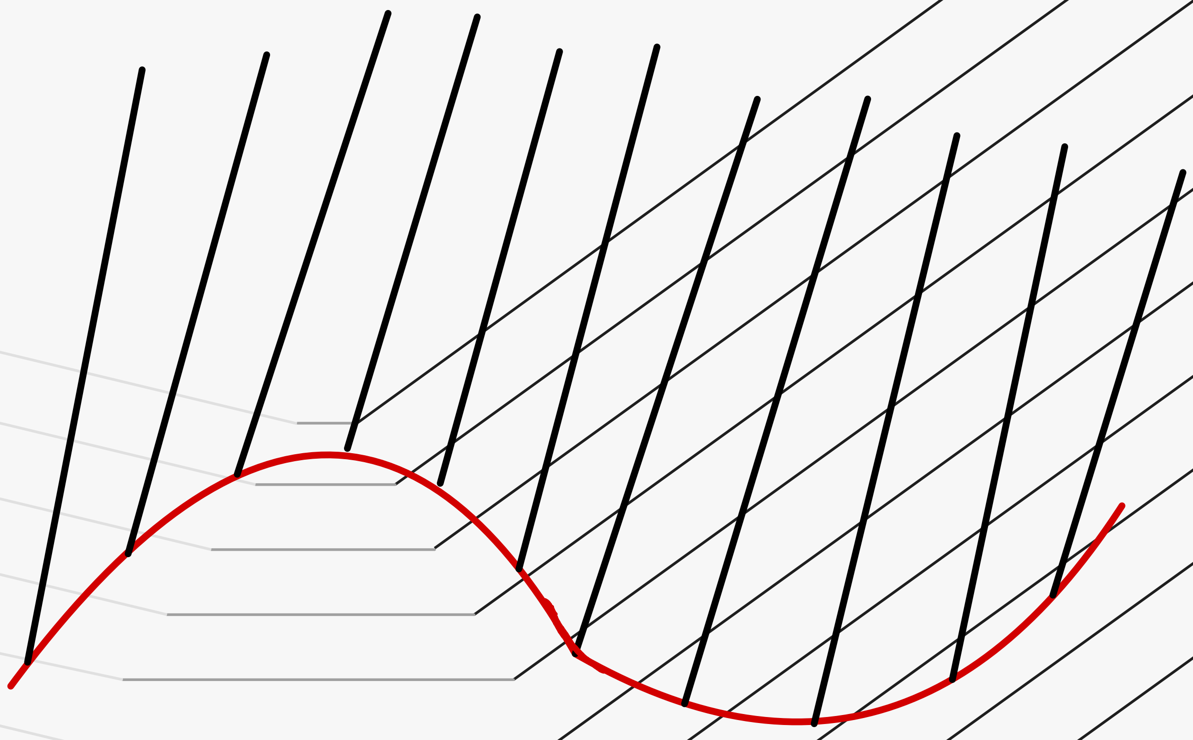
Lecture 5

Complex Vector Bundles



Outline

- Definition of complex vector bundles
- Examples
- $\Omega^{p,q}$ as a bundle & Dolbeault cohomology.
- Hermitian connections & curvature
- Chern classes

not enough time
Move to the next lecture!

- **Def** (Complex vector bundle) A cplx vector bundle of rank r over a differentiable mfd X is a differentiable mfd E together with a smooth surjective map $E \xrightarrow{\pi} X$ s.t.

  ① $\forall\, p \in X$, $\pi^{-1}(p)$ has the structure of r-dim vector space over $\mathbb{C}$

  Write $E_p := \pi^{-1}(p)$, which is called the fiber over p.

  ② $\exists$ open cover $\{U_i\}$ of X s.t. $\pi^{-1}(U_i)$ is diffeomorphic say via φ_i, to $U_i \times \mathbb{C}^r$ & on each overlap $U_i \cap U_j$ the induced map $(U_i \cap U_j) \times \mathbb{C}^r \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (U_i \cap U_j) \times \mathbb{C}^r$ can be identified w/ a smooth map $U_i \cap U_j \to GL(r, \mathbb{C})$

  The pair (U_i, φ_i) is called a local trivialization.

