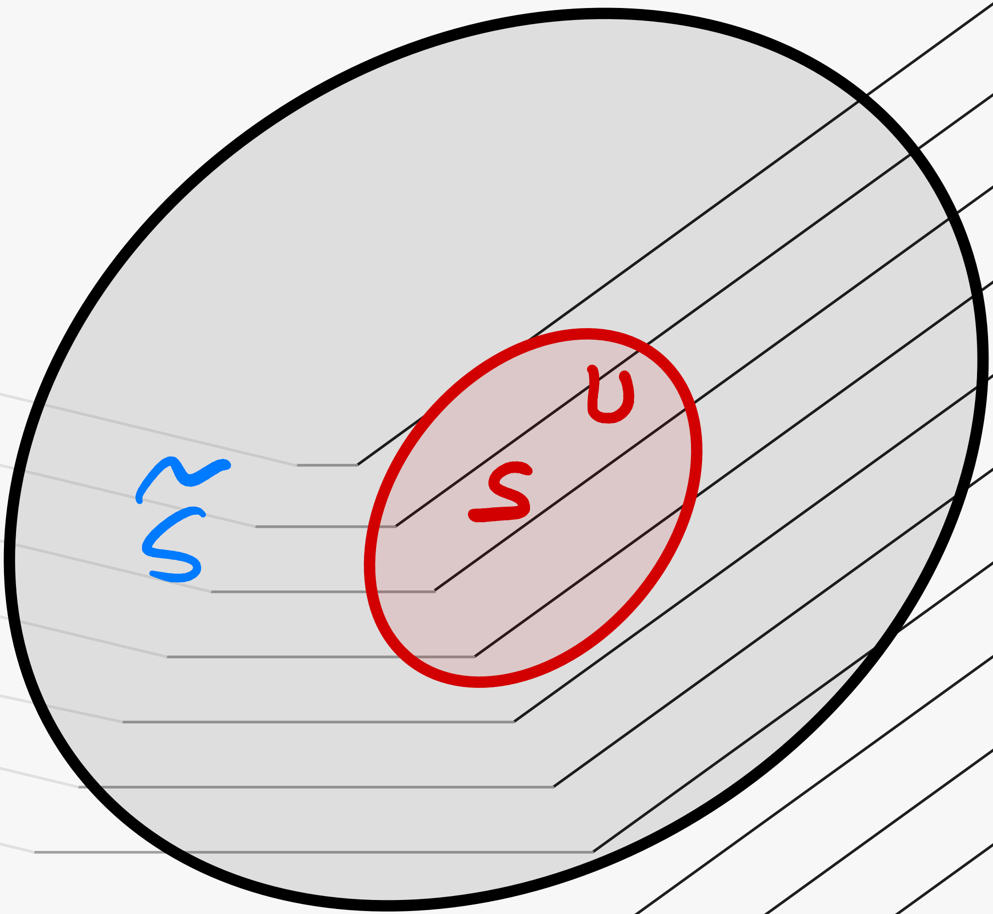
Lecture 3

Sheaf cohomology.



$$s = \tilde{s}\big|_U$$

Outline ① exact sequence & examples.

② resolution of sheaves

③ cohomology & examples.

④ Čech cohomology

- Recall Given $\varphi: \mathcal{F} \to \mathcal{G}$

$\begin{cases} ① \ \varphi \text{ is inj.} \\ ② \ \varphi_x \text{ is inj. for } \forall x \in X \\ ③ \ \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \text{ inj. for } \forall U. \end{cases}$

$\begin{cases} ① \ \varphi \text{ is surj.} \\ ② \ \varphi_x \text{ is surj. for } \forall x \in X. \\ ③ \ \forall \tau \in \mathcal{G}(U) \ \exists \text{ open cover } U = \cup U_i \ \& \ s_i \in \mathcal{F}(U_i) \text{ s.t. } \tau|_{U_i} = \varphi_{U_i}(s_i). \end{cases}$

- We call $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ exact

if α inj, β surj. & $\ker\beta = \operatorname{Im}\alpha$.

i.e. $0 \to \mathcal{A}_x \xrightarrow{\alpha_x} \mathcal{B}_x \xrightarrow{\beta_x} \mathcal{C}_x \to 0$ is an exact sequence.

- e.g. ① Let $X := \mathbb{C}$. $\mathcal{A} := \mathbb{Z}$, $\mathcal{B} := \mathcal{O}_X$ & $\mathcal{C} := \mathcal{O}_X^*$.

$\mathcal{A}$: constant sheaf. $\mathcal{B}$:= sheaf of holomorphic functions

$\mathcal{C}$:= sheaf of holomorphic functions that is no-where vanishing

then we have $0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$.

