$f^k = g^{kl} f_l$

Then, $\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \frac{1}{2}\left(f_i \delta_{jk} + f_j \delta_{ik} - f_l \cdot g^{kl} g_{ij}\right)$. ✓

$\tilde{R}_{ijkl} = \cdots$. See below.

- $\tilde{R}_{ijkl} = -\frac{1}{2}\left(\partial_i \partial_k \tilde{g}_{jl} + \partial_j \partial_l \tilde{g}_{ik} - \partial_i \partial_l \tilde{g}_{jk} - \partial_j \partial_k \tilde{g}_{il}\right)$ ...... ①

$- \tilde{g}_{rs}\left(\tilde{\Gamma}^r_{ik} \tilde{\Gamma}^s_{jl} - \tilde{\Gamma}^r_{il} \tilde{\Gamma}^s_{jk}\right)$ ...... ②

$① = -\frac{1}{2}\left[\partial_i\left(f_k \tilde{g}_{jl} + e^f \partial_k g_{jl}\right) + \cdots\right]$

$= -\frac{1}{2}\left[f_{ik} \tilde{g}_{jl} + f_i f_k \tilde{g}_{jl} + e^f\left(f_i \partial_k g_{jl} + f_k \partial_i g_{jl} + \partial_i \partial_k g_{jl}\right) + \cdots\right]$

$= -\frac{e^f}{2}\left(\partial_i \partial_k g_{jl} + \cdots\right)$

$- \frac{e^f}{2}\left(f_i \partial_k g_{jl} + f_k \partial_i g_{jl} + \cdots\right)$

$- \frac{e^f}{2}\left(f_{ik} g_{jl} + f_i f_k g_{jl} + \cdots\right)$.

$② = -e^f g_{rs}\left[\left(\Gamma^r_{ik} + \frac{1}{2}(f_i \delta_{kr} + f_k \delta_{ir} - f^r g_{ik})\right)\left(\Gamma^s_{jl} + \frac{1}{2}(f_j \delta_{ls} + f_l \delta_{js} - f^s g_{jl})\right) + \cdots\right.$

$= -e^f g_{rs} \Gamma^r_{ik} \Gamma^s_{jl}$

$- \frac{e^f}{2} g_{rs}\left[\Gamma^r_{ik}(f_j \delta_{ls} + f_l \delta_{js} - f^s g_{jl}) + \Gamma^s_{jl}(f_i \delta_{kr} + f_k \delta_{ir} - f^r g_{ik}) + \cdots\right]$ ...... ③

$- \frac{e^f}{4} g_{rs}\left[(f_i \delta_{kr} + f_k \delta_{ir} - f^r g_{ik})(f_j \delta_{ls} + f_l \delta_{js} - f^s g_{jl}) + \cdots\right]$ ...... ④

$$③ = -\frac{e^f}{2}\cdot\left[\Gamma^r_{ik}\left(f_j g_{lr} + f_l g_{jr} - f_r g_{jl}\right) + \Gamma^s_{jl}\left(f_i g_{ks} + f_k g_{is} - f_s g_{ik}\right) + \cdots\right]$$

$$= -\frac{e^f}{2}\cdot\Big[f_i\, g_{kr}\Gamma^r_{jl} + \underline{f_k\, g_{ir}}\Gamma^r_{jl} - f_r\, g_{ik}\Gamma^r_{jl}$$

$$+ f_j\, g_{lr}\Gamma^r_{ik} + \underline{f_l\, g_{jr}\Gamma^r_{ik}} - f_r\cdot g_{jl}\cdot\Gamma^r_{ik} + \cdots\Big]$$

$$= -\frac{e^f}{2}\cdot\Big[f_i\left(\underline{g_{kr}\Gamma^r_{jl} - g_{lr}\Gamma^r_{jk}}\right) + f_k\left(g_{ir}\Gamma^r_{jl} - g_{jr}\Gamma^r_{il}\right)$$

$$+ f_j\left(g_{lr}\Gamma^r_{ik} - g_{kr}\Gamma^r_{il}\right) + f_l\cdot\left(g_{jr}\Gamma^r_{ik} - g_{ir}\Gamma^r_{jk}\right)$$

$$- f_r\cdot\left(g_{ik}\Gamma^r_{jl} + g_{jl}\Gamma^r_{ik} - g_{il}\Gamma^r_{jk} - g_{jk}\Gamma^r_{il}\right)\Big]$$