  $\varphi_i \circ \varphi_j^{-1}$ is called transition matrix

- **Def.** (Holomorphic vector bundle) The definition is similar as above. Just replace everywhere "differentiable/smooth" with "complex/holomorphic". Ex. Make the definition precise.

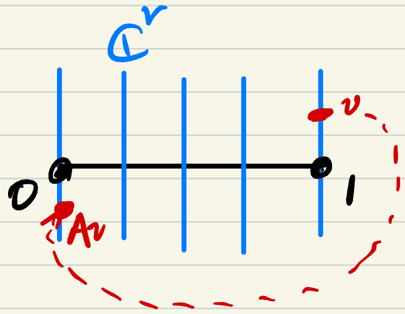
- **Example**.

  ① $E = S^1 \times \mathbb{C}^r$, $X = S^1$. Then E is a (trivial) complex vector bundle of rank r over S^1.

  But this is not a holomorphic vector bundle as S^1 is not a cplx mfd.

② Consider $[0,1] \times \mathbb{C}^r$ together with an element $A \in GL(r,\mathbb{C})$ then we can construct a non-trivial cplx vector bundle over S^1 by identifying $\{0\} \times \mathbb{C}^r$ w/ $\{1\} \times \mathbb{C}^r$ via A.

This is a generalization of the Mobius band.

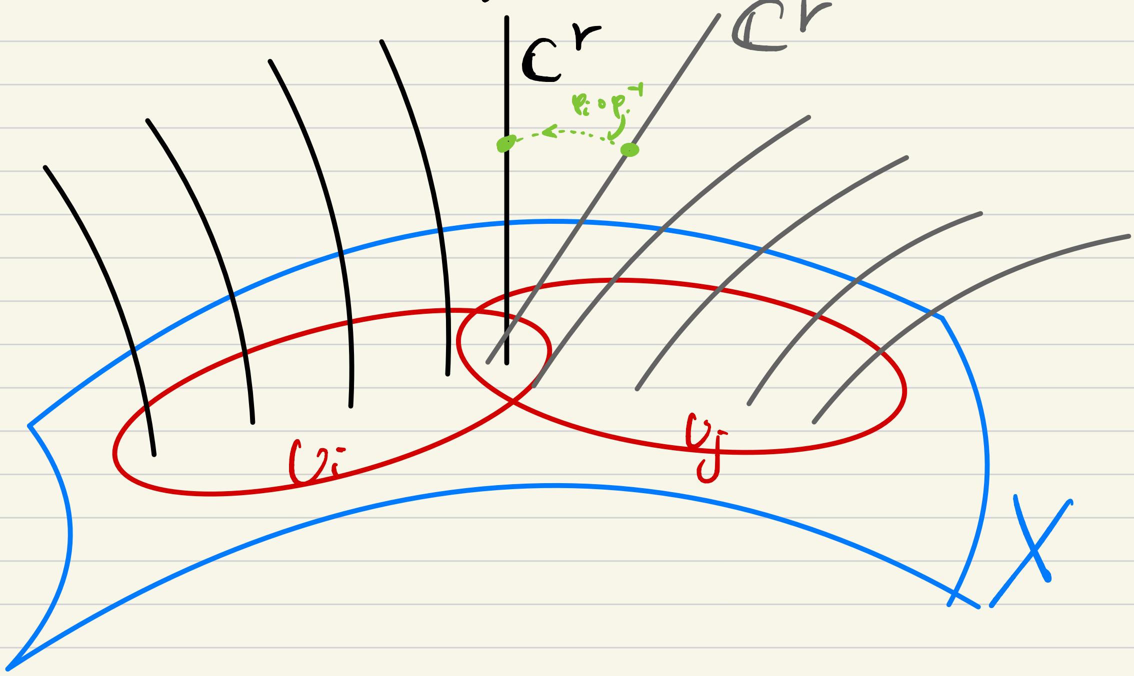


③ Let E be a real vector bundle over a diff. mfd X. Then one can "complexify" it using the following construction.

Let $\{(U_i, \varphi_i)\}$ be a local trivialization of E.

