§1. Differential manifold (微分流形).

§1.1

<u>Definition 1</u> (n-dim diff mfd). A set M, together with a family of injective mappings $\varphi_\alpha: U_\alpha \to M$, $U_\alpha \subset \mathbb{R}^n$ open,

s.t.: (D_1). $\cup \varphi_\alpha(U_\alpha) = M$,

下面坐标系, 采用 φ_α, 不用 X_α

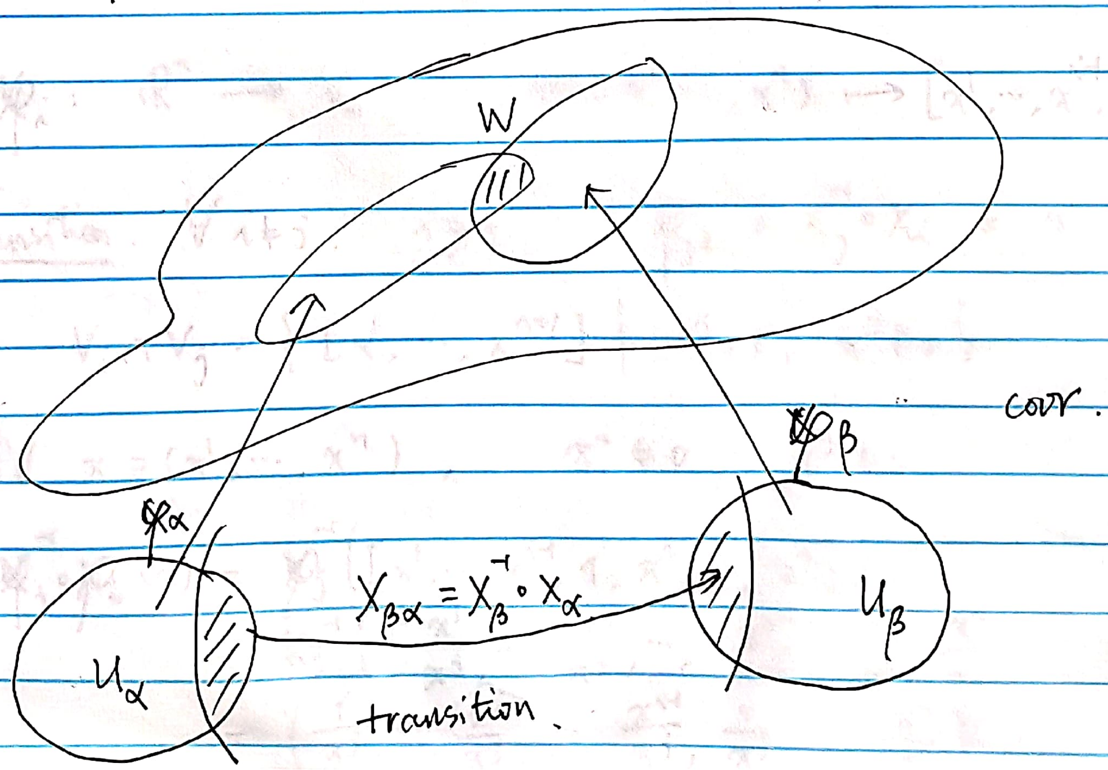
(D_2). If $W = \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \phi$, then:

① $\varphi_\alpha^{-1}(W)$, $\varphi_\beta^{-1}(W)$ are open sets of $\mathbb{R}^n$,

② the transition $\varphi_{\beta\alpha} = \varphi_\beta^{-1} \circ \varphi_\alpha : X_\alpha^{-1}(W) \to X_\beta^{-1}(W)$ is ~~differ~~ differentiable.

(D_3). The family $\{(U_\alpha, \varphi_\alpha)\}$ satisfying $(D_1) + (D_2)$ is maximal. #.

- $\varphi_\alpha : U_\alpha \to M$ is a coordinate of M
- $\{(U_\alpha, \varphi_\alpha)\}$ in the definition is called a diff structure on M.