Verify that this is exact.

② If $\mathcal{A}$ is a subsheaf of $\mathcal{B}$, then

$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{A}/\mathcal{B} \to 0$ is exact.

③ e.g. $X := \mathbb{C}$, $\mathcal{B} := \mathcal{O}_X$. $\mathcal{A} := \mathcal{I}_0$ ideal sheaf of 0.

then $(\mathcal{B}/\mathcal{A})_x = \begin{cases} \mathbb{C}, & x=0 \\ 0, & x\neq 0. \end{cases}$ this is a skyscraper sheaf.

④ $X = \mathbb{C}^2$, $Y := \{z_1 = 0\}$ let $\mathcal{I}_Y :=$ sheaf of holo. functions vanish along Y.

then $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Y \to 0$.

Observe that $\mathcal{O}_X/\mathcal{I}_Y \cong \mathcal{O}_Y$.

This exact sequence gives information for X & Y.

- More generally, we say

$$\cdots \to \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^1 \xrightarrow{d_1} \mathcal{F}^2 \xrightarrow{d_2} \cdots$$

is a complex of sheaves if $d_{i+1} \circ d_i = 0$.

We say $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$

is a resolution of $\mathcal{F}$ if this complex is exact.

- e.g. let M be a Riem mfd.

$A^i :=$ sheaf of C^∞ i-forms on M.

then $A^0 =$ sheaf of C^∞ functions on M.

Then $0 \to \mathbb{Z} \xrightarrow{i} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$

is a resolution of the constant sheaf $\mathbb{Z}$.

& $0 \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$ is a complex.

it is not exact at Ω^0.

exactness at A^i for $i \geq 1$ follows from Poincaré Lemma

⊙ e.g. M C^∞ mfd. For $\forall$ U open let

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$S_p(U) := \{ C^\infty$ chains : linear combinations of C^∞ maps $f: \Delta^p \to U \}$

C^∞ in a nbhd of Δ^p.

Note that $\{S_p(U)\}_U$ doesn't give rise to sheaves as there is no naturation map $S_p(U) \to S_p(V)$ whenever $V \subseteq U$. But there is indeed an inclusion

$S_p(V) \hookrightarrow S_p(U)$ so it induces a restriction

$$\mathrm{Hom}(S_p(U), \mathbb{R}) \longrightarrow \mathrm{Hom}(S_p(V), \mathbb{R}).$$

Let $S^p(U) := \mathrm{Hom}(S_p(U), \mathbb{R})$.

Then $\{S^p(U)\}_U$ actually gives rise to a sheaf on M.

The usual coboundary map $\delta: S^p \to S^{p+1}$ gives a resolution of the constant sheaf $\mathbb{R}$.

$$0 \to \mathbb{R} \xrightarrow{i} S^0 \xrightarrow{\delta} S^1 \to S^2 \to \cdots$$

for each pt $x \in X$ associate it to a constant value.

- Now, what is cohomology?

Given a short exact sequence

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

then it is straightforward to show that the sequence

e.x.

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0$$

is exact at $\mathcal{A}(X)$ & $\mathcal{B}(X)$ but not at $\mathcal{C}(X)$.

The cohomology group measures the inexactness!

▲ Another example.

① Given a surjection between two groups

$$\varphi: G \to H.$$

For another group K, it is easy to see that φ induces an injection $\varphi^*: \mathrm{Hom}(H,K) \to \mathrm{Hom}(G,K)$.

② Now given an injection $\varphi: G \hookrightarrow H$
it is not necessarily true that
$\varphi^*: \mathrm{Hom}(H,K) \to \mathrm{Hom}(G,K)$ is surjection.
So in general there are obstructions for a morphism $G \to K$ to extend to a morphism $H \to K$.

③ So in general, whether something can be extended
or whether something is the restriction of something from a bigger space
are both questions about "obstructions".
These are often related to cohomology theory!

- Axioms of sheaf cohomology.

Let X be a paracompact Hausdorff space X. (so X has partition of unity)
↳ $\forall$ open cover admits a locally finite refinement.

Then for $\forall$ sheaf $\mathcal{F}$ of abelian groups one can associate a sequence of groups $H^q(X, \mathcal{F})$ for $q \geq 0$ s.t.

① $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

② If $\mathcal{F}$ soft, then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$

③ For $\forall$ sheaf morphism $h: \mathcal{A} \to \mathcal{B}$ $\exists$ for $\forall q \geq 0$ a group morphism $h_q: H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B})$ s.t.