$$\cancel{= -\frac{e^f}{2}\cdot\Big[2f_i\,\partial_l g_{jk} + 2f_k\,\partial_j g_{il} - 2f_j\,\partial_l g_{ik} - 2f}$$

$$= -\frac{e^f}{2}\cdot\Big[f_i\left(\partial_l g_{jk} - \partial_k g_{jl}\right) + f_j\left(\partial_k g_{il} - \partial_l g_{ik}\right)$$

$$+ f_k\left(\partial_j g_{il} - \partial_i g_{jl}\right) + f_l\cdot\left(\partial_i g_{jk} - \partial_j g_{ik}\right)$$

$$- f_r\cdot\left(g_{ik}\Gamma^r_{jl} + g_{jl}\Gamma^r_{ik} - g_{il}\Gamma^r_{jk} - g_{jk}\Gamma^r_{il}\right)\Big]$$

$$④ = -\frac{e^f}{4}\left[(f_i g_{ks} + f_k g_{is} - f_s g_{ik})(f_j \delta_{ls} + f_l \delta_{js} - f^s g_{jl}) + \cdots\right]$$

$$= -\frac{e^f}{4}\left[\cancel{f_i f_j g_{kl}} + f_i f_l g_{jk} - f_i f_k g_{jl} + f_j f_k g_{il} + \cancel{f_k f_l g_{ij}} - f_i f_k g_{jl}\right.$$

$$\left. - f_j f_l g_{ik} - f_j f_l g_{ik} + f_s f^s g_{ik} g_{jl} + \cdots\right]$$

$$= -\frac{e^f}{4}\left[-2 f_i f_k g_{jl} - 2 f_j f_l g_{ik} + f_i f_l g_{jk} + f_j f_k g_{il} + f_s f^s g_{ik} g_{jl}\right.$$

$$\left. + 2 f_i f_l g_{jk} + 2 f_j f_k g_{il} - f_i f_k g_{jl} - f_j f_l g_{ik} - f_s f^s g_{il} g_{jk}\right]$$

$$= -\frac{1}{4} e^f \left[3(f_i f_l g_{jk} + f_j f_k g_{il} - f_i f_k g_{jl} - f_j f_l g_{ik}) + f_s f^s (g_{ik} g_{jl} - g_{il} g_{jk})\right]$$

$\Rightarrow$

$$\tilde{R}_{ijkl} = e^f \cdot R_{ijkl}$$

$$- \frac{e^f}{2}\left[g_{ik}(f_{jl} - f_r \Gamma^r_{jl}) + g_{jl}(f_{ik} - f_r \Gamma^r_{ik}) - g_{il}(f_{jk} - f_r \Gamma^r_{jk}) - g_{jk}(f_{il} - f_r \Gamma^r_{il})\right]$$

$$- \frac{e^f}{4}\left[3(f_i f_l g_{jk} + f_j f_k g_{il} - f_i f_k g_{jl} - f_j f_l g_{ik}) + f_s f^s (g_{ik} g_{jl} - g_{il} g_{jk})\right]$$

$\Rightarrow$

$$\tilde{R} = e^f \left[R - \frac{1}{2} g \otimes \mathrm{Hess} f + \frac{1}{4} g \otimes (df \otimes df) - \frac{|\nabla f|^2}{4} g \otimes g\right]$$

$$\tilde{R} = e^f \left[R - \frac{1}{2}\left(\mathrm{Hess} f - \frac{df \otimes df}{2}\right) \otimes g - \frac{|\nabla f|^2}{8} g \otimes g\right]$$

where • $\owedge$ is the Kulkarni-Nomizu product.

for symmetric, $(2,0)$ tensors α, β, define

$\alpha \owedge \beta$, a $(4,0)$ tensor, by

$$\alpha \owedge \beta(u,v,w,z) = \alpha(u,w)\cdot\beta(v,z) + \alpha(v,z)\cdot\beta(u,w) - \alpha(u,z)\cdot\beta(v,w) - \alpha(v,w)\cdot\beta(u,z)$$

• ∇f is the gradient field of f, defined by

$$\langle \nabla f, X\rangle = Xf, \quad \forall\, X \in \Gamma(TM).$$

Locally, $\nabla f = g^{ij}\dfrac{\partial f}{\partial x^i}\dfrac{\partial}{\partial x^j}$.