Remark[1] **(Rmk)**: Given any family of coordinates $\{(U_\alpha, \varphi_\alpha)\}$ satisfying $(D_1)+(D_2)$, there exist diff structure compatible with each $(U_\alpha, \varphi_\alpha)$. The smallest diff structure compatible is called the diff. structure determined by $\{(U_\alpha, \varphi_\alpha)\}$. #.

Example[1]: Open sets of $\mathbb{R}^n$. Canonical diff. structure induced from $\mathbb{R}^n$.

$x=(x^1,\cdots,x^n)$.

Exp 2: Real projective space $\mathbb{R}P^n$.

Exp 3: 1 dim curves

line, circle

Exp 4: Conn Sum

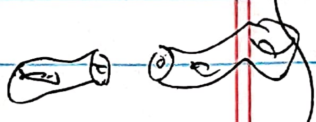
2 dim:

Σ_g.

Exp 5: 3 dim

Conn. Sum.

Toridal sum.



As set: $\mathbb{R}P^n = \{\text{lines of } \mathbb{R}^{n+1} \text{ passing through } 0\}$

$= \mathbb{R}^{n+1}\setminus\{0\} / \sim$. $\quad x \sim \lambda x, \ \lambda \in \mathbb{R}, \lambda \neq 0$.

Denote by $[x^1, \cdots, x^{n+1}]$, a pt of $\mathbb{R}P^n$.

Homogeneous coor. $\quad V_i = \{[x^1,\cdots,x^{n+1}] \mid x^i \neq 0\}$.

$\varphi_i : \mathbb{R}^n \longrightarrow V_i, \quad x=(x^1,\cdots,x^n) \longmapsto [x^1,\cdots,x^{i-1},1,x^i,\cdots,x^n]$

Transition: $\forall\, i\neq j,\ i>j,\quad \varphi_{ji} = \varphi_j^{-1}\circ\varphi_i = ?$

$V_i \cap V_j = \{[x^1,\cdots,x^{n+1}] \mid x^i \neq 0,\ x^j \neq 0\}$

$\forall\ x=(x^1,\cdots,x^n),\quad x^i\neq 0\quad x^j \neq 0$.

$$\varphi_j^{-1}\circ\varphi_i(x) = \varphi_j^{-1}([x^1,\cdots,x^{i-1},1,x^i,\cdots,x^n])$$
$$= \varphi_j^{-1}\left(\left[\frac{x^1}{x^j},\cdots,\frac{x^j}{x^j},\cdots,\frac{x^{i-1}}{x^j},\frac{1}{x^j},\frac{x^i}{x^j},\cdots,\frac{x^n}{x^j}\right]\right)$$
$$= \left(\frac{1}{x^j},\cdots,\widehat{\frac{x^j}{x^j}},\cdots,\frac{x^{i-1}}{x^j},\frac{1}{x^j},\cdots,\frac{x^n}{x^j}\right)$$

Def² (Diff mapping). M^m, N^n: diff. mfds.

A map $f: M \to N$ is called differentiable at $p \in M$, if $\exists$ coor at p, and $f(p)$, say $\varphi: \mathcal{U} \to M$ & $\psi: V \to N$, s.t.:

$$\psi^{-1} \circ f \circ \varphi: \mathcal{U} \longrightarrow V \subset \mathbb{R}^n$$

is differentiable at $\varphi^{-1}(p)$.

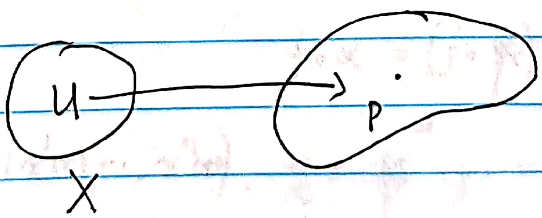
Rmk²: If f is diff at p, restricted on a coordinate, then it is diff. restricted on any coordinate.