Functorality
- $h_0 = h_X : \mathcal{A}(X) \to \mathcal{B}(X)$.
- $h_q = id$ if $h = id$ for $q \geq 0$
- given $\mathcal{A} \xrightarrow{h} \mathcal{B} \xrightarrow{g} \mathcal{C}$ one has $g_q \circ h_q = (g \circ h)_q$.

④ For $\forall$ short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

$\exists$ group homomorphism $\delta^q : H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A})$

s.t. the induced sequence

$$0 \to H^0(X,\mathcal{A}) \to H^0(X,\mathcal{B}) \to H^0(X,\mathcal{C}) \to H^1(X,\mathcal{A}) \to H^1(X,\mathcal{B})$$
$$\to \cdots \to H^q(X,\mathcal{A}) \to H^q(X,\mathcal{B}) \to H^q(X,\mathcal{C}) \to H^{q+1}(X,\mathcal{A}) \to \cdots$$

⑤ A commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \mathcal{C} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{A}' & \to & \mathcal{B}' & \to & \mathcal{C}' & \to & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(X,\mathcal{A}) & \to & H^0(X,\mathcal{B}) & \to & H^0(X,\mathcal{C}) & \to & H^1(X,\mathcal{A}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(X,\mathcal{A}') & \to & H^0(X,\mathcal{B}') & \to & H^0(X,\mathcal{C}') & \to & H^1(X,\mathcal{A}') & \to \cdots \end{array}$$

★ Thm. Such cohomology theory exists & is unique.

- Soft sheaf.
  - Def. A sheaf $\mathcal{F}$ is called soft if for $\forall$ closed subset $K \subseteq X$, the restriction $\mathcal{F}(X) \to \mathcal{F}(K) = \varinjlim_{U \supset S} \mathcal{F}(U)$ is surjective.
  - e.g. The sheaf of continuous / C^1 / C^∞ functions is soft.
    But the sheaf of holomorphic functions is not soft!
  - Let $\mathcal{R}$ be a sheaf of rings & $\mathcal{M}$ a sheaf of $\mathcal{R}$-modules.
    Namely, each $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module.
    If $\mathcal{R}$ is soft then $\mathcal{M}$ is also soft.
  - Cor. The sheaf Ω^p of differential p-forms is soft.

• **Thm.** Let $0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{F}^1 \xrightarrow{\partial^1} \cdots$

computation of sheaf.

be a resolution of $\mathcal{F}$ s.t. $\mathcal{F}^i$ is soft for all $i \geq 0$

Then consider the complex

$$0 \to \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{F}^1 \xrightarrow{\partial^1} \mathcal{F}^2 \xrightarrow{\partial^2} \cdots$$

One has $\ker \partial^q / \operatorname{Im} \partial^{q-1} \cong H^q(X, \mathcal{F})$. $\quad (\partial^{-1} = 0)$.

• Examples of sheaf cohomology groups.

We end this lecture by computing $H^q(X, \mathbb{R})$. (diff. mfd.)

$$0 \to \mathbb{R} \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \cdots \quad \text{(Soft resolution)}$$

then $H^q(X, \mathbb{R}) \cong \dfrac{\ker(A^q \xrightarrow{d} A^{q+1})}{\operatorname{Im}(A^{q-1} \xrightarrow{d} A^q)}$.

On the other hand one also has

$$0 \to \mathbb{R} \to S^0 \xrightarrow{\delta} S^1 \xrightarrow{\delta} \cdots \quad \text{(this is also soft!)}$$

then one finds that

$$H^q(X, \mathbb{R}) \cong \frac{\ker(S^q \xrightarrow{\delta} S^{q+1})}{\operatorname{Im}(S^{q-1} \xrightarrow{\delta} S^q)}$$

This is exactly the de Rham theorem.

★• **Čech Cohomology.** Let X be a topological space & $\mathcal{F}$ a sheaf.

Let us fix an open covering $X = \bigcup_{i \in I} U_i$, w/ I an ordered set.

Put $U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$ &

$$C^p(\{U_i\}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p}).$$

Given $\alpha_{i_0 \cdots i_p} \in U_{i_0 \cdots i_p}$, define $\alpha_{\sigma(i_0) \cdots \sigma(i_p)} := \operatorname{Sign}(\sigma) \alpha_{i_0 \cdots i_p} \in U_{\sigma(i_0) \cdots \sigma(i_p)}$

One can define a coboundary operator $d : C^p \to C^{p+1}$ by:

for $\forall\, \alpha = \prod \alpha_{i_0 \cdots i_p} \in C^p$, define $d\alpha$ by putting

$$(d\alpha)_{i_0 \cdots i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i_k} \cdots i_{p+1}} \Big|_{U_{i_0 \cdots i_{p+1}}}$$

e.x. Check that $d \circ d = 0$.