But then let $E^{\mathbb{C}} := \bigsqcup_\alpha U_i \times \mathbb{C}^r / \sim$ where $\sim$ is given as: $(x,v) \sim (y,w)$ for $\begin{cases}(x,v) \in U_i \times \mathbb{C}^r \\ (y,w) \in U_j \times \mathbb{C}^r\end{cases}$ iff

$x = y$ & $v = \varphi_i \circ \varphi_j^{-1}(w)$.



④ Let X be a cplx mfd of dim n.

Then TX is a real vector bundle of rank $2n$ over X. We may look at $TX^{\mathbb{C}}$. Then it's a cplx v.b. of rank $2n$ over X. In each coordinate chart, say $(U,(z^1,\cdots,z^n))$, let $z^i = x^i + \sqrt{-1}\,y^i$.

Then $\begin{cases} TX|_U = U \times \mathrm{Span}_{\mathbb{R}}\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\rangle \\ TX^{\mathbb{C}}|_U = U \times \mathrm{Span}_{\mathbb{C}}\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\rangle. \end{cases}$

But remember that we have a cplx structure J on X:

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i} \quad \& \quad J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}.$$

So $TX^{\mathbb{C}}|_U = TX^{1,0}|_U \oplus TX^{0,1}|_U$, where

$$\begin{cases} TX^{1,0}|_U = U \times \mathrm{Span}_{\mathbb{C}}\langle \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i})\rangle = U \times \mathrm{Span}_{\mathbb{C}}\langle \frac{\partial}{\partial z^i}\rangle \\ TX^{0,1}|_U = U \times \mathrm{Span}_{\mathbb{C}}\langle \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})\rangle = U \times \mathrm{Span}_{\mathbb{C}}\langle \frac{\partial}{\partial \bar{z}^i}\rangle. \end{cases}$$

These $TX^{1,0}|_U$ patch together to a holomorphic v.b. of rank n over X. Indeed, choose another chart, say $(V,(w^1,\cdots,w^n))$, then one has

$$\frac{\partial}{\partial z^i} = \frac{\partial w^j}{\partial z^i}\frac{\partial}{\partial w^j}.$$

Note that the map
$$U \cap V \longrightarrow GL(n,\mathbb{C})$$
$$p \longmapsto \left(\frac{\partial w^j}{\partial z^i}\right)(p)$$
is holomorphic. The resulting bundle $TX^{1,0}$ is called the holomorphic tangent bundle.

Rmk. $TX^{0,1}$ is called "anti-holomorphic" vector bundle.

⑤ The dual bundle of $TX^{1,0}$ is also holomorphic. In fact, in $(U,(z^1,\cdots,z^n))$,

$$(TX^{1,0})^*|_U = U \times \mathrm{Span}_{\mathbb{C}}\langle dz^1,\cdots,dz^n\rangle.$$

$$dz^i = \frac{\partial z^i}{\partial w^j}dw^j.$$ So the transition function is holo. as well.

⑥ Given a cplx/holo. v.b. E, one can construct new cplx/holo. v.b. using

$$\begin{cases} \text{duality} : * \text{ or } \vee \\ \text{tensor product} : \otimes \\ \text{wedge product} : \Lambda^p \text{ \& } \det \\ \text{direct sum} : \oplus \end{cases}$$

Warning! These are only defined up to isomorphism. See the end.

So in particular, $\Lambda^p((TX^{1,0})^*) = \boxed{\Omega^p_X}$ is also holo.

⑦ The tautological line bundle of $\mathbb{C}P^n$.

Define $\mathcal{O}(-1) := \{(p,z)\in \mathbb{C}P^n\times\mathbb{C}^{n+1} \mid z\in p\}$.

e.x. Prove that $\mathcal{O}(-1) \xrightarrow{\pi} \mathbb{C}P^n$, $(p,z) \longmapsto p$ is a holo. v.b. of rank 1 over $\mathbb{C}P^n$.

let $\mathcal{O}(1)$ be the dual of $\mathcal{O}(-1)$.