Exp³: diff functions: $f: M \to \mathbb{R}$.

diff. curves: $c: I \to M$, $I \subset \mathbb{R}$ open.

• $\mathcal{D}_p = \{ f: M \to \mathbb{R}, \text{ diff at } p \in M \}$.

$\mathcal{D}_p \neq \phi$:



coor. $u \subset \mathbb{R}^n$.

Take $\eta \in C_0^\infty(\mathcal{U})$, then

$$f(x) = \begin{cases} \eta \circ X^{-1}(x), & x \in X(\mathcal{U}), \\ 0, & x \notin X(\mathcal{U}) \end{cases}$$

is diff. on M.

§1.2 Tangent bundle

Def³ (Tangent space).

Let $\alpha: (-\varepsilon, \varepsilon) \to M$, $\alpha(0) = p$, be a diff curve.

The tangent vector, at $t=0$, of α is a map

$$\alpha'(0) : \mathcal{D}_p \to \mathbb{R}, \qquad f \mapsto \frac{d}{dt}(f\circ\alpha)\Big|_{t=0}.$$

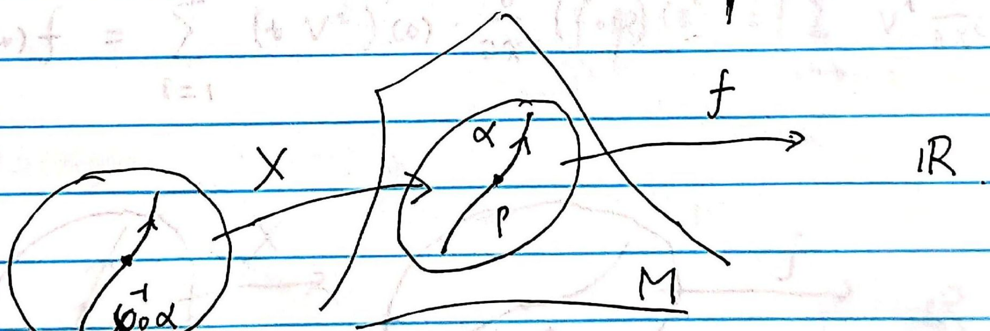
Denote $T_pM = \{ \alpha'(0) \mid \alpha: (-\varepsilon,\varepsilon) \to M \text{ diff curve, } \alpha(0) = p \}$.

Each element is called a tangent vector at p.

- $\mathcal{D}_p$ is a linear space over $\mathbb{R}$, $\alpha'(0) : \mathcal{D}_p \to \mathbb{R}$ is linear.

  i.e., $\alpha'(0) \in \mathcal{D}_p^*$, dual space.

- Representation in local coor. $\qquad X: U \to M$, coor. at p.



$$f\circ\alpha = (f\circ X)\circ(X^{-1}\circ\alpha).$$

Write $X^{-1}\circ\alpha(t) = (x^1(t), \cdots, x^n(t))$. $q = X^{-1}(p) = X^{-1}\circ\alpha(0)$.

$$\alpha'(0)f = \frac{d}{dt}\Big|_{t=0}(f\circ\alpha) = \frac{d}{dt}\Big|_{t=0}(f\circ X)(x^1(t), \cdots, x^n(t)),$$

$$= \sum_{i=0}^{n} (x^i)'(0) \cdot \frac{\partial}{\partial x^i}(f\circ X)(q).$$

Lemma 2: T_pM is a linear space of dim n.

Pf: Take a coor. $\varphi : U \to M$, $p \in \varphi(U)$, $q = \varphi^{-1}(p)$.

Let $x = (x^1, \cdots, x^n)$ be coor. on $U \subset \mathbb{R}^n$.

Define: $\frac{\partial}{\partial x^i} : D_p \to \mathbb{R}$, $f \mapsto \frac{\partial}{\partial x^i}(f\circ\varphi)(q)$.