$\forall \alpha \in C^p(\{U_i\}, \mathcal{F})$ w/ $d\alpha$ is called a **p-cochain**

$\{$all the p-cochains$\} =: Z^p(\{U_i\}, \mathcal{F})$.

$\forall \beta \in C^p(\{U_i\}, \mathcal{F})$ w/ $\beta = d\alpha$ for some $\alpha \in C^{p-1}$ is **p-coboundary**

$\{$all p-coboundary$\} =: B^p(\{U_i, \mathcal{F}\})$.

Then define $\check{H}^p(\{U_i\}, \mathcal{F}) := \dfrac{\mathrm{Ker}(C^p \xrightarrow{d} C^{p+1})}{\mathrm{Im}(C^{p-1} \xrightarrow{d} C^p)} = \dfrac{Z^p}{B^p}$.

Note that this construction dpd on the choice of $\{U_i\}$.

If $\{V_j\}$ is a refinement of $\{U_i\}$, then there is a natural map

$$\check{H}^p(\{U_i\}, \mathcal{F}) \longrightarrow \check{H}^p(\{V_j\}, \mathcal{F}).$$

Then one defines the Čech cohomology w/o specifying an open cover as

$$\check{H}^p(X, \mathcal{F}) := \varinjlim \check{H}^p(\{U_i\}, \mathcal{F}).$$

- For $\forall$ open cover $\{U_i\}$, there is a natural homomorphism

$$\check{H}^p(\{U_i\}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}),$$

which is not necessarily bijective. But passing to the limit often result in isomorphisms (at least when X is **paracompact**)

The upshot is
① One always has $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ when $i = 0, 1$
② One has $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for $i > 1$ when X paracpt.

So when X is a diff. mfd, one has $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$, $\forall i \geq 0$.

- When X is a diff. mfd, one can always find a "good cover" $\{U_i\}$ s.t. $\forall$ $U_{i_0 \cdots i_p}$ is contractible. In this case,

$$\check{H}^p(\{U_i\}, \mathbb{R}) \cong H^p(X, \mathbb{R}).$$

More generally, for a sheaf $\mathcal{F}$ on X, if there is an open cover $\{U_i\}$ s.t. $H^q(U_{i_0 \cdots i_p}, \mathcal{F}) = 0$ for $\forall q \geq 1$, $\forall$ $U_{i_0 \cdots i_p}$.
then one has $\check{H}^q(\{U_i\}, \mathcal{F}) \cong H^q(X, \mathcal{F})$.

Such a cover is called **Leray cover**, which is quite useful to compute cohomology.

- Thm. Let X be a diff. mfd w/ the constant sheaf $\mathbb{R}$. Then $\check{H}^p(X,\mathbb{R}) \cong H^p_{dR}(X,\mathbb{R})$.

pf: We will look at the case $p=2$ to illustrate the ideas. The general cases follow in a similar manner.

① We will fix a good cover $\{U_i\}$ as above.
For $\forall$ representative α w/ $[\alpha]\in H^2_{dR}(X,\mathbb{R})$, since α is d-closed on each U_i, one can find a 1-form $\theta_i \in \Omega^1(U_i,\mathbb{R})$ s.t. $\alpha|_{U_i} = d\theta_i$.
Since $d(\theta_i-\theta_j)=0$ on U_{ij}, one finds $f_{ij}\in C^\infty(U_{ij},\mathbb{R})$ s.t. $\theta_i-\theta_j = df_{ij}$ on U_{ij}.
Note that $d(f_{ij}+f_{jk}+f_{ki}) = \theta_i-\theta_j+\theta_j-\theta_k+\theta_k-\theta_i=0$ on U_{ijk}.
So $\exists$ constant $a_{ijk}\in\mathbb{R}$ s.t. $f_{ij}+f_{jk}+f_{ki}=a_{ijk}$ on U_{ijk}.
It is easy to check that $a_{jkl}-a_{ikl}+a_{ijl}-a_{ijk}=0$.
So $\{a_{ijk}\}$ defines a cochain in $B^2(\{U_i\},\mathbb{R})$.
Different choices of α, θ_i, f_{ij} only differ $\{a_{ijk}\}$ by a coboundary.
So one get a map from $H^2_{dR}(X,\mathbb{R})$ to $\check{H}^2(\{U_i\},\mathbb{R})$.

