topology coincides with the original topology.

Def (Complete Rie. mfd)

A Rie. mfd is called a complete Rie. mfd if it is complete as a metric space.

Thm (Hopf - Rinow)

For a ~~Rie.~~ Connected Rie. mfd, the followings are equivalent:

(a). (M^n, g) is complete (as a metric space).

(b). (M^n, g) is complete in the sense of geodesic, such that any geodesic can be extended over $\mathbb{R}$.

(c). The exponential map $\exp_p : TM \to M$ can be defined over the whole T_pM, for any $p \in M$.

(d). Any two points p, q can be joined by a minimal geodesic

(e). Any bounded, closed set is compact.

- In particular, if M is complete, then $\exp_p$ is onto.

Example: $\mathbb{R}^n$, $\exp_p v = p+v$

$\Omega \subset \mathbb{R}^n$, domain with g_E, is not complete, only if $\Omega = \mathbb{R}^n$.

*: From now on, a Rie. mfd is always assumed complete.

§3.3 2^{nd} variation formula.

Let $\gamma : [0,\ell] \times (-\varepsilon, \varepsilon) \to M$ be a variation, with γ_0 a unit speed geodesic. Then

Prop: $$\frac{d^2}{dt^2}\Big|_{t=0} L(\gamma_t) = \langle \nabla_Y Y, X\rangle \Big|_0^\ell + \int_0^\ell \left(\left|(\nabla_X Y)^\perp\right|^2 - R(X,Y,X,Y)\right) ds.$$

Pf: At $t=0$:

$$\frac{d^2}{dt^2} L(\gamma_t) = \frac{d}{dt}\int_0^\ell \frac{Y\langle X,X\rangle}{2\sqrt{\langle X,X\rangle}}\, ds = \frac{d}{dt}\int_0^\ell \frac{\langle \nabla_X Y, X\rangle}{|X|}\, ds$$

$$= \int_0^\ell \left(\frac{Y\langle \nabla_X Y, X\rangle}{|X|} - \frac{\langle \nabla_X Y, X\rangle \langle \nabla_X Y, X\rangle}{|X|^3}\right) ds$$

$|X| \equiv 1$ at $t=0$

$$= \int_0^\ell \left(\langle \nabla_Y \nabla_X Y, X\rangle + \langle \nabla_X Y, \nabla_X Y\rangle - \langle \nabla_X Y, X\rangle^2 \right) ds$$

$$\int_0^\ell \langle \nabla_Y \nabla_X Y, X\rangle ds = \int_0^\ell \left(\langle \nabla_X \nabla_Y Y, X\rangle - \langle R_{X,Y} Y, X\rangle \right) ds$$

$$= \underbrace{\int_0^\ell X\langle \nabla_Y Y, X\rangle\, ds}_{=\ \langle \nabla_Y Y, X\rangle |_0^\ell} - \int_0^\ell R(X,Y,X,Y)\, ds$$

& $\langle \nabla_X Y, \nabla_X Y\rangle - \langle \nabla_X Y, X\rangle^2 = \left| \nabla_X Y - \langle \nabla_X Y, X\rangle X\right|^2 = |(\nabla_X Y)^{\perp}|^2$. $\square$

<u>Thm (Bonnet - Myers)</u> (M^n, g), complete, connected.

$Ric \geq (n-1) K_0 g$, for some $K_0 > 0$.

Then, $\mathrm{diam}(M) = \sup\{d(p,q) \mid p, q \in M\} \leq \dfrac{\pi}{\sqrt{K_0}}$.

Pf: Let $p, q \in M$ be any two pts, with $l = d(p,q)$.

Let $\gamma : [0, l] \to M$ be a minimal geodesic connecting p & q.