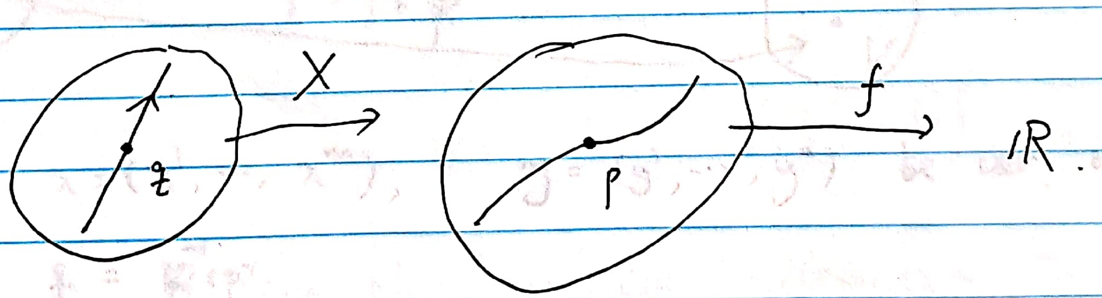
Claim: $T_pM = \mathrm{Span}\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$.

- $T_pM \subset \mathrm{Span}\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$, according to local rep, above
- $T_pM \supset \mathrm{Span}\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$.

For each $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$, $v^i \in \mathbb{R}$, define a curve

$\alpha(t) = \varphi(q + t\bar{v})$, $t \in (-\varepsilon, \varepsilon)$, $\bar{v} = (v^1, \cdots, v^n) \in \mathbb{R}^n$

$$\alpha'(0) f = \sum_{i=1}^n (t v^i)'(0) \cdot \frac{\partial}{\partial x^i}(f\circ\varphi)(q) = \left(\sum_{i=0}^n v^i \frac{\partial}{\partial x^i}\right) f = vf$$



- In coor., $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ forms a basis of T_pM.

Proposition 2. (definition of tangent map / differential).

Suppose $f: M^m \to N^n$ is diff at $p \in M$.

Define $df_p : T_pM \to T_{f(p)}N$ as follows:

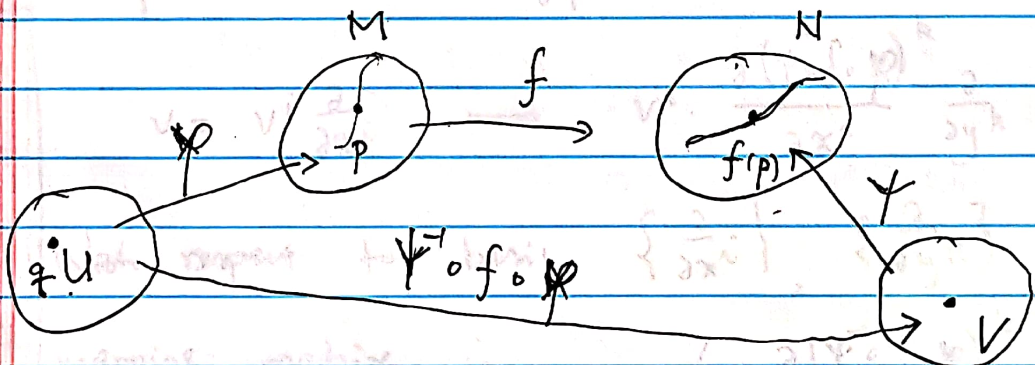
$\forall\, v \in T_pM$, choose a curve $\alpha: (-\varepsilon, \varepsilon) \to M$, $\alpha(0) = p$, $\alpha'(0) = v$, then put $df_p(v) = (f\circ\alpha)'(0)$

$df_p(v): \mathcal{D}_{f(q)} \to \mathbb{R}$, $\quad df_p(v)\, h = \left.\dfrac{d}{dt}\right|_{t=0} h\circ f\circ\alpha(t)$.