• Hessian tensor: $\mathrm{Hess} f$, defined by

$$\mathrm{Hess} f(X,Y) = XYf - (\nabla_X Y)f, \quad \forall\, X,Y \in \Gamma(TM)$$

•• $\mathrm{Hess} f$ is C^∞-linear in X & Y

•• $\mathrm{Hess} f$ is symmetric: $\mathrm{Hess} f(X,Y) = \mathrm{Hess} f(Y,X)$.

• $f_s f^s = g^{rs} f_r f_s = \langle \nabla f, \nabla f\rangle = |\nabla f|^2$.

Prop 15. Under conformal change $\tilde{g} = e^{-2f} g$,

(i). $\tilde{R} = e^{-2f} \cdot \left[R + (\mathrm{Hess} f + df \otimes df) ⓧ g - \frac{|\nabla f|^2}{2} g ⓧ g \right]$ ······ (*1)

or, $\tilde{R} = e^{-2f} \cdot R + e^{-3f} \cdot \mathrm{Hess}\, e^f ⓧ g - e^{-4f} \cdot |\nabla e^f|^2 \cdot \frac{g ⓧ g}{2}$. ······ (*2)

where $|\nabla f|^2 = |\nabla f|_g^2$.

(ii). For plane $P \subset T_pM$, sectional curvature

$\tilde{K} = e^{2f} \cdot K - |\nabla e^f|^2 + e^f \cdot \mathrm{tr}_P \mathrm{Hess}\, e^f$ ·········· (*3)

where $\mathrm{tr}_P \mathrm{Hess}\, e^f$ is the trace of $\mathrm{Hess}\, e^f$ on P:

$\mathrm{tr}_P \mathrm{Hess}\, e^f = \mathrm{Hess}\, e^f(e_1, e_1) + \mathrm{Hess}\, e^f(e_2, e_2)$

$\{e_1, e_2\} \subset P$ is any orthonormal base.

(iii). Ricci curvature

$\widetilde{Ric} = Ric - (n-1)|\nabla f|^2 \cdot g + e^{-f} \cdot [\Delta e^f \cdot g + (n-2)\,\mathrm{Hess}\, e^f]$.

$= Ric + [\Delta f - (n-2)|\nabla f|^2]\, g + (n-2) \cdot (\mathrm{Hess} f + df \otimes df)$.

(iv). Scalar curvature

$\tilde{s} = e^{2f} \cdot [s + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2]$, Pf: (i). Replace f by $-2f$. Apply

$$\mathrm{Hess}\, e^f = e^f \cdot (\mathrm{Hess} f + df \otimes df)$$

& $\nabla e^f = e^f \cdot \nabla f$

(ii). Take orthonormal bases $u, v \in P$, $|u| = |v| = 1$, $u \perp v$.

Then, $|u \wedge v|^2_{\tilde{g}} = e^{-4f} \cdot |u \wedge v|^2_g = e^{-4f}$.

$$\tilde{K} = \frac{\tilde{R}(u,v,u,v)}{|u \wedge v|^2_{\tilde{g}}} = e^{4f} \cdot \tilde{R}(u,v,u,v)$$

$$= e^{2f} \cdot K + e^f \cdot \left[\mathrm{Hess}\, e^f (u,u) + \mathrm{Hess}\, e^f (v,v) \right]$$

$$- |\nabla e^f|^2$$

(iii). Let $\{e_i\}$ be an orthonormal basis of g, at p,

then $\underset{\tilde{e}_i}{\underset{\parallel}{\{e^f \cdot e_i\}}}$ $\cdots \cdots$ $\tilde{g}$, at p.

$$\widetilde{Ric}(X,Y) = \sum_i \tilde{R}(X, \tilde{e}_i, Y, \tilde{e}_i) = e^{2f} \cdot \sum_i \tilde{R}(X, e_i, Y, e_i)$$

$$= \sum_i R(X, e_i, Y, e_i)$$

$$+ e^{-f} \cdot \sum_i \Big[\mathrm{Hess}\, e^f (X,Y) + \mathrm{Hess}\, e^f (e_i, e_i) \cdot \langle X, Y \rangle$$

$$- \mathrm{Hess}\, e^f (X^i e_i, Y) - \mathrm{Hess}\, e^f (X, Y^i e_i) \Big]$$

$$+ e^{-2f} \cdot |\nabla e^f|^2 \sum_i \left[\langle X, Y \rangle - X^i Y^i \right]$$

where $X^i = \langle X, e_i \rangle$, $Y^i = \langle Y, e_i \rangle$

$$\Rightarrow \widetilde{Ric}(X,Y) = Ric(X,Y) + e^{-f}\cdot\Big[n\cdot \mathrm{Hess}\, e^{f}(X,Y) + \Delta e^{f}\cdot\langle X,Y\rangle - 2\,\mathrm{Hess}\, e^{f}(X,Y)\Big] + (n-1)\, e^{-2f}\cdot |\nabla e^{f}|^2 \langle X,Y\rangle .$$

where $\Delta e^{f} = \sum_i \mathrm{Hess}\, e^{f}(e_i, e_i) = \mathrm{tr}\,\mathrm{Hess}\, e^{f}$

is the Laplacian of e^{f}.

Apply $\Delta e^{f} = e^{f}(\Delta f + |\nabla f|^2)$ to get the 2^{nd} identity.

(iv). $\tilde{S} = \sum \widetilde{Ric}(\tilde{e}_i, \tilde{e}_i) = e^{2f}\cdot \sum \widetilde{Ric}(e_i, e_i)$. $\square$

<u>Typical example</u> (Riemann's model of constant curvature)

Let $\Omega \subset \mathbb{R}^n$, g_E be Euclidean metric.

Let $\tilde{g} = \varphi^{-2} g_E$ on Ω. ($\varphi = e^{f}$ in above notion)

$\tilde{K} = -|\nabla\varphi|^2 + \varphi\cdot \mathrm{tr}_P\, \mathrm{Hess}\,\varphi$, for any plane $P \subset T\Omega$.

If $\varphi(x) = 1 + \frac{K_0}{4}|x|^2$, $K_0 \in \mathbb{R}$. then

$\tilde{K} \equiv K_0$.

<u>I.e.,</u> $\tilde{g} = \dfrac{g_E}{(1+\frac{K_0}{4}|x|^2)^2}$ has const curvature K_0.

§3. Geodesics, Variation formula, Jacobi fields

§3.1 Geodesics, 1^{st} variation formula

Def 1 (Length of curve). Let $\gamma : [a,b] \to M$ be a diff curve, define

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

- The length is invariant under reparametrization: if $\eta : [c,d] \to [a,b]$ is a diffeomorphism, then

$$L(\gamma \circ \eta) = L(\gamma).$$

Def 2 (Variation of curves)

$$\gamma : [a,b] \times (-\varepsilon, \varepsilon) \to M, \quad \gamma_t(s) = \gamma(s,t), \quad t \in (-\varepsilon, \varepsilon).$$

$$X = \frac{\partial}{\partial s}\gamma_t = \frac{\partial}{\partial s}\gamma : \quad \text{tangent field of } \gamma$$

$$Y = \frac{\partial}{\partial t}\gamma : \quad \text{variation field}.$$

Prop 3 (1^{st} variation formula)

$$\frac{d}{dt} L(\gamma_t) = \left\langle Y, \frac{X}{|X|} \right\rangle \Big|_0^{\ell} - \int_0^{\ell} \frac{\langle Y, (\nabla_X X)^{\perp} \rangle}{|X|}\, ds.$$

where $(\nabla_X X)^{\perp} = \nabla_X X - \left\langle \nabla_X X, \frac{X}{|X|} \right\rangle \frac{X}{|X|}$ is the normal part of $\nabla_X X$ along γ_t.

$$[X,Y]f = XYf - YXf = \frac{\partial^2(f\circ\gamma)}{\partial s\,\partial t} - \frac{\partial^2(f\circ\gamma)}{\partial t\,\partial s} = 0$$

Pf: Notice that $[X,Y]=0$. Then,

$$\frac{d}{dt}L(\gamma_t) = \frac{d}{dt}\int_0^\ell \sqrt{\langle X,X\rangle}\,ds = \int_0^\ell \frac{\frac{\partial}{\partial s}\langle X,X\rangle(\gamma)}{2\sqrt{\langle X,X\rangle}}\,ds$$

$$= \int_0^\ell \frac{Y\langle X,X\rangle}{2|X|}\,ds = \int_0^\ell \frac{\langle \nabla_Y X, X\rangle}{|X|}\,ds = \int_0^\ell \frac{\langle \nabla_X Y, X\rangle}{|X|}\,ds$$

$$\frac{\langle \nabla_X Y, X\rangle}{|X|} = \frac{X\langle X,Y\rangle - \langle Y, \nabla_X X\rangle}{|X|} = \frac{\partial}{\partial s}\left(\frac{\langle X,Y\rangle}{|X|}\right) - \langle X,Y\rangle \frac{\partial}{\partial s}\frac{1}{|X|} - \frac{\langle Y,\nabla_X X\rangle}{|X|}$$

where $\displaystyle \frac{\partial}{\partial s}\frac{1}{|X|} = -\frac{X\langle X,X\rangle}{2|X|^3} = -\frac{\langle \nabla_X X, X\rangle}{|X|^3}$.

So: $$\frac{d}{dt}L(\gamma_t) = \left\langle Y, \frac{X}{|X|}\right\rangle\Big|_0^\ell + \int_0^\ell \left(\frac{\langle \nabla_X X, X\rangle\langle X,Y\rangle}{|X|^3} - \frac{\langle Y,\nabla_X X\rangle}{|X|}\right)ds.$$

□

Corollary: Suppose γ_0 is of unit speed.

(i). Assume $\gamma_t(0)\equiv p$, $\gamma_t(\ell)\equiv q$. Then,

$$\frac{d}{dt}\Big|_{t=0} L(\gamma_t) = -\int_0^\ell \langle Y, \nabla_X X\rangle\,ds.$$

γ_0 is a critical pt of any variation, iff

$$\nabla_X X = 0 \quad \text{along } \gamma_0.$$

(ii). If $\nabla_X X = 0$ along γ_0, then

$$\frac{d}{dt}\Big|_{t=0} L(\gamma_t) = \langle X,Y\rangle(q) - \langle X,Y\rangle(p).$$

Pf: $|X|\equiv 1 \Rightarrow \nabla_X X = (\nabla_X X)^\perp$. □

- A curve γ satisfing $\nabla_{\gamma'}\gamma' = 0$, is called a <u>geodesic</u>.
- When γ is a geodesic, the speed must be constant.

$$\frac{d}{dt}|\gamma'(t)|^2 = \gamma' \cdot \langle \gamma', \gamma' \rangle = 2\langle \nabla_{\gamma'}\gamma', \gamma' \rangle = 0.$$

- $\nabla_{\gamma'}\gamma' = 0$ is an ODE.

For any $p \in M$, $v \in T_pM$, $\exists!$ local geodesic $\gamma_v(s)$, with $\gamma_v(0) = p$, $\gamma_v'(0) = v$. $s \in (-\varepsilon, \varepsilon)$.

·· In coor, $x = (x^1, \cdots, x^n)$.

$\gamma(s) = (\gamma^1(s), \cdots, \gamma^n(s))$.

γ is a geodesic $\Leftrightarrow$ $\frac{d^2}{dt^2}\gamma^k + \Gamma_{ij}^k \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} \equiv 0$.

- If $\gamma : [a,b] \to M$ is a geodesic, then, for $\lambda > 0$,

$\tilde{\gamma}(s) = \gamma(\lambda s) : [a/\lambda, b/\lambda] \to M$, is also a geodes

with $\tilde{\gamma}(a/\lambda) = \gamma(a)$, $\tilde{\gamma}'(a/\lambda) = \lambda \cdot \gamma'(a)$

By uniqueness, $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$, $\forall\, v \in T_pM$, $\lambda > 0$.

Def (Exponential map) $\forall\, p \in M$, $v \in T_pM$,

Let $\exp_p v = \gamma_v(1)$, once the right side exists.

- ODE $\Rightarrow$ $\exp_p v$ can be defined whenever $|v| \ll 1$.

Lemma: $d(\exp_p)_0 = id : T_pM \to T_pM$.

Pf: $\forall\, v \in T_pM$, choose $\sigma(t) = tv$, $t \in \mathbb{R}$, a curve in T_pM, with $\sigma'(0) = v$.

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(\sigma(t)) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{tv}(1) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v. \qquad \square$$

- Injectivity radius

$$i_p = \sup\{r>0 \mid \exp_p : B_r(0) \ (\subset T_pM) \to M \text{ diff onto its image}\}$$

Conjugate radius

$$r_p = \sup\{r>0 \mid \cdots \quad \cdots \text{ non-degenerate}\}$$

i_p: maximal radius so that $B_r(p)$ is topological trivial.