§2. Riemannian metric, Levi-Civita connection, Riemann Curvature

§2.1 Rie. metric.

M^n: n-dim diff mfd. $\quad C^\infty(M) = \{ f : M \to \mathbb{R},\ \text{diff} \}$.

Def 1 (Rie metric). A Rie metric on M, is given by an inner product $\langle \cdot, \cdot \rangle_p$ on each T_pM, which varies smoothly in $p \in M$.

I.e., $\quad \langle X, Y \rangle \in C^\infty(M), \quad \forall\ X, Y \in \Gamma(TM)$.

- The Rie metric determines a positive, symmetric (2,0) tensor

$$g \in \Gamma(T^*M \otimes T^*M), \quad g(u,v) = \langle u, v \rangle_p, \quad \forall u, v \in T_pM.$$

g is also called the Rie metric.

- In local coor., $\quad \varphi : U \to M, \quad x = (x^1, \cdots, x^n)$.

put $\quad g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$

Then $(g_{ij})_{n\times n}$ is positive, symmetric matrix, varies smoothly, & $\quad g(X,Y) = g_{ij} X^i Y^j, \quad \forall X, Y \in \Gamma(TM)$

$$g = g_{ij}\, dx^i \otimes dx^j$$

Exp 2. Euclidean space $\quad g_{ij} = \delta_{ij}$

$$g = \delta_{ij}\, dx^i \otimes dx^j = \sum_i dx^i \otimes dx^i.$$

Lemma 3 (existence): Any diff mfd M admits a Rie. metric.

pf: Take a countable, locally finite coor. covering $\{(U_\alpha, \varphi_\alpha)\}$. Let $\{f_\alpha\}$ be a partition of unity subordinated to $\{(U_\alpha, \varphi_\alpha)\}$.

On each U_α, let g_α be Euclidean metric on U_α,

Put $g = \sum f_\alpha g_\alpha$. $\square$

- Norm: $|v| = \sqrt{\langle v, v\rangle}$, $\forall\, v \in T_pM$.

Let $\gamma : (a,b) \to M$ be a curve, define its length

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

§2.2 Levi-Civita connection.

Prop 4 (definition). On any Rie. mfd (M,g), $\exists!$ map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad (X,Y) \mapsto \nabla_X Y$$

Satisfying: (LC1): $\nabla_{fX+hY} Z = f\nabla_X Z + h\nabla_Y Z$

$$\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$$

$$\nabla_X (fY) = X(f)\,Y + f\,\nabla_X Y$$

(LC2) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ compatible with metric

(LC3) $\nabla_X Y - \nabla_Y X = [X, Y]$ torsion free

The map ∇ is called Levi-Civita connection.

Pf: • $2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle$

$$- \langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle + \langle Z, [X,Y]\rangle. \quad (*)$$

By
$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad ①$$
$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \quad ②$$
$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad ③$$
①+②−③

• check that in (*), $\nabla_X Y$ satisfies (LC1).

②: Trivial.

③: $2\langle \nabla_X (fY), Z\rangle = X\langle fY, Z\rangle + fY\langle Z, X\rangle - Z\langle X, fY\rangle$

$$- \langle X, [fY, Z]\rangle + \langle fY, [Z,X]\rangle + \langle Z, [X, fY]\rangle$$

$$= X(f)\langle Y,Z\rangle + fX\langle Y,Z\rangle + fY\langle Z,X\rangle - Z(f)\langle X,Y\rangle - fZ\langle X,Y\rangle$$
$$- \langle X, f[Y,Z]\rangle + \langle X, Z(f)Y\rangle + f\langle Y, [Z,X]\rangle$$
$$+ f\langle Z, [X,Y]\rangle + Xf\langle Z, Y\rangle$$
$$= f\cdot[\cdots] + 2Xf\cdot\langle Y,Z\rangle$$
$$= 2f\cdot\langle \nabla_X Y, Z\rangle + 2Xf\cdot\langle Y,Z\rangle. \quad \checkmark$$

①: ② + ③ + torsion free □

Calculate in coor. $\varphi : U \to M$, $x = (x^1, \cdots, x^n)$

Introduce Christoffel symbols. $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}$.