The definition does not dep. on choice of α.

pf: Choose local coor. $\varphi: U \to M$, $\psi: V \to N$.

$p \in \varphi(U)$, $f(p) \in \psi(V)$.



Let $x = (x^1, \cdots, x^m)$, $y = (y^1, \cdots, y^n)$ be coor. on U, V.

$q = \varphi^{-1}(p)$.

$\left\{\dfrac{\partial}{\partial x^i}\right\}_{i=1}^{m}$, $\left\{\dfrac{\partial}{\partial y^k}\right\}_{k=1}^{n}$: basis of T_pM, $T_{f(p)}N$.

Claim: $\forall\, v = \sum_i v^i \dfrac{\partial}{\partial x^i}$, $df_p(v) = \sum_{i,k} v^i \dfrac{\partial(\psi^{-1}\circ f\circ\varphi)^k}{\partial x^i}(q)\, \dfrac{\partial}{\partial y^k}$

So $df_p(v)$ does not dep. on choice of α.

Proof of the claim: Take α, curve on M, $\alpha(0)=p$, $\alpha'(0)=$

Then, $\forall h \in \mathcal{D}_{f(p)}$, by chain rule,

$$(f\circ\alpha)'(0)\,h = \left.\frac{d}{dt}\right|_{t=0} h\circ f\circ\alpha(t) = \left.\frac{d}{dt}\right|_{t=0} (h\circ\psi)\circ\underline{(\psi^{-1}\circ f\circ\varphi)\circ(\varphi^{-1}\circ\alpha)(t)}$$

$$= \sum_k \left[(\psi^{-1}\circ f\circ\varphi)^k\circ(\varphi^{-1}\circ\alpha)\right]'(0)\cdot\frac{\partial}{\partial y^k}(h\circ\psi)$$

$$\boxed{\begin{gathered}\alpha'(0) = v = v^i\frac{\partial}{\partial x^i}.\\ (x^{-1}\circ\alpha)'(0) = (v^1,\cdots,v^m)\end{gathered}}$$

$$= \sum_k\sum_i v^i\frac{\partial(\psi^{-1}\circ f\circ\varphi)^k}{\partial x^i}\cdot\frac{\partial}{\partial y^k}(h\circ\psi)$$

$$= \left(\sum_{i,k} v^i\cdot\frac{\partial(\psi^{-1}\circ f\circ\varphi)^k}{\partial x^i}(q)\cdot\frac{\partial}{\partial y^k}\right)\cdot h \qquad \square$$

- $df_p : T_pM \to T_{f(p)}N$ is linear.

$$v = v^i\frac{\partial}{\partial x^i} \longmapsto v^i\cdot\frac{\partial(\psi^{-1}\circ f\circ\varphi)^k}{\partial x^i}\frac{\partial}{\partial y^k}.$$

$$\boxed{\begin{gathered}\text{In particular, in diff cor.}\\ \varphi: U\to M\\ \psi: V\to M\\ \frac{\partial}{\partial x^i} = \frac{\partial(\psi^{-1}\circ\varphi)^k}{\partial x^i}\cdot\frac{\partial}{\partial y^k}\\ \text{on } \varphi(U)\cap\psi(V)\end{gathered}}$$

With respect to basis $\left\{\frac{\partial}{\partial x^i}\right\}$, $\left\{\frac{\partial}{\partial y^k}\right\}$, the associated mapping matrix is $\left(\frac{\partial(\psi^{-1}\circ f\circ\varphi)^k}{\partial x^i}(q)\right)_{m\times n}$.

- If $f: M\to N$ is diff everywhere, we call f a diff map. If $m=n$, $df_p : T_pM \to T_{f(p)}N$ is an isomorphism, then $\exists$ neighborhood $W_1\subset M$, $W_2\subset N$, $f|_{W_1}: W_1\to W_2$ a <u>diffeomorphism</u>.

Def[4] (Diffeomorphism) 微分同胚.