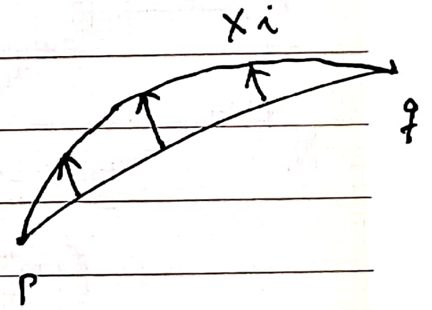
Take $\{e_1, \cdots, e_{n-1}, e_n = \gamma'(0)\} \subset T_pM$, an orthonormal basis.

Let $e_i(s)$, be the parallel translation along γ, $i = 1, \cdots, n-1$.

<u>Define</u> varition fields $X_i(s) = \sin\left(\frac{\pi}{l} s\right) \cdot e_i(s)$, $0 \leq s \leq l$.

<u>For $\forall i$:</u> $\exists$ variation

$\gamma_{i,t}(s) = \exp_{\gamma(s)}(t X_i(s))$



with variation field $X_i(s)$

<u>Now:</u>

$$\frac{d^2}{dt^2}\Big|_{t=0} L(\gamma_{i,t}) = \int_0^l \left(|\nabla_{\gamma'} X_i|^2 - R(\gamma', X_i, \gamma', X_i)\right) ds \geq 0$$

$|\nabla_{\gamma'} X_i(s)|^2 = \left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi}{l} s\right)$. & $R(\gamma', X_i, \gamma', X_i) = \sin^2\left(\frac{\pi}{l} s\right) \cdot R(\gamma', e_i, \gamma', e_i)$

<u>Summing up</u>:

$$0 \leq \int_0^l \left[(n-1) \cdot \left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi}{l} s\right) - \sin^2\left(\frac{\pi}{l} s\right) \cdot Ric(\gamma', \gamma')\right] ds$$

By assumption, $\quad Ric \geq (n-1) K_0\, g$,

$$0 \leq \int_0^{\ell} \left[(n-1) \left(\frac{\pi}{\ell}\right)^2 \cdot \cos^2\left(\frac{\pi}{\ell} s\right) - (n-1) K_0 \cdot \sin^2\left(\frac{\pi}{\ell} s\right) \right] ds$$

$$= (n-1) \cdot \left[\left(\frac{\pi}{\ell}\right)^2 - K_0 \right] \cdot \int_0^{\ell} \sin^2\left(\frac{\pi}{\ell} s\right) ds$$

$$\Rightarrow \quad \left(\frac{\pi}{\ell}\right)^2 - K_0 \geq 0 \,, \quad \text{or}, \quad \ell \leq \frac{\pi}{\sqrt{K_0}} \qquad \square .$$

<u>thm (Synge)</u>: M^{2n}, $K > 0$, compact, Rie. mfd.

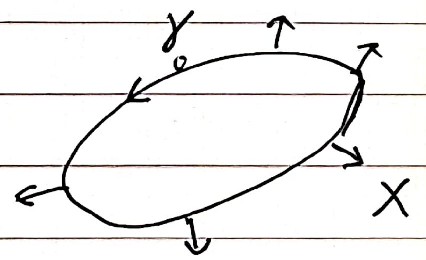
If M is orientable, then M is simply-connected.

证: • Suppose $\exists$ non-trivial homotopy class $\alpha \in \pi_1(M)$.

Then $\exists$ shortest representation $\gamma_0 \in \alpha$, a closed geodesic

• Construct a parallel vector field X along γ_0.

Fix $\gamma_0(0) = p \in M$, consider parallel translation



$$P_{\gamma_0} : T_pM \to T_pM. \qquad P_{\gamma_0} \in SO(T_pM)$$

$P_{\gamma_0}(\gamma_0'(0)) = \gamma_0'(0)$, $\Rightarrow$ $P_{\gamma_0} : \gamma_0'(0)^{\perp} \longrightarrow \gamma_0'(0)^{\perp}$, isomorphism

$\because$ $\dim \gamma_0'(0)^{\perp} = 2n-1$ is odd. $\Rightarrow$ $\exists$ fixed pt, $v \in \gamma_0'(0)^{\perp}$.

$P_{\gamma_0}(v) = v$.

Let $X(s)$ be the parallel vector field on γ_0.

Then, variation formula for $X(s)$, say $\gamma_t(s)$, gives

$$\frac{d^2}{dt^2}\Big|_{t=0} L(\gamma_t) = \int_0^\ell \left(|\nabla_{\gamma_0'} X|^2 - R(\gamma_0', X, \gamma_0', X)\right) ds$$

$$= -\int_0^\ell R(\gamma_0', X, \gamma_0', X)\, ds \quad < 0.$$

Contradicts with shortest assumption of $\gamma_0 \in \alpha$. □

§3.4. Jacobi fields.

Let $\gamma : [0,\ell] \times (-\varepsilon, \varepsilon) \to M$ be a variation of geodesics.

i.e., each $\gamma_t = \gamma(\cdot, t)$ is a geodesic.

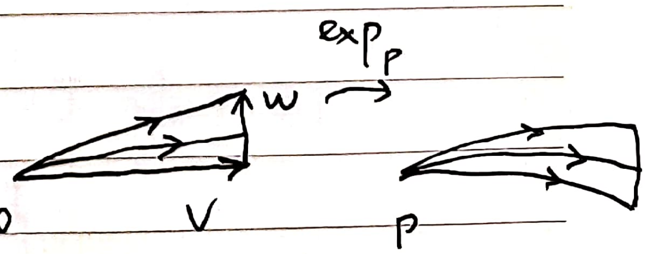
Then, $\quad 0 = \nabla_Y \nabla_X X = \nabla_X \nabla_X Y - R_{X,Y} X$

Def: Let γ be a geodesic. A vector field J along γ is called a Jacobi field, if

$$\nabla_{\gamma'} \nabla_{\gamma'} J - R_{\gamma', J}\gamma' = 0. \qquad (*)$$

- $(*)$ is a 2^{nd} order ODE along γ. It can be solved uniquely according to initial value $J(0)$ & $J'(0) = \nabla_{\gamma'} J(0)$.

- Construction of Jacobi field



Take $v, w \in T_pM$.

$\gamma(s,t) = \exp_p(s(v+tw))$

$J(s) = \frac{\partial}{\partial t}\Big|_{t=0} \gamma(s,t)$, Jacobi field along γ.

∴ $J(0) = 0$.

∴ $J'(0) = \nabla_{\gamma'} J(0) = w$.

In fact, $\nabla_{\gamma'} J(s) = \nabla_{\gamma'} \left[d(\exp_p)_{sv}(sw) \right] = \nabla_{\gamma'} \left[s \cdot d(\exp_p)_{sv}(w) \right]$

$$= d(\exp_p)_{sv}(w) + s \cdot \nabla_{\gamma'} \left[d(\exp_p)_{sv}(w) \right].$$

<u>Lemma</u>: For any Jacobi field J along geodesic γ,

$$\langle J, \gamma' \rangle (s) = \langle J, \gamma' \rangle (0) + \langle J', \gamma' \rangle (0) \cdot s$$

Pf: $\langle J, \gamma' \rangle'' = \langle J'', \gamma' \rangle = \langle R_{\gamma', J} \gamma', \gamma' \rangle = 0$.

& $\langle J, \gamma' \rangle' (0) = \langle J', \gamma' \rangle (0)$. □

• Notice that $(as+b)\gamma'(s)$ is always a Jacobi field,

It follow that, any Jacobi field J admits a decomposition

$$J = J^{\perp} + J^{T}$$

where $J^{T} = \langle J, \gamma' \rangle \cdot \gamma' = \left[\langle J, \gamma' \rangle (0) + \langle J', \gamma' \rangle (0) \cdot s \right] \cdot \gamma'(s)$

& $J^{\perp}$ is orthogonal to γ pointwisely.

• <u>Let</u> $\mathcal{J} = \mathcal{J}_{\gamma} = \{ \text{Jacobi field } J \text{ along } \gamma \}$.

$\mathcal{J}^{\perp} = \{ J \in \mathcal{J},\ J \perp \gamma' \}$.

$\mathcal{J}_0 = \{ J \in \mathcal{J},\ J(0) = 0 \}$. $\mathcal{J}_0^{\perp} = \mathcal{J}^{\perp} \cap \mathcal{J}_0$.

$\therefore\ \mathcal{J} = \mathcal{J} \oplus \{ (as+b)\cdot\gamma'(s) \mid a, b\in\mathbb{R}\}$.

$\mathcal{J}_0 = \mathcal{J}_0^{\perp} \oplus \{ as\cdot\gamma'(s) \mid a\in\mathbb{R}\}$.

$\therefore\ \dim_{\mathbb{R}} \mathcal{J} = n^2$. $\dim \mathcal{J}_0 = n$, $\dim \mathcal{J}_0^{\perp} = n-1$.

<u>Def</u> (Conjugate point) Let $\gamma : [0, l] \to M$ be a unit speed geodesic, $\gamma(0) = p$. A point $q = \gamma(s_0)$, $t_0 \neq 0$, is called a conjugate point of p along γ, if $\exists$ nontrivial Jacobi field J along γ, s.t., $J(0) = J(s_0) = 0$.

- q conjugate to p along $\gamma \Leftrightarrow p$ is conjugate to q along γ^{-1}.
- q conjugate to $p \Leftrightarrow \exists\, J \in \mathcal{J}$, $J \not\equiv 0$, $J(0) = J(s_0) = 0$

Let $q = \exp_p(t_0 v)$. Let $w = J'(0) \neq 0$. then

$\gamma = \gamma_v$ $\Leftrightarrow \exists\, \overset{\neq 0}{w} \in T_pM$, $d(\exp_p)_{s_0 v}(w) = 0$.

<u>i.e.,</u> $\exp_p$ is degenerate at $s_0 v$.

$\boxed{\text{~~If } q = \gamma(t_0) \text{ is not conjugate to } p \text{ along } \gamma\text{~~}}$.

- Suppose p has no conjugate pt on time interval $[0, s_0]$.

Let $J_1, \cdots, J_{n-1} \in \mathcal{J}_0^{\perp}$ be a basis. Then $\{J_i(s)\}_{i=1}^{n-1}$ is a basis of $\gamma'(s)^{\perp}$, $\forall\ s \in [0, s_0]$.

• Jacobi field on space form.

Suppose (M^n, g) has constant sectional curvature $K \equiv k_0$

Then $R(x, y, z, w) = k_0 \left(\langle x, z\rangle \langle y, w\rangle - \langle x, w\rangle\langle y, z\rangle \right)$.

$$R_{x,y} z = k_0 \left(\langle y, z\rangle x - \langle x, z\rangle y \right).$$

Jacobi eq: $\quad J'' + k_0 \left(\langle J, \gamma'\rangle \gamma' - J \right) = 0$

If $J \in \gamma_0^\perp$, then $\quad J'' + k_0 J = 0$

$$J(s) = a \cdot \frac{\sin \sqrt{k_0} s}{\sqrt{k_0}} \cdot e(s)$$

for some unit parallel vector field e. $\quad \left(e(0) \,/\!/\, J'(0) \right)$.

§4. Index of vector fields. Rauch comparison.

§4.1 Index of variation fields.

Let $\gamma : [0, \ell] \to M$ be a unit speed geodesic.

For vector field X along γ, define the index

$$I(x, x) = \int_0^\ell \left(|\nabla_{\gamma'} X|^2 - R(\gamma', X, \gamma', X) \right) ds.$$

• I defines a symmetric form on the space of vector fields on γ:

$$I(X, Y) = \int_0^\ell \left(\langle \nabla_{\gamma'} X, \nabla_{\gamma'} Y\rangle - R(\gamma', X, \gamma', Y) \right) ds.$$