For, $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^j \frac{\partial}{\partial x^j}$,

$$\nabla_X Y = \nabla_{X^i \frac{\partial}{\partial x^i}} \left(Y^j \frac{\partial}{\partial x^j}\right) = X^i \nabla_{\frac{\partial}{\partial x^i}} \left(Y^j \frac{\partial}{\partial x^j}\right)$$

$$= X^i \left(\frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + Y^j \Gamma_{ij}^{k} \frac{\partial}{\partial x^k} \right)$$

$$= X^i \left(\underline{\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma_{ij}^{k}} \right) \frac{\partial}{\partial x^k}.$$

Def 5 (Covariant derivative). $X \in \Gamma(TM)$. $\forall v \in T_pM$,

Define $\nabla_v X$ as follows.

Extend v to be a vector field V, then put

$$\nabla_v X = \nabla_V X (p).$$

- Let $\gamma : I \to M$ be a curve. $X \in \Gamma(TM)$.

Then $\nabla_{\gamma'} X$ dep only on values of X along γ.

•• $\langle \nabla_{\gamma'} X, Y \rangle = \gamma' \langle X, Y \rangle - \langle X, \nabla_{\gamma'} Y \rangle$

$$= \frac{d}{dt} \langle X, Y \rangle (\gamma(t)) - \langle X, \nabla_{\gamma'} Y \rangle. \quad \#$$

Lemma 6: In coor. $\varphi : U \to M$, $x = (x^1, \cdots, x^n)$.

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right),$$

$$\partial_k g_{ij} = g_{il} \Gamma_{jk}^{l} + g_{jl} \Gamma_{ik}^{l},$$

where $(g^{ij}) = (g_{ij})^{-1}$, inverse matrix.

$g_{kl}\Gamma_{ij}^{k}$.
$\parallel$

Pf: ⊛: $\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l} \rangle = \frac{1}{2}\Big(\frac{\partial}{\partial x^i}\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l}\rangle + \frac{\partial}{\partial x^j}\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^l}\rangle$

$- \frac{\partial}{\partial x^l}\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \Big).$ ✓

$$\left.\begin{aligned} \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} &= 2 g_{kl}\Gamma_{ij}^{k} \\ \partial_i g_{jl} + \partial_l g_{ij} - \partial_j g_{il} &= 2 g_{kj}\Gamma_{il}^{k}. \end{aligned}\right\} \Rightarrow \text{second identity.}$$

□

§2.3 Curvature.

Def 7 (Rie. Curvature). $R: \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$.

$$(X,Y,Z) \mapsto R_{X,Y}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Or: $R: \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \to C^\infty(M)$.

$$(X,Y,Z,W) \mapsto R(X,Y,Z,W) = \langle R_{X,Y}W, Z\rangle.$$

Prop 8: ① R is $C^\infty(M)$-linear in each variable.

② $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y)$

③ 1^{st} Bianchi identity:

$$R_{X,Y}Z + R_{Z,X}Y + R_{Y,Z}X = 0$$

or, $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$.

Pf: ①. $R_{fX,Y}Z = \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z$

$= f\cdot R_{X,Y}Z - Y(f)\nabla_X Z + \nabla_{Yf\cdot X}Z = f\cdot R_{X,Y}Z.$

Others: by ②.

3. $R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y$

$= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X$

$+ \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y$

$= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y$

$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$ ✓

2. • $R(X,Y,Z,W) = -R(Y,X,Z,W).$ ✓

• $R(X,Y,Z,W) + R(X,Y,W,Z)$

$= \langle \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W, Z\rangle$

$+ \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle$

$= X\langle\nabla_Y W, Z\rangle - \cancel{\langle\nabla_Y W, \nabla_X Z\rangle} - Y\langle\nabla_X W, Z\rangle + \cancel{\langle\nabla_X W, \nabla_Y Z\rangle} - \langle\nabla_{[X,Y]}W, Z\rangle$

$+ X\langle\nabla_Y Z, W\rangle - \cancel{\langle\nabla_Y Z, \nabla_X W\rangle} - Y\langle\nabla_X Z, W\rangle + \cancel{\langle\nabla_X Z, \nabla_Y W\rangle} - \langle\nabla_{[X,Y]}Z, W\rangle$

$= XY\langle Z,W\rangle - YX\langle Z,W\rangle - [X,Y]\langle Z,W\rangle = 0.$ ✓

• Bianchi identity.