A diff map $\varphi: M \to N$ is a diffeomorphism, if φ is bijective and $\varphi^{-1}: N \to M$ is diff. #.

Denote: $M_1 \overset{\text{diff}}{\cong} M_2$

- If $f_1: M_1 \to M_2$, $f_2: M_2 \to M_3$

Chain rule f_1 diff at $p \in M_1$, f_2 diff at $f_1(p) \in M_2$,

then $f_2 \circ f_1: M_1 \to M_3$ diff at p

&: $d(f_2 \circ f_1)_p = d(f_2)_{f_1(p)} \circ d(f_1)_p : T_p M_1 \to T_{f_2 \circ f_1(p)} M_3$

- If $\varphi: M \to N$ is diffeomorphism, then

$$d(\varphi^{-1} \circ \varphi)_p = d\varphi^{-1}_{\varphi(p)} \circ d\varphi_p = id: T_pM \to T_pM.$$

$\Rightarrow$ $d\varphi_p$ is isomorphic at each $p \in M$.

Exercise: Check that: ① $(0,1) \overset{\text{diff}}{\cong} \mathbb{R}$.

② $(0,1)$ 与单位圆 $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ 不微分同胚.

Def[5] (Tangent bundle) TDK.

Let M^n be an n-dim diff mfd, with diff structure $\{(U_\alpha, \varphi_\alpha)\}$.

Put $TM = \coprod_{p\in M} T_pM$. For $\forall \alpha$, define corr.

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow TM,$$

$$(x_\alpha, v) \longmapsto \xi = v^i \frac{\partial}{\partial x_\alpha^i} \in T_{\varphi_\alpha(x_\alpha)}M.$$

where $x_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$ is corr. on U_α,

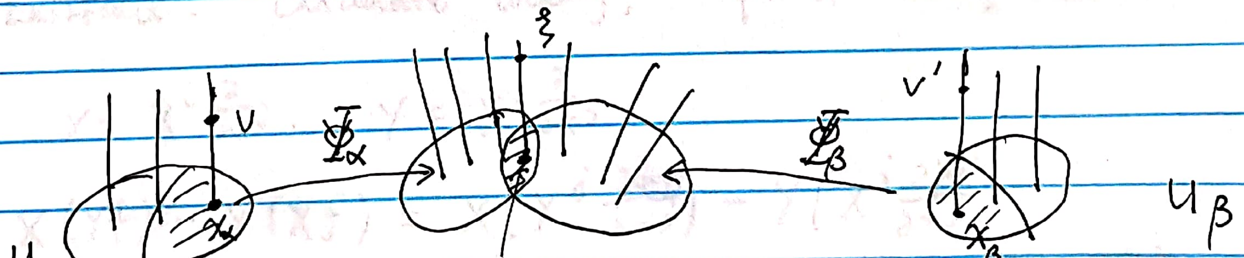
$\left\{ \frac{\partial}{\partial x_\alpha^i} \right\}$ is basis of T_pM, $\forall p \in \varphi_\alpha(U_\alpha)$.

The diff mfd TM, with diff structure determined by $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$ is called tangent bundle of M.

T_pM is called the fibre over p of the bundle.

$\pi : TM \to M$, $v \mapsto p$, $\forall v \in T_pM$,
is called the projection of the bundle.

- **Transition**: $\Phi_{\beta\alpha} = \Phi_\beta^{-1} \circ \Phi_\alpha = (\varphi_{\beta\alpha}, d\varphi_{\beta\alpha})$.



$p = \varphi_\alpha(x_\alpha) = \varphi_\beta(x_\beta)$

$$(x_\alpha, v) \longmapsto \xi = v^i \frac{\partial}{\partial x_\alpha^i} = v^i \frac{\partial(\varphi_\beta^{-1}\circ\varphi_\alpha)^k}{\partial x_\alpha^i} \cdot \frac{\partial}{\partial x_\beta^k} \mapsto (x_\beta, \underset{\overset{\|}{v'}}{d\varphi_{\beta\alpha}(v)})$$

§1.3.