② Conversely, for $\forall$ cochain $\{a_{ijk}\}$ one can recover a 2-form α in the following magical way. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define $f_{ij}\in C^\infty(U_{ij},\mathbb{R})$ by letting $f_{ij} := \sum_k a_{ijk}\psi_k$. Then one has

$$f_{ij}+f_{jk}+f_{ki} = \sum_l (a_{ijl}+a_{jkl}+a_{kil})\psi_l$$
$$= \sum_l a_{ijk}\psi_l = a_{ijk} \text{ on } U_{ijk}.$$

So $df_{ij}+df_{jk}+df_{ki}=0$.
Now define $\theta_i\in\Omega^1(U_i,\mathbb{R})$ by $\theta_i := \sum_j df_{ij}\psi_j$
then $\theta_i-\theta_j = \sum_l (df_{il}-df_{jl})\psi_l = \sum_l df_{ij}\psi_l = df_{ij}$ on U_{ij}
thus $d\theta_i - d\theta_j = 0$ on U_{ij}. So $\alpha := d\theta_i$ defines a global d-closed 2-form. So we have $\check{H}^2(X,\mathbb{R}) \to H^2_{dR}(X,\mathbb{R})$

The above two constructions are inverse to each other and generalizes to $\forall p\geq 0$.

$\square$

※ Appendix ① de Rham cohomology

- Let X^m be a differentiable manifold. Consider the de Rham complex:
$$0 \to A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots \xrightarrow{d} A^m(X) \to 0,$$
where $A^i(X) := \{ C^\infty \ i\text{-form on } X \}$. $\leftarrow$ $\mathbb{R}$-vector space.
Then the i-th de Rham cohomology group of X is defined by
$$H^i_{dR}(X, \mathbb{R}) := \frac{\operatorname{Ker}(A^i(X) \xrightarrow{d} A^{i+1}(X))}{\operatorname{Im}(A^{i-1}(X) \xrightarrow{d} A^i(X))} \quad \leftarrow \text{this is an } \mathbb{R}\text{-vector space.}$$
which measures the inexactness of the above complex.

- For $\forall$ d-closed i form $\alpha \in A^i(X)$, its equivalent class $[\alpha] \in H^i_{dR}(X, \mathbb{R})$ is called the cohomology class of α. By definition, for $\forall \alpha' \in A^i(X)$ of the form $\alpha' = \alpha + d\beta$ for some $\beta \in A^{i-1}(X)$, one has $[\alpha'] = [\alpha]$, namely, they lie in the same cohomology class. Conversely, $\forall \alpha' \in [\alpha]$ must be of the form $\alpha' = \alpha + d\beta$ for some $\beta \in A^{i-1}(X)$.

- Analogously, one can also consider $\mathbb{C}$-valued forms:
$$A^i_{\mathbb{C}}(X) := \{ \mathbb{C}\text{-valued } C^\infty \ i\text{-form on } X \} \leftarrow \mathbb{C}\text{-vector space}$$
So $\forall \alpha \in A^i_{\mathbb{C}}(X)$ can be written as $\alpha = \operatorname{Re}\alpha + \sqrt{-1}\operatorname{Im}\alpha$, w/ $\operatorname{Re}\alpha$ & $\operatorname{Im}\alpha \in A^i(X)$.
Then can define $H^i_{dR}(X, \mathbb{C})$ similarly as above, and $H^i_{dR}(X, \mathbb{C})$ is nothing but the complexification of $H^i_{dR}(X, \mathbb{R})$.

- de Rham thm: $H^i_{dR}(X, K) \cong H^i(X, K)$, $K = \mathbb{R}$ or $\mathbb{C}$
Here $H^i(X, K)$ is the (topological) cohomology group of X.

※ Appendix ② Let X be a paracpt Hausdorff space.

Let $\mathcal{C}$ be the sheaf of continuous functions on X.
Then $\mathcal{C}$ is soft.

Pf: We need to use that X has partition of unity subordinate to $\forall$ locally finite covering.

For $\forall f \in \mathcal{C}(K)$, K closed, one can find open $U \supseteq K$ & $f \in \mathcal{C}(U)$. Consider $X = X \setminus K \cup U$.

let $X \setminus K = \bigcup_{\alpha \in A} U_\alpha$ & $U = \bigcup_{\beta \in B} U_\beta$ be a locally finite refinement. let $\{\theta_\alpha, \theta_\beta\}$ be partition of unity subordinate to the covering.

Then $\hat{f} := \sum_\alpha 0 \cdot \theta_\alpha + \sum_\beta f \theta_\beta \in C(X)$

s.t. $\tilde{f}|_{U'} = f|_{U'}$ for some U' w/ $K \subseteq U' \subseteq U$.