$R(X,\cancel{Y},\cancel{Z},W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

$R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) = 0$

$R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) = 0$

$R(W,X,Y,Z) + R(Y,W,X,Z) + R(X,Y,\cancel{W},Z) = 0.$

Summing up the four identities. □

In local cor, denote

$$R_{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} = R_{ij\ \ l}^{\ \ k} \frac{\partial}{\partial x^k}$$

Then, $R_{ijkl} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = g_{ks} R_{ij\ \ l}^{\ \ s}$.

Satisfies: $R_{ijkl} + R_{jkil} + R_{kijl} = 0$

& $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$.

For vector fields, $R(X, Y, Z, W) = R_{ijkl} X^i Y^j Z^k W^l$.

- $C^\infty(M)$-linearity implies that R defines a tensor, $(3,1)$ tensor, or $(4,0)$ tensor.

Def 9 (Sectional curvature)

Let $P \subset T_pM$ be a plane, $u, v \in P$ linear indep vectors

Define the sectional curvature of P,

$$K = \frac{R(u,v,u,v)}{|u \wedge v|^2}, \quad |u\wedge v|^2 = |u|^2|v|^2 - \langle u, v\rangle^2$$

$$= \det\begin{pmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle x,y\rangle & \langle y,y\rangle \end{pmatrix}.$$

K does NOT dep. choice of u & v.

Pf: choose orthonormal base $e_1, e_2 \in P$.

$\forall\ u = u^1 e_1 + u^2 e_2, \quad v = v^1 e_1 + v^2 e_2$

$$R(u,v) = R_{1212}(u^1)^2(v^2)^2 + R_{1221} u^1u^2v^1v^2 + R_{2121}(u^2)^2(v^1)^2 + R_{2112} u^1u^2v^1v^2 = R_{1212}\cdot|u\wedge v|^2.$$

□

Lemma [10]: Sectional curvature determines Rie curvature.

$$6\,R(x,y,z,w) = \frac{\partial^2}{\partial s\,\partial t}\bigg|_{s=t=0}\Big(R(x+sz,\, y+tw,\, x+sz,\, y+tw) - R(x+sw,\, y+tz,\, x+sw,\, y+tz)\Big) \quad (**)$$

Pf: By linearity, coefficient of $s\cdot t$ on RHS,

$$R(z,w,x,y) + R(z,y,x,w) + R(x,w,z,y) + R(x,y,z,w)$$
$$- R(w,z,x,y) - R(w,y,x,z) - R(x,z,w,y) - R(x,y,w,z)$$
$$= 4\,R(x,y,z,w) - R(y,z,x,w) - R(z,x,y,w)$$
$$- R(w,x,z,y) - R(x,z,w,y)$$
$$= 6\,R(x,y,z,w). \qquad \square$$

Corollary [11]: If $K \equiv K_0$ at $p \in M$, then,

$$R(x,y,z,w) = K_0 \cdot \big(\langle x,z\rangle \langle y,w\rangle - \langle x,w\rangle\langle y,z\rangle \big).$$

Pf. $R(x,y,x,y) = K_0 \cdot |x \wedge y|^2 = K_0 \cdot \big(|x|^2 |y|^2 - \langle x,y\rangle^2 \big)$.