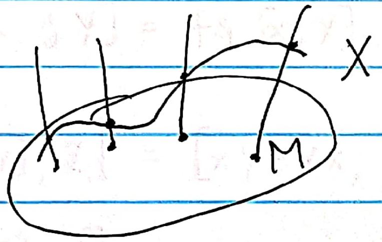
Def[6] (Vector field) 向量场,

A vector field on a diff mfd M^n, is a diff map

$$X : M \to TM, \quad \text{s.t.,} \quad X(p) \in T_pM, \quad \forall p \in M. \qquad \#$$

- In local cor. $\varphi : U \to M$,

$$X = X^i \frac{\partial}{\partial x^i}, \quad \text{each } X^i \text{ diff on } U.$$



- $\varphi : M \to N$ diffeomorphism. for $\forall$ vector field X on M,

$d\varphi(X)$ is a vector field on N.

Lemma[4]: For any Vector fields X, Y. $\exists!$ vector field Z,

s.t., $\quad Zf = X(Yf) - Y(Xf), \quad \forall$ diff funct. f.

Pf: • Uniqueness: trivial.

• Existence: calculate locally, $\varphi : U \to M$, $x = (x^1, \cdots, x^n)$.

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}.$$

$$X(Yf) - Y(Xf) = X\left(Y^j \frac{\partial (f\circ\varphi)}{\partial x^j}\right) - Y\left(X^j \frac{\partial f\circ\varphi}{\partial x^j}\right)$$

$$= X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f\circ\varphi}{\partial x^j}\right) - Y^i \frac{\partial}{\partial x^i}\left(X^j \frac{\partial f\circ\varphi}{\partial x^j}\right)$$

$Z = Z^i \frac{\partial}{\partial x^i}$.

$$= \underbrace{\left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)}_{Z^j} \frac{\partial}{\partial x^i}(f\circ\varphi). \qquad \square$$

Def[7] (Lie bracket) 李括积.

$[X, Y] = XY - YX$, $\forall$ vector fields X, Y.

Prop: ① $[X, Y] = -[Y, X]$

② $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

③ $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ Jacobi iden.

④ $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

Pf: ③ $[[X,Y], Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$.

$[[Y, Z], X] = \cdots = YZX - ZYX - XYZ + XZY$

$[[Z, X], Y] = \cdots = ZXY - XZY - YZX + YXZ$.

④ $\forall$ funct. h,

$[fX, gY]h = (fX)(gYh) - gY(fXh)$

$= f[X(g)Y(h) + gXYh] - g[Yf \cdot Xh + fYXh]$

$= fg(XYh - YXh) + fX(g)Y(h) - gY(f)X(h)$. □

§1.4 Partition of unity.

Countable axiom & partition of unity. 单位分解.

In the course, mfds satisfy the countable axiom:

$\exists$ countable coordinate corving.

- A covering $\{U_\alpha\}$ of M is said to be locally finite, if any $p \in M$ has a neighborhood W which intersects finite U_α's.

- Let $\{U_\alpha\}$ be an open, locally finite covering of M.

A partition of unity, subordinated to $\{U_\alpha\}$, is a family of diff functions, $\{f_\alpha\}$, s.t:

①. $\operatorname{supp} f_\alpha \subset U_\alpha$,

②. $\sum_\alpha f_\alpha \equiv 1$ on M.

Prop 3: Any diff mfd satisfies partion of unity property. #.

Exp 4: $\mathbb{R}^2 = \coprod_{x \in \mathbb{R}} \{ (x, y) \mid y \in \mathbb{R} \}$.

Exercise 2: Construct $\eta \in C_0^\infty(B_2(0))$, $\eta \equiv 1$ on $B_1(0)$, $\eta \geq 0$.

$$B_r(0) = \{ x \in \mathbb{R}^n \mid |x| < r \}.$$