More generally, put

$$\begin{cases} \mathcal{O}(k) := \underbrace{\mathcal{O}(1)\otimes\cdots\otimes\mathcal{O}(1)}_{k} \\ \mathcal{O}(-k) := \underbrace{\mathcal{O}(-1)\otimes\cdots\otimes\mathcal{O}(-1)}_{k}. \end{cases}$$

⑧ Let X be a cplx mfd. Then put

$$K_X := \Lambda^n((TX^{1,0})^*) = \Omega^n_X$$

In local coordinate $(z^1,\cdots,z^n)$, $K_X|_U = U\times \mathbb{C}.dz^1\wedge\cdots\wedge dz^n$.

Then $dz^1\wedge\cdots\wedge dz^n = \det\left(\frac{\partial z^i}{\partial w^j}\right) dw^1\wedge\cdots\wedge dw^n$.

e.x. Show that $K_{\mathbb{C}P^n} = \mathcal{O}(-n-1)$.

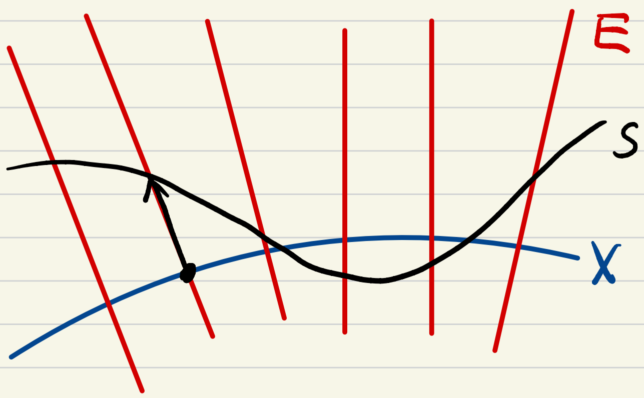
⑨ The bundle of (p,q)-forms: $A^{p,q}$ $\quad p,q\in\{0,\dots,n\}$

Locally it is given by:

$$A^{p,q}\big|_U = U\times \mathrm{Span}_{\mathbb{C}}\left\langle dz^I\wedge d\bar{z}^{\bar{J}} \;\middle|\; \begin{matrix} i_1<i_2<\dots<i_p \\ j_1<\dots<j_q \end{matrix}\right\rangle$$

This is cplx v.b. but usually not holomorphic.

- <u>Def.</u> A smooth/holo $\boxed{\text{section}}$ of a cplx/holo. v.b E over $U\subseteq X$ is a smooth/holo map $s: U\to E$ s.t. $\pi\circ s = \mathrm{id}|_U$.



▲ We say $s(p)=0$ if $s(p)\in E_p\cong\mathbb{C}^r$ is zero.

▲ If E is trivialized over U, so that $\pi^{-1}(U)\cong U\times\mathbb{C}^r$ then $\forall$ section s over U is given by an $\mathbb{C}^r$-valued smooth/holo. function on U.

So a section is a generalization of multivalued functions on U.

▲ If $U=X$, then a section called "global section".

If X is cpt & $E=X\times\mathbb{C}$. Then $\forall$ holo. global section of E is a holo. function on X & hence has to be a const. So this case is not interesting.

This is because the v.b. is trivial. However if we look at non-trivial holo. v.b. over X, it is possible that there exist non-trivial holo. global sections of E.

e.x. Show that there exist notrivial global hol. sections of $\mathcal{O}(1)$. What are they?
How about $\mathcal{O}(k)$, $k>0$?

- Each cplx/holo. v.b. E over X can be naturally identified w/ a sheaf by putting

$$E(U) := \{ \text{smooth/holo. sections } s: U \to E \}.$$

For this reason, the space of global sections of E is usually denoted as $\Gamma(X,E)$ or $H^0(X,E)$.

ex. Check that the above def. indeed gives a sheaf

- Let E be a cplx/holo. v.b. over X. Let $U \subseteq X$ be an open. Say $\mathrm{rk}\, E = r$. We say smooth/holo. sections $s_1, \dots, s_r : U \to E$ is a smooth/holo. $\boxed{\text{frame}}$ if $s_1(p), \dots, s_r(p)$ is linearly indpd in E_p for $\forall\, p \in U$. Using this frame we can identify $\pi^{-1}(U) \cong U \times \mathbb{C}^r$. So in particular a frame gives rise to a local trivialization. Of course, conversely, $\forall$ local trivialization gives a local frame of E. Using this frame, $\forall$ section $s: U \to E$ can be written as

$$s = \sum f^i s_i \quad \text{where } f^i \in C^\infty(U,\mathbb{C}) \text{ or } \mathcal{O}(U).$$

- Frames can help us do computations locally.