On right hand side of (**), coefficients of $s\cdot t$:

$$|(x+sz)\wedge(y+tw)|^2 - |(x+sw)\wedge(y+tz)|^2$$
$$= |x+sz|^2 |y+tw|^2 - |x+sw|^2 |y+tz|^2$$
$$- \langle x+sz, y+tw\rangle^2 + \langle x+sw, y+tz\rangle^2$$
$$= s\cdot t \cdot \big[4\langle x,z\rangle\langle y,w\rangle - 4\langle x,w\rangle\langle y,z\rangle - 2\langle x,y\rangle\langle z,w\rangle - 2\langle x,w\rangle\langle y,z\rangle$$
$$+ 2\langle x,y\rangle\langle z,w\rangle + 2\langle x,z\rangle\langle y,w\rangle \big] + \cdots \qquad \square$$

Prop 12. In coor. $x=(x^1,\cdots,x^n)$,

$$R_{ij}{}^{k}{}_{l}=\frac{\partial\Gamma_{jl}^{k}}{\partial x^i}-\frac{\partial\Gamma_{il}^{k}}{\partial x^j}+\Gamma_{jl}^{r}\Gamma_{ir}^{k}-\Gamma_{il}^{r}\Gamma_{jr}^{k}.$$

$$R_{ijkl}=-\frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}+\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}-\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}-\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}\right)$$

$$-g_{rs}\left(\Gamma_{ik}^{r}\Gamma_{jl}^{s}-\Gamma_{il}^{r}\Gamma_{jk}^{s}\right).$$

pf: • By definition

$$R_{ij}{}^{k}{}_{l}\frac{\partial}{\partial x^k}=\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}$$

$$=\nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma_{jl}^{k}\frac{\partial}{\partial x^k}\right)-\nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma_{il}^{k}\frac{\partial}{\partial x^k}\right)$$

$$=\left(\frac{\partial\Gamma_{jl}^{k}}{\partial x^i}-\frac{\partial\Gamma_{il}^{k}}{\partial x^j}+\Gamma_{jl}^{r}\Gamma_{ir}^{k}-\Gamma_{il}^{r}\Gamma_{jr}^{k}\right)\frac{\partial}{\partial x^k}.$$

• $$R_{ijkl}=\left\langle\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k}\right\rangle$$

$$=\frac{\partial}{\partial x^i}\left\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k}\right\rangle-\frac{\partial}{\partial x^j}\left\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k}\right\rangle\quad\cdots ①$$

$$-\left\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right\rangle+\left\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l},\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}\right\rangle\cdots ②$$

②: Γ terms

①: $$=\frac{\partial}{\partial x^i}\left(g_{kr}\Gamma_{jl}^{r}\right)-\frac{\partial}{\partial x^j}\left(g_{kr}\Gamma_{il}^{r}\right)$$

$$=\frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{\partial g_{jk}}{\partial x^l}+\cancel{\frac{\partial g_{kl}}{\partial x^j}}-\frac{\partial g_{jl}}{\partial x^k}\right)-\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial g_{ik}}{\partial x^l}+\cancel{\frac{\partial g_{kl}}{\partial x^i}}-\frac{\partial g_{il}}{\partial x^k}\right)$$

$$=\frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}-\frac{\partial^2 g_{jl}}{\partial x^k\partial x^i}+\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}-\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}\right).\quad\square$$

• Ricci curvature.

$$\mathrm{Ric}(X,Y) = \sum R(X, e_i, Y, e_i) = g^{ij} R\left(X, \frac{\partial}{\partial x^i}, Y, \frac{\partial}{\partial x^j}\right).$$

where $\{e_i\}$ is orthonormal frame.

In coor., write $R_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

then, $R_{ik} = g^{jl} R_{ijkl}$.

A Rie mfd (M, g) satisfying $\mathrm{Ric} \equiv \lambda \cdot g$, for some $\lambda \in \mathbb{R}$, is called an Einstein mfd.

• Scalar curvature.

$$s = \sum \mathrm{Ric}(e_i, e_i) = g^{ik} R_{ik}.$$

• Ric is contraction of R & g

s ... ... Ric & g.

Example 13: Euclidean space

$g_{ij} = \delta_{ij}$, $\Gamma_{ij}^k \equiv 0$, $R \equiv 0$.

Example 14 (Conformal change).

g: Rie metric. $\tilde{g} = e^f \cdot g$, for some $f \in C^\infty(M)$.

In coor. $x = (x^1, \cdots, x^n)$

Let g_{ij}, $\tilde{g}_{ij}$, Γ, $\tilde{\Gamma}$, R, $\tilde{R}$ be metric, ...