- We end this lecture by introducing the Dolbeault cohomology. Using the $\bar{\partial}$-operator, one has a complex

$$0 \to \Gamma(X, A^{p,0}) \xrightarrow{\bar{\partial}} \Gamma(X, A^{p,1}) \xrightarrow{\bar{\partial}} \cdots \to \Gamma(X, A^{p,n}) \to 0.$$

where each $\Gamma(X, A^{p,q})$ denotes the space of global smooth (p,q) forms.

Put

$$H^{p,q}(X) := \frac{\operatorname{Ker}(\bar{\partial}: \Gamma(X, A^{p,q}) \to \Gamma(X, A^{p,q+1}))}{\operatorname{Im}(\bar{\partial}: \Gamma(X, {}^{p,q-1}) \to \Gamma(X, A^{p,q}))}$$

(p,q)-Dolbeault cohomology.

Note that

$$0 \to \underbrace{\Omega_X^p}_{\text{hol. v.b.}} \to \underbrace{A^{p,0}}_{\text{cplx v.b.}} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} \cdots$$

is a soft resolution of Ω_X^p so one has

$$H^q(X, \Omega_X^p) \cong H^{p,q}(X)$$

<u>Rmk</u>. $H^0(X, \Omega_X^p) = \operatorname{Ker}(\bar{\partial}: \Gamma(X, A^{p,0}) \to \Gamma(X, A^{p,1}))$

$= \{\text{global holo. section of } \Omega_X^p\}$.

- In general, for a holo. v.b. E, one can naturally think of it as a differentiable cplx v.b. (make it "soft") which we denote as $\mathcal{E}$. Then one has soft resolution of E:

$$0 \to E \to \mathcal{E} \xrightarrow{\bar{\partial}} A^{0,1} \otimes E \xrightarrow{\bar{\partial}} A^{0,2} \otimes E \to \cdots$$

☆ Here $\bar{\partial}$ operator is well-defined using locally the holo. frames of E (Explain this)

Then by sheaf cohomology theory, one has

$$H^q(X, E) \cong \frac{\operatorname{Ker}(\bar{\partial}: A^{0,p} \otimes E \to A^{0,p+1} \otimes E)}{\operatorname{Im}(\bar{\partial}: A^{0,p-1}(E) \to A^{0,p}(E))}.$$

In particular $H^0(X, E) = \{\text{holo. global sections of } E\}$.

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- Def. Let $\pi_E : E \to X$ & $\pi_F : F \to X$ be two cplx/holo. v.b. A vector bundle homomorphism from E to F is a smooth/holo. map $\varphi : E \to F$ s.t. $\pi_E = \pi_F \circ \varphi$ & the induced map $\varphi_x : E_x \to F_x$ is linear s.t. $\mathrm{rk}(\varphi(x))$ is constant in x.

Two vector bundles are isomorphic if φ is bijective.

Bijection automatically implies that φ^{-1} is also a bundle homo. Since in any common local trivialization $\varphi|_U$ is of the form

$$\varphi|_U : \begin{array}{ccc} U \times \mathbb{C}^r & \longrightarrow & U \times \mathbb{C}^r \\ (x, v) & \longmapsto & (x, A(x) v) \end{array}$$

s.t. $A : \begin{array}{ccc} U & \longrightarrow & GL(r, \mathbb{C}) \\ x & \longmapsto & A(x) \end{array}$

Then $A^{-1} : \begin{array}{ccc} U & \longmapsto & GL(r, \mathbb{C}) \\ x & \longmapsto & A^{-1}(x) \end{array}$ is also holomorphic in x

So φ^{-1} is also holom. bundle morphism from F to E.

☆ Given a holo. v.b. let $\{U_i \cap U_j, g_{ij}\}$ be its transition functions. One can construct a "new" holo. v.b. $\tilde{E}$ by gluing $U_i \times \mathbb{C}^r$ via g_{ij}.

Then $E \cong \tilde{